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AN INTEGRAL EQUATION FOR THE PROBLEM OF SMOOTH INDENTATION OF ORTHOTROPIC BEAMS

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Abstract—An integral equation for the problem of smooth contact between a rigid indenter and an orthotropic beam is formulated using an approximate Green's function for surface displacements in the beam, which is obtained as the sum of half-plane solutions for surface displacements, and beam theory deflections. The left and right Green's functions for beam slope are approximated as a single function with continuous derivatives using a least squares error procedure. A closed form solution is obtained for the integral equation. Solutions are obtained for two cases: symmetric indentation of simply supported orthotropic beams and indentation of cantilever beams. Closed form expressions are derived for contact stresses and the contact force–contact length relation in terms of a nondimensional beam parameter B and a nondimensional contact parameter β.

NOTATION

B	nondimensional beam parameter $= \dfrac{\pi}{32}\dfrac{D_2}{D_1}\dfrac{l^3}{h^3}$
b	beam width
$2c$	contact length
$\tilde{c}$	c/l
D_1, D_2	stiffness coefficients (functions of elastic constants)
E_1, E_2	Young's modulus in 1 and 2 directions
$g(x, \xi)$	Green's function
g'_b	Green's function for beam slope
g'_h	Green's function for half-plane boundary slope
G_{12}	shear modulus
h	beam thickness
l	beam length
P	contact force
$\hat{P}$	nondimensional contact force $= (4PT)/(\pi D_2 b l^2)$
p	contact stresses
$\bar{p}$	nondimensional contact stress $= (pR)/(D_2 l)$
$\hat{p}$	nondimensional contact stress $= (\pi bcp)/(2P)$
q_n	coefficients in the Chebyshev polynomial for $\bar{p}$
R	indenter radius of curvature
T_n	Chebyshev polynomials of first kind
U_n	Chebyshev polynomials of second kind
x, y	coordinate axes
$\bar{x}$	$(x - x_c)/c$
x_i	x-coordinate of indenter center
x_c	x-coordinate of contact center
$\tilde{x}_i$	x_i/l
$\tilde{x}_c$	x_c/l
β	nondimensional contact parameter $= 8.75\, B\tilde{c}^4$
Δ	indenter y-displacement
θ_{h0}	boundary slope of half-plane at $x = 0$
ν_{12}	Poisson's ratio
ξ	dummy variable
$\bar{\xi}$	$(\xi - x_c)/c$

1. INTRODUCTION

The problem of smooth indentation of beams of finite length by a rigid cylindrical indenter has been studied by several authors. Keer and Ballarini (1983), Keer and Miller (1983) and Keer and Schonberg (1986) approached the problem via a local–global technique. Their methods of analysis superpose an infinite-layer solution, derived through the use of integral transforms, on a pure-bending beam-theory solution. An integral equation is obtained for

The integral equation for the symmetric contact problem is

$$b\int_{-c}^{+c} p(\xi)g(x,\xi)\,\mathrm{d}\xi = \Delta - \frac{x^2}{2R}, \tag{1}$$

where $p(x) = -\sigma_{yy}(x,0)$ is the unknown contact stress beneath the indenter, $2c$ is the contact length, $g(x,\xi)$ is the Green's function for surface displacements in a beam, R is the indenter radius of curvature and Δ is the y-displacement of the indenter. It should be noted that eqn (1) assumes that the indenter has a parabolic profile. If the indenter is circular, eqn (1) is valid only for $c/R \ll 1$. The unknown displacement Δ can be eliminated by differentiating eqn (1) with respect to x. Thus the integral equation takes the form

$$b\int_{-c}^{+c} p(\xi)g'(x,\xi)\,\mathrm{d}\xi = -x/R, \tag{2}$$

where a prime denotes differentiation with respect to x. It was shown in Sankar (1987b) that an approximate $g(x,\xi)$ can be obtained by adding $g_h(x,\xi)$, the Green's function for surface displacements in an orthotropic half-plane, and $g_b(x,\xi)$, the Green's function for beam deflections. Thus eqn (2) can be written as

$$b\int_{-c}^{+c} p(\xi)[g_h'(x,\xi)+g_b'(x,\xi)]\,\mathrm{d}\xi = -x/R, \tag{3}$$

where g_h' is given by (Sankar, 1987b):

