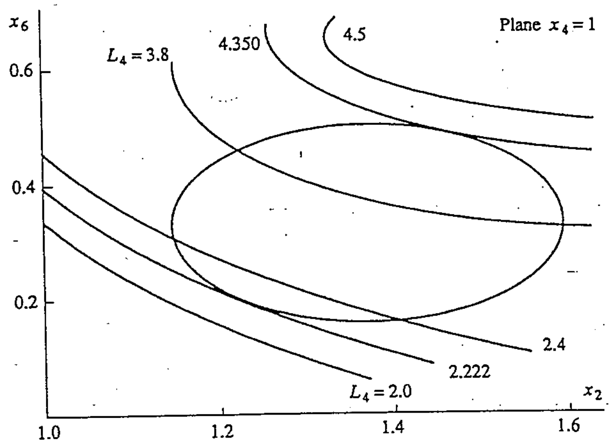

Fig. 2 Intersection patterns in the plane $x_4 = 1$ representing the loci of the subsystem displacements

immobile and possesses only first-order virtual mobility; it is a first-order compound infinitesimal mechanism with three virtual degrees-of-freedom and no kinematic ones.

(2) The two cones touch along a generator, thus determining the displacement ratio and the corresponding direction of compatible displacements of the two subsystems. This outcome is inconclusive in that it only identifies a second-order virtual displacement, but leaves the question on kinematic mobility unresolved. At this stage it is only clear that the constraints are compatible to at least the second order and a higher-order analysis is required for evaluating the system mobility.

(3) The cones intersect along two generators, thereby determining two directions (displacement ratios) along which the displacements of the two singular subsystems are compatible. With these displacements possible, the system is kinematically mobile, it is a finite mechanism with one kinematic degree-of-freedom in each to the two distinct deformation modes characterized by the two displacement ratios.

All three of the described situations are observed in the course of varying the length L_4 in the example system. As discussed above, the problem size is reduced by switching to a plane intersecting one of the cones over an ellipse. A suitable plane is given by $x_4 = 1$; introducing this into (9) and (10) and setting the resulting forms to zero yields

$$q_1 = 3x_2^2/2 - 2x_2 + 3/2 - x_6 + (1/2 - 1/L_4)x_6^2 = 0, \qquad (11)$$

$$q_2 = 3x_2^2/2 - x_2 + 3/2 - 2x_6 + (1 - 1/3.7)x_6^2 = 0. \qquad (12)$$

The length $L_8 = 3.7$ has been chosen such that the graph of Eq. (12), defining the displacement domain for the second subsystem, is a relatively small ellipse (Fig. 2). This enables the curve given by (11) to be shifted relative to the ellipse by changing L_4. At $L_4 = 2$ the curve is a parabola not intersecting the ellipse, which produces outcome (1). With an increasing L_4, (11) becomes an ellipse that first touches (12) (outcome 2); intersects it (3); again touches it (2) and, finally, disengages once again (1).

Note in conclusion that unprestressable compound infinitesimal mechanisms, along with the only class of unprestressable systems with infinitesimal mobility—even-order mechanisms—would require perfect geometric precision for their implementation. A slight deviation from the nominal bar lengths, while not precluding the possibility of assembling these systems, would restore them to one of the two generic types—geometrically invariant (with an ill-conditioned stiffness matrix) or variant (with a very small domain of finite displacements).

Conclusions

(1) Prestressability, i.e., the existence of stable virtual self-stress, as a statical criterion of immobility in underconstrained systems, is only sufficient. All even-order infinitesimal mechanisms and some compound infinitesimal mechanisms are unprestressable, yet kinematically immobile.

(2) The class of unprestressable systems with *first-order* infinitesimal mobility is confined to compound infinitesimal mechanisms. Remarkably, these include some topologically inadequate underconstrained systems (i.e., ones, with the number of constraints *smaller* than the total number of degrees-of-freedom, $C < N$).