$$g_h'(x,\xi) = \frac{2}{\pi b D_2(\xi - x)}. \tag{4}$$

For the case of plane stress parallel to the x–y plane, $D_2 = 2E_2/(\lambda_1+\lambda_2)$, where λ_1 and λ_2 are the roots of the characteristic equation $S_{11}\lambda^4-(2S_{12}+S_{66})\lambda^2+S_{22} = 0$, $S_{11} = 1/E_1$, $S_{22} = 1/E_2$, $S_{66} = 1/G_{12}$, $S_{12} = -\nu_{12}/E_1$, E_1 and E_2 are the Young's moduli in the 1 and 2 directions, G_{12} is the shear modulus in the 1–2 plane and ν_{12} is the Poisson's ratio. For the case of plane strain, D_2 will be slightly different (Lekhnitskii, 1981).

The beam Green's function for the slope is

$$g_b'(x,\xi) = (l^2/D_1bh^3)[2(\xi/l)^3+6(\xi/l)(x/l)^2-3(x/l)+(\xi/l)+3\phi(x,\xi)], \tag{5}$$

where $D_1 = E_1$ for plane stress and $D_1 = E_1/(1-\nu_{12}^2)$ for plane strain. The function $\phi(x,\xi)$ is defined as

$$\phi(x,\xi) = -\left(\frac{x-\xi}{l}\right)^2, \quad x<\xi,$$

and

$$\phi(x,\xi) = +\left(\frac{x-\xi}{l}\right)^2, \quad x>\xi. \tag{6}$$

It may be noted that ϕ is an odd function of the argument $(x-\xi)$, and can be expanded in terms of odd powers of $(x-\xi)$. We shall approximate ϕ by a single function of the type $c_1(x-\xi)+c_2(x-\xi)^3$. The constants c_1 and c_2 depend upon the degree of accuracy and the range of $(x-\xi)$ over which the approximation is sought. In the present study, the maximum contact length is assumed to be given by $2c = 0.5l$. We will therefore approximate $\phi(x,\xi)$ such that the error is a minimum over the range $-0.5 < (x-\xi)/l < +0.5$. Using the least

$$\{P\}\frac{1}{\pi}\int_{-1}^{+1}\frac{(1-t^2)^{-1/2}T_n(t)}{(t-s)}\,\mathrm{d}t = U_{n-1}(s), \quad n = 0,1,2,\ldots, \text{ and } |s|<1,$$

where $\{P\}$ denotes the Cauchy principal value and $U_n(s)$ are the Chebyshev polynomials of the second kind defined by $U_{-1}(s) = 0$, $U_0(s) = 1$, $U_1(s) = 2s$, $U_2(s) = 4s^2-1$, $U_3(s) = 8s^3-4s$ and $U_4(s) = 16s^4-12s^2+1$. The second term of the integral in eqn (11) can be easily evaluated using the orthogonality condition

$$\int_{-1}^{+1}(1-t^2)^{-1/2}T_m(t)T_n(t)\,\mathrm{d}t = \begin{cases} 0, & n \neq m \\ \pi/2, & n = m \neq 0 \\ \pi, & n = m = 0. \end{cases}$$

Thus eqn (11) takes the form

$$2q_1+4q_2\bar{x}+q_3(8\bar{x}^2-2)+q_4(16\bar{x}^3-8\bar{x})+B\{q_0(-81\tilde{c}^2\bar{x}+140\tilde{c}^4\bar{x}^3+210\tilde{c}^4\bar{x}) \\ +q_1(8.5\tilde{c}^2-114\tilde{c}^4\bar{x}^2-28.5\tilde{c}^4)+q_2(105\tilde{c}^4\bar{x})+q_3(-9.5\tilde{c}^4)\} = -\tilde{c}\bar{x}. \quad (13)$$

By equating the coefficients of $\bar{x}^0,\ldots,\bar{x}^3$ on both sides of eqn (13), we obtain four equations (14)–(17) in the unknowns $q_0,\ldots,q_4$.

$$\{2+B(8.5\tilde{c}^2-28.5\tilde{c}^4)\}q_1+(-9.5B\tilde{c}^4-2)q_3 = 0 \quad (14)$$

$$B(210\tilde{c}^4-81\tilde{c}^2)q_0+(4+105B\tilde{c}^4)q_2-8q_4 = -\tilde{c} \quad (15)$$

$$(-114B\tilde{c}^4)q_1+8q_3 = 0 \quad (16)$$

$$104B\tilde{c}^4q_0+16q_4 = 0. \quad (17)$$

The fifth equation (18) is obtained from the fact that the contact stresses vanish at the ends of contact zone, i.e.