(3) Capitalizing on a geometric interpretation of indefinite quadratic forms as convex conical surfaces, analysis of compound mechanisms reduces to finding the common intersection of several cones with one (hyper)plane in the space of virtual displacements.

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A Beam Theory for Laminated Composites and Application to Torsion Problems

Bhavani V. Sankar[22]

A theory for laminated composite beams is derived from the shear deformable laminated plate theory. The displacement field in the beam is derived by retaining the first-order terms in the Taylor series expansion for the plate midplane deformations in the width coordinate. The displacements in the beam are expressed in terms of three deflections, three rotations, and one warping term. The equilibrium equations are assumed to be satisfied in an average sense over the width of the beam. This introduces a new set of force and moment resultants for the beam. The principle of minimum potential energy is applied to derive the equilibrium equations and boundary conditions. A closed-form solution is derived for the problem of torsion of a specially orthotropic laminated beam.

Derivation of a Composite Beam Theory

Consider a laminated composite beam shown in Fig. 1. The midplane displacements u_0, v_0, w_0, and rotations ψ_x and ψ_y can be expanded in the form of a Taylor series in y. Retaining only the first-order terms in y, we obtain

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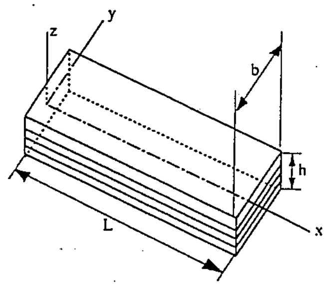

Fig. 1 Laminated composite beam

$$u_0(x,y) = U(x) + yF(x) \quad (1)$$

$$v_0(x,y) = V(x) + yG(x) \quad (2)$$

$$w_0(x,y) = W(x) + y\theta(x) \quad (3)$$

$$\psi_x(x,y) = \phi(x) + y\alpha(x) \quad (4)$$

$$\psi_y(x,y) = \Psi(x) + yH(x). \quad (5)$$

The terms U, V, and W are displacements of points on the longitudinal axis of the beam (x-axis). Similarly, ϕ and Ψ are rotations along the x-axis. From the above kinematic assumptions the expressions for normal strain ϵ_{yy} and shear strain γ_{yz} take the form

$$\epsilon_{yy} = G + zH \quad (6)$$

$$\gamma_{yz} = \Psi + yH + \theta. \quad (7)$$

We will further assume that the normal strain ϵ_{yy} and shear strain γ_{yz} vanish. This can be accomplished by setting $G = 0$, $H = 0$, and $\Psi = -\theta$. These assumptions, along with the plate theory assumption $\epsilon_{zz} = 0$, imply that beam cross-sections normal to the x-axis do not undergo andy in-plane deformations. From the assumed displacement the midplane deformations **E** can be expressed as

$$\mathbf{E} = \overline{\mathbf{E}} + y\hat{\mathbf{E}}. \quad (8)$$

where

$$\mathbf{E}^T = [u_{0,x} v_{0,y} (u_{0,x} + v_{0x}) \psi_{x,x} \psi_{y,y} (\psi_{x,y} + \psi_{y,x})(\psi_y + w_{,y})(\psi_x + w_{,x})] \quad (9)$$

$$\overline{\mathbf{E}}^T = [U' \;\; 0 \;\; (V' + F) \;\; \phi' \;\; 0 \;\; (a - \theta') \;\; 0 \;\; (\phi + W')] \quad (10)$$

$$\hat{\mathbf{E}}^T = [F' \;\; 0 \;\; 0 \;\; a' \;\; 0 \;\; 0 \;\; 0 \;\; (a + \theta')] \quad (11)$$

and a prime denotes differentiation with respect to x. From (8) the constitutive relations $\mathbf{F} = \mathbf{CE}$ become

$$\mathbf{F} = C(\overline{\mathbf{E}} + y\hat{\mathbf{E}}) \quad (12)$$

where **F** is the vector of force and moment resultants and **C** is the laminate stiffness given by (Whitney, 1987)

$$\mathbf{F}^T = [N_x \; N_y \; N_{xy} \; M_x \; M_y \; M_{xy} \; V_y \; V_x] \quad (13)$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & 0 \\ \mathbf{B} & \mathbf{D} & 0 \\ 0 & 0 & \mathbf{K} \end{bmatrix}. \quad (14)$$

A new set of force and moment resultants for the beam are defined as follows:

$$\overline{\mathbf{F}}(x) = \int_{-b/2}^{+b/2} \mathbf{F}(x,y)\,dy \quad (15)$$

$$\hat{\mathbf{F}}(x) = \int_{-b/2}^{+b/2} y\mathbf{F}(x,y)\,dy. \quad (16)$$

Then the laminate constitutive relations (12) take the form

$$\overline{\mathbf{F}} = b\mathbf{C}\overline{\mathbf{E}} \quad (17)$$

$$\hat{\mathbf{F}} = \left(\frac{b^3}{12}\right)\mathbf{C}\hat{\mathbf{E}}. \quad (18)$$

In deriving (17) and (18), **C** is assumed to be constant along y. Explicit forms of the laminate constitutive relations are presented in the following four expressions (19)–(22) as they will be used in the next section:

$$\begin{Bmatrix} \overline{N}_x \\ \overline{N}_{xy} \\ \overline{M}_x \\ \overline{M}_{xy} \end{Bmatrix} = b \begin{bmatrix} A_{11} A_{16} B_{11} B_{16} \\ A_{16} A_{66} B_{16} B_{66} \\ B_{11} B_{16} D_{11} D_{16} \\ B_{16} B_{66} D_{16} D_{66} \end{bmatrix} \begin{Bmatrix} U' \\ V' + F \\ \phi' \\ a - \theta' \end{Bmatrix} \quad (19)$$

$$\overline{V}_x = b\kappa^2 A_{55}(\phi + W') \quad (20)$$

$$\begin{Bmatrix} \hat{N}_x \\ \hat{M}_x \end{Bmatrix} = \left(\frac{b^3}{12}\right) \begin{bmatrix} A_{11} B_{11} \\ B_{11} D_{11} \end{bmatrix} \begin{Bmatrix} F' \\ a' \end{Bmatrix} \quad (21)$$

$$\hat{V}_x = \left(\frac{b^3}{12}\right)\kappa^2 A_{55}(a + \theta') \quad (22)$$

where the bar and hat accents associate the resultants in the obvious way. In Eqs. (20) and (22) κ^2 is the shear correction factor. The strain energy per unit area of the laminate, Φ_L, can be derived as

$$\Phi_L = \frac{1}{2}\mathbf{E}^T\mathbf{C}\mathbf{E}. \quad (23)$$

From Eqs. (8) and (23) the strain energy per unit length of the beam, Φ_B, can be derived as

$$\Phi_B = \int_{-b/2}^{+b/2} \Phi_L dy = \left(\frac{1}{2}\right)\left(b\overline{\mathbf{E}}^T\mathbf{C}\overline{\mathbf{E}} + \left(\frac{b^3}{12}\right)\hat{\mathbf{E}}^T\mathbf{C}\hat{\mathbf{E}}\right). \quad (24)$$

The principle of minimum potential energy (Reddy, 1984) is applied to derive the equations of equilibrium and the boundary conditions. For the purpose of illustration we will consider only transvers loading, $q(x,y)$, in the z-direction acting on the beam surface. The equilibrium equations and boundary conditions are:

Variable	Equilibrium Equations	Boundary Conditions	
δU:	$\frac{d\overline{N}_x}{dx} = 0$	$\overline{N}_x \delta U = 0$	(25)
δV:	$\frac{d\overline{N}_{xy}}{dx} = 0$	$\overline{N}_{xy}\delta V = 0$	(26)
δW:	$\frac{d\overline{V}_x}{dx} + \overline{q} = 0$	$\overline{V}_x \delta w = 0$	(27)
δF:	$\overline{N}_{xy} - \frac{d\hat{N}_x}{dx} = 0$	$\hat{N}_x \delta F = 0$	(28)
$\delta\theta$:	$\frac{d}{dx}(\hat{V}_x - \overline{M}_{xy}) + \hat{q} = 0$	$(\hat{V}_x - \overline{M}_{xy})\delta\theta = 0$	(29)
$\delta\phi$:	$\overline{V}_x - \frac{d\overline{M}_x}{dx} = 0$	$\overline{M}_x \delta\phi = 0$	(30)
δa:	$\frac{d\hat{M}_x}{dx} = \hat{V}_x + \overline{M}_{xy}$	$\hat{M}_x \delta a = 0.$	(31)

In (27) and (29), $\overline{q}$ and $\hat{q}$ are given by

$$\overline{q}(x) = \int_{-b/2}^{+b/2} q(x,y)\,dy \quad (32)$$

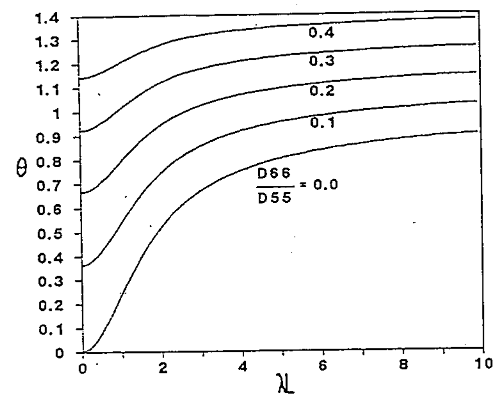

Fig. 2 Nondimensional tip rotation Θ (Eq. (36))

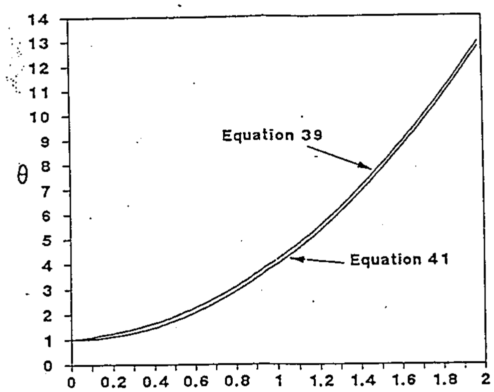

Fig. 3 . Nondimensional tip rotation Θ for long beams

$$\hat{q} = \int_{-b/2}^{+b/2} yq(x,y)\,dy. \tag{33}$$

The solution to the problem of a cantilever beam sujected to end loads is derived in Sankar (1991).

Torsion of Specially Orthotropic Laminated Beams

For specially orthotropic beams, $\mathbf{B} = 0$, $A_{16} = A_{26} = D_{16} = D_{26} = 0$. We assume that an end torque T is the only external force acting on the beam. The solution for the angle of twist is (Sankar, 1991)

$$\theta(x) = \frac{T}{D'_{66}}x - \left(\frac{D'^2_{55}T}{4D_{55}D_{66}D'_{66}}\right) \times \left(\frac{\sinh\lambda x}{\lambda} - x - \frac{\tanh\lambda L}{\lambda}(\cosh\lambda x - 1)\right) \tag{34}$$

where $D_{55} = (b^2/12)\kappa^2 A_{55}$, $D'_{55} = (D_{55} - D_{66})$ and $D'_{66} = (D_{66} + D_{55})$.

For the purpose of comparison with available results we will introduce a nondimensional tip rotation Θ defined as

$$\Theta = \frac{4bD_{66}\theta(L)}{TL}. \tag{35}$$

Then from (34) and (35) the solution for the tip rotation takes the form

$$\Theta = 1 + \frac{D_{66}}{D_{55}} - \frac{\left(1 - \frac{D_{66}}{D_{55}}\right)^2}{\left(1 + \frac{D_{66}}{D_{55}}\right)} \frac{\tanh\lambda L}{\lambda L}. \tag{36}$$

In the above result, the first term on the right-hand side corresponds to the classical theory solution for isotropic beams. The shear deformation effects are reflected in the second and third terms. The third term represents the effect of the restrained end $x = 0$, where warping is prevented, i.e., $\alpha(0) = 0$. In Fig. 2, Θ is plotted as a function of λL for various values of (D_{66}/D_{55}). It may be seen that the restrained end effects are felt only for $\lambda L < 10$. Further, the restrained end effects are less pronounced as the ratio of the shear stiffness coefficients (D_{66}/D_{55}) increases. The effect of transverse shear flexibility $(1/D_{55})$ is to increase the angle of twist significantly.