$$q_0+q_1+q_2+q_3+q_4 = 0. \quad (18)$$

The solution of eqns (14)–(18) is as follows:

$$q_0 = \tilde{c}/(4+81B\tilde{c}^2-210B\tilde{c}^4-918.75B^2\tilde{c}^8), \quad (19)$$

$$q_1 = q_3 = 0, \quad (20)$$

$$q_2 = (\beta-1)q_0, \quad (21)$$

and

$$q_4 = -\beta q_0, \quad (22)$$

where β is a nondimensional parameter defined as $\beta = 8.75B\tilde{c}^4$.

Contact stresses

The contact force is given by

$$P = b\int_{-c}^{+c}p(x)\,\mathrm{d}x,$$

and, using eqn (12), we obtain

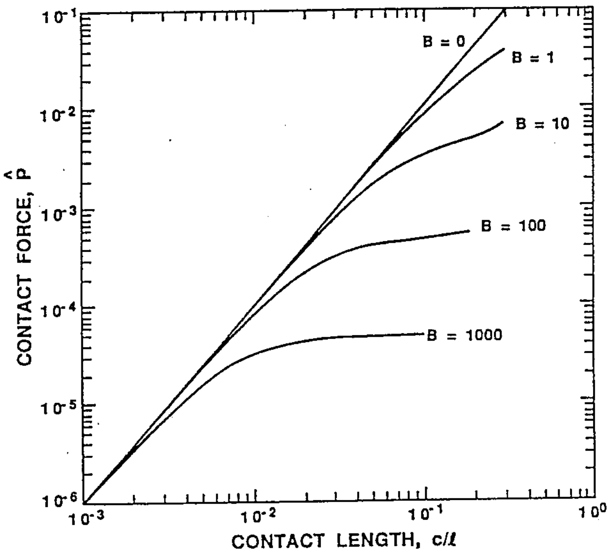

Fig. 4. Contact force–contact length relation in simply supported beams.

3. INDENTATION OF AN ORTHOTROPIC CANTILEVER BEAM

In this section we consider the case of a cantilever beam, as shown in Fig. 5. The indenter location is given by x_i. Initial contact will be a line contact at $x = x_i$. As the load is applied, c will increase, and the center of contact defined by x_c will move towards the fixed end of the beam. Thus an additional unknown x_c is introduced. However, the contact stresses will be unsymmetric about the center of contact length, and so we have one more equation which states that the contact stresses vanish at the left end of the contact region too.

There is another important difference between symmetric and nonsymmetric cases. The solution for y-displacements in the half-plane contains arbitrary terms for translation and rotation, which means that the expression for the boundary slope of the half-plane will contain an arbitrary constant. In the case of symmetric contact, the rotation term can be assumed to be zero. In the case of the cantilever beam problem, this difficulty can be overcome by subtracting $g_h'(0, \xi)$ from $g_h'(x, \xi)$ in the integral equation (3). This means that we are measuring the boundary slope relative to the slope at $x = 0$. Thus eqn (3) will become

$$b \int_{x_c-c}^{x_c+c} p(\xi)[g_h'(x, \xi) - g_h'(0, \xi) + g_b'(x, \xi)] \, d\xi = -(x - x_i)/R. \tag{27}$$

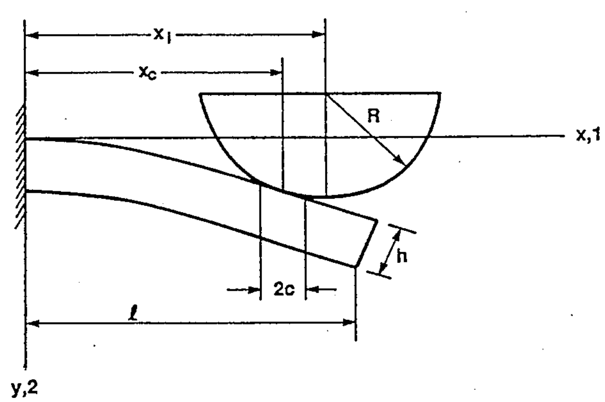

Fig. 5. Indentation of an orthotropic cantilever beam.