We will compare our results with two available results. If we ignore the shear deformation, i.e., let $D_{55} \rightarrow \infty$ in Eq. (36), we will obtain

$$\Theta = 1 + \frac{\tanh\lambda L}{\lambda L}, \tag{37}$$

which is identical to the result for an isotropic beam (Boresi, Sidebottom, Seely, and Smith, 1978). If we ignore the restrained end effects by letting $\lambda L \rightarrow \infty$ in result (36), then we obtain

$$\theta = 1 + \frac{D_{66}}{D_{55}}. \tag{38}$$

The above results can be compared with that of (Tsai, Daniel, and Yaniv, 1990) for a 0 deg unidirectional composite beam. Let us denote their result by Θ_1. In their notation

$$\Theta_1 = \left(1 - \frac{\tanh\beta}{\beta}\right)^{-1} \tag{39}$$

where

$$\beta = \left(\frac{b}{2h}\right)\sqrt{10\frac{G_{13}}{G_{12}}}. \tag{40}$$

In the same notation, result (38) obtained in the present study can be written as

$$\Theta = 1 + \frac{3}{\beta^2}. \tag{41}$$

In deriving (39) and (41) the shear correction factor κ^2 is assumed to be 5/6. The two results, Θ and Θ_1, are compared for some practical range of $1/\beta$ in Fig. 3. The agreement is quite good. The maximum difference is about 11 percent which occurs at $1/\beta \simeq 0.288$. It is interesting to note that

$$\lim_{\beta\rightarrow 0}\beta^2\left(1 - \frac{\tanh\beta}{\beta}\right)^{-1} = 3 = \lim_{\beta\rightarrow 0}\beta^2\left(1 + \frac{3}{\beta^2}\right) \tag{42}$$

so that $(\Theta/\Theta_1) \rightarrow 1$ as $1/\beta \rightarrow \infty$.

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Fjortoft's Theorem for a Parallel Flow With a Free Surface

John P. McHugh[23]

1 Introduction

The stability of inviscid parallel flows has been intensively studied in the past. The theory is reviewed by Drazin and Howard (1966), Drazin and Reid (1981), and Craik (1985). Several comprehensive and important theorems which have been proven for flows between rigid boundaries are Rayleigh's theorem, Fjortoft's theorem, and Howard's semicircle theorem. Rayleigh's theorem and Howard's theorem have been extended to include a free surface by Yih (1972). Yih (1972) also found a variety of results concerning the neutral modes. The equivalent to Fjortoft's theorem for a parallel flow with a free surface is presented in this Note.

2 Basic Equations

The flow is considered to be inviscid and incompressible. All variables are assumed to be the sum of a primary flow and a disturbance quantity, and the primary flow is a parallel flow, $U(y)$. The equation governing stability is Rayleigh's equation,

$$\phi'' - \alpha^2\phi - \frac{U''}{U-c}\,\phi = 0, \tag{1}$$

where ϕ is the stream function, α is the wave number, c is the wave speed, and the prime denotes differentiation.

The free-surface boundary condition is

$$\phi'' - \left(\frac{U'}{U-c} + \frac{1}{F^2(U-c)^2}\right)\phi = 0, \tag{2}$$

which holds on the mean free surface. Equations (1) and (2) have been made nondimensional using the depth of the layer, d, as the length scale, and the velocity range, ΔU, i.e., the difference between the maximum and minimum values of U, as the velocity scale. The Froude number is $F = \Delta U/\sqrt{gd}$. Equation (2) appears in Yih (1972) (Eq. (11) in Yih's paper) in dimensional form.