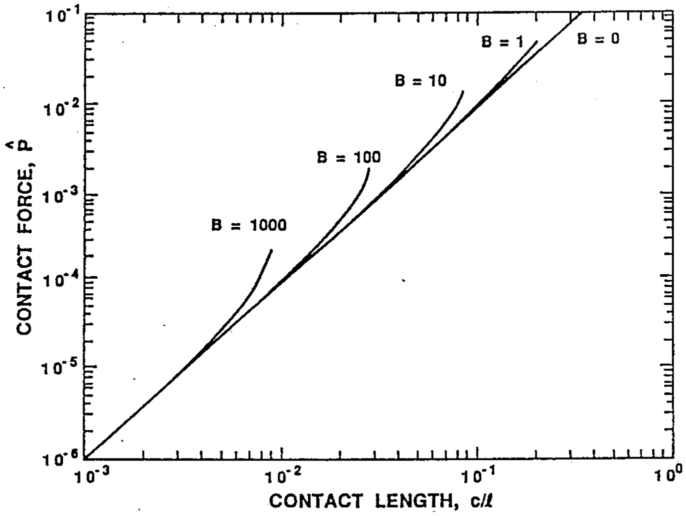

Fig. 7. Contact force–contact length relation in cantilever beams.

$$B\tilde{c}^2(15+210\tilde{c}^2)q_0+96B\tilde{c}^3q_1+(4+105B\tilde{c}^4)q_2-8q_4=-\tilde{c} \tag{32}$$

$$-96B\tilde{c}^3q_0-210B\tilde{c}^4q_1+8q_3=0 \tag{33}$$

$$140B\tilde{c}^4q_0+16q_4=0 \tag{34}$$

$$q_0+q_1+q_2+q_3+q_4=0 \tag{35}$$

$$q_0-q_1+q_2-q_3+q_4=0. \tag{36}$$

Equations (33)–(36) can be used to solve for $q_1,\ldots,q_4$ in terms of q_0. The results are: $q_1=(-12B\tilde{c}^3)q_0/(1+3\beta)$, $q_2=(-1+\beta)q_0$, $q_3=-q_1$, and $q_4=-\beta q_0$.

Contact force–contact length relation

Substituting for $q_1,\ldots,q_4$ in terms of q_0 in eqn (32), one can obtain a relation between q_0 and $\tilde{c}$. In terms of the nondimensional contact force $\hat{P}$, the $\hat{P}$–$\tilde{c}$ relation takes the form

$$\hat{P}=\tilde{c}^2\bigg/\left\{1-3.75B\tilde{c}^2-6\beta+\frac{228B^2\tilde{c}^6}{(1+3\beta)}-3\beta^2\right\}. \tag{37}$$

The $\hat{P}$–$\tilde{c}$ relations for various values of B are plotted in Fig. 7. $B=0$ corresponds to the half-plane. It is interesting to note that unlike the simply supported beam, as the contact length increases, the load required for a given contact length is more than that in the half-plane. This is because of the convex shape of the deformed beam. However, the beam curvature effect is not as pronounced as in the case of a simply supported beam (see Fig. 4).

Contact stresses

Substituting for $q_1,\ldots,q_4$ in terms of q_0 in eqn (12), the nondimensional contact stresses can be written as

$$\hat{p}=\sqrt{1-\bar{x}^2}\left[1-\beta(1-4\bar{x}^2)-\frac{24B\tilde{c}^3\bar{x}}{1+3\beta}\right]. \tag{38}$$

The contact stress distribution is unsymmetric about the contact center. A sample contact

$$\tilde{x}_c = (-1+\sqrt{1+192\hat{P}B\tilde{x}_i})/(96\hat{P}B). \tag{39}$$

In Fig. 9, solid circles represent the relation between $\hat{P}$ and $\tilde{x}_c$ obtained from eqn (39). It may be seen that eqn (39) provides a simple method of finding $\tilde{x}_c$ for a given contact force.

4. SUMMARY

The approximate Green's function method described in this paper provides a closed form solution for the problem of contact between a rigid indenter and an orthotropic beam. The dimensionless beam parameter B and the contact parameter β seem to reflect the effects of beam dimensions, degree of orthotropy of the beam material and contact length to beam length ratio on the contact behavior of the beam. Equations (24) and (25) describe the contact behavior of a simply supported orthotropic beam. In the case of cantilever beams, eqn (39) provides a simple expression for determining the contact center, and eqns (37) and (38) can be used to determine the contact length and the contact stresses. Extension of the present method to other types of beam support is straightforward.

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