The boundary condition on the flat rigid bottom is

$$\phi = 0. \tag{3}$$

3 Derivation of the Theorem

The coordinate system is chosen to be moving with the velocity of the primary flow at the free surface. In this way, the value of U is zero at the free surface. The difference between this case and a primary velocity profile with a nonzero value at the free surface is merely a Galilean transformation by the amount U_f, where U_f is the primary velocity at the free surface.

Multiply (1) by the conjugate of ϕ, denoted by ϕ^*, and integrate over the domain, integrating the first term in (1) by parts, to obtain

$$\int \phi'\phi'^*dy + \alpha^2\int \phi\phi^* dy + \int \frac{U''\phi\phi^*}{U-c}\,dy = \phi_f^*\phi_f', \tag{4}$$

where again the subscript f refers to the value on the free surface. The term on the right-hand side of (4) contains ϕ_f', which is evaluated using the free-surface boundary condition. Equation (4) then becomes

$$J_1 + \int \frac{U''\phi\phi^*}{U-c}\,dy = \left(-\frac{U_f'}{c} + \frac{1}{F^2c^2}\right)\phi_f^*\phi_f, \tag{5}$$

where J_1 is the sum of the first two terms in (4). Note that U does not appear in the right-hand side of (5), since $U_f = 0$. The imaginary part of (5) is

$$c_i\int \frac{U''|\phi|^2}{(U-c_r)^2+(c_i)^2}\,dy = c_i\left(\frac{U_f'}{c_r^2+c_i^2} - \frac{1}{F^2}\frac{2c_r}{[c_r^2+c_i^2]^2}\right)|\phi_f|^2, \tag{6}$$

where c_r and c_i are the real and imaginary parts of c, respectively. For unstable flows, c_i cannot be zero, and the remaining terms in (6) must balance.

The real part of (5) is

$$J_1 + \int \frac{U''|\phi|^2(U-c_r)}{(U-c_r)^2+(c_i)^2}\,dy = \left(\frac{c_rU_f'}{c_r^2+c_i^2} + \frac{1}{F^2}\frac{c_r^2-c_i^2}{[c_r^2+c_i^2]^2}\right)|\phi_f|^2, \tag{7}$$

The integral on the left-hand side of (7) may be split into two integrals. One of the new integrals is the negative of the left-hand side of Eq. (6), and may be replaced by the right-hand side of (6). After some cancellation and rearranging, (7) reduces to

$$J_1 + \int \frac{UU''|\phi|^2}{(U-c_r)^2+(c_i)^2}\,dy = -\frac{1}{F^2}\frac{|\phi_f|^2}{c_r^2+c_i^2}. \tag{8}$$

The right-hand side of (8) without the negative sign is assigned the value J_2, and it can be seen from (8) that J_2 is positive definite. Equation (8) becomes

$$J_1 + J_2 + \int \frac{UU''|\phi|^2}{(U-c_r)^2+(c_i)^2}\,dy = 0. \tag{9}$$

The important conclusion from (9) may now be explained as follows. Since the sum of J_1 and J_2 is positive definite, then the integral must be negative definite. Since the integrand other than $U''U$ is positive definite, then the product $U''U$ must be negative somewhere in the domain of flow for instability. Note that this conclusion is necessary for instability, but not sufficient. A velocity profile may obey (9) and still be stable.

The coordinate system could be chosen so that the value of U at the free surface, U_f, is not zero. The conclusion would then be that $U''(U-U_f)$ must be negative somewhere in the domain of flow. Consider monotonic velocity profiles, for which the quantity $U-U_f$ is either negative or positive throughout. The above result implies that U'' and $U-U_f$ must have opposite sign for instability.

This version of Fjortoft's theorem with monotonic profiles distinguishes between flows which are driven by a shear force, such as wind on the free surface, and flows driven by a pressure gradient or body force.[24] The shear driven flows may be unstable while the others cannot. This result implies that the

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