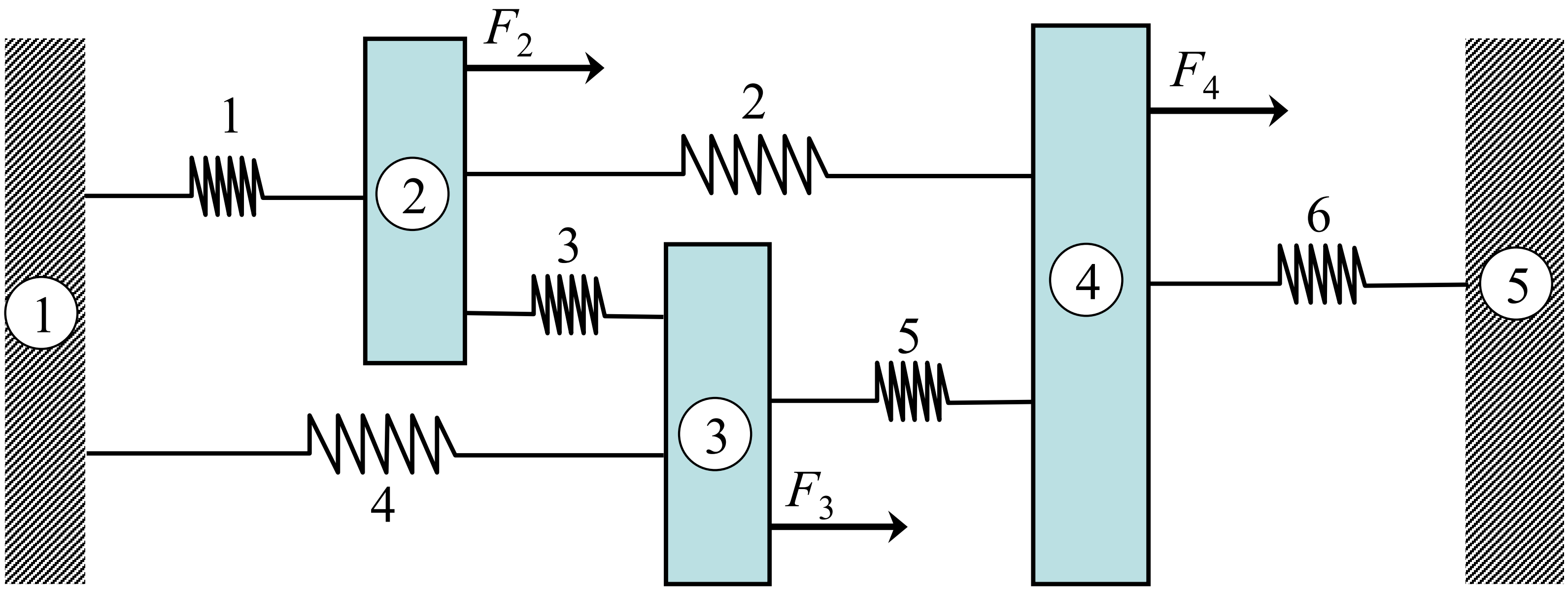
Chapter 1: Direct Method

Springs, Bars and Truss Elements

1.1 DIRECT METHOD

1-D SYSTEM OF SPRINGS

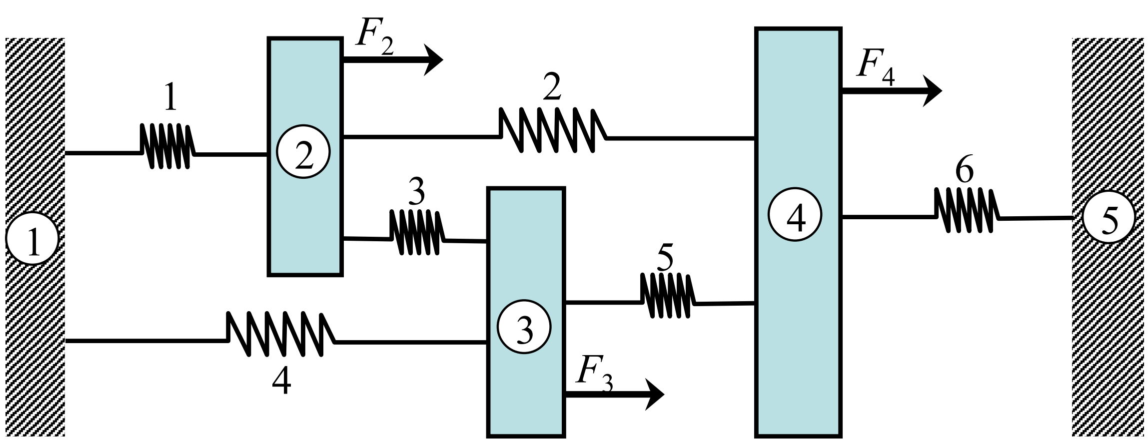


- Bodies move only in horizontal direction
- External forces, F_2, F_3, and F_4, are applied
- No need to discretize the system (it is already discretized!)
- Rigid body (including walls) ➡ **NODE**
- Spring ➡ **ELEMENT**

Connectivity Table

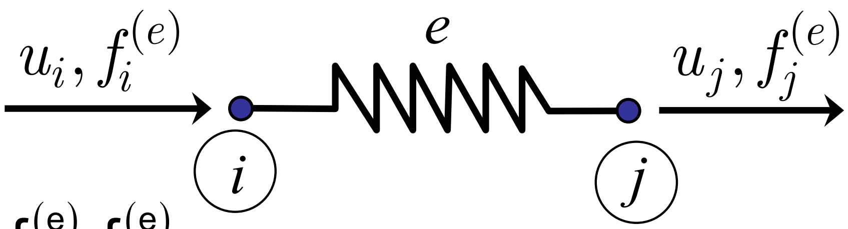
- Mesh: system of connected elements
- Connectivity: Local node 1 (i) → Local node 2 (j)

Element	LN1 (i)	LN2 (j)
1	1	2
2	2	4
3	2	3
4	1	3
5	3	4
6	4	5



SPRING ELEMENT

- Element e
  - Consist of Nodes i and j
  - Spring constant $k^{(e)}$
  - Internal force applied to the nodes: $f_i^{(e)}, f_j^{(e)}$
  - Displacement u_i and u_j
  - Elongation: $\Delta^{(e)} = u_j - u_i$
  - Element force in the spring: $P^{(e)} = k^{(e)}\Delta^{(e)} = k^{(e)}(u_j - u_i)$
  - Relation b/w element force and nodal forces: $f_j^{(e)} = P^{(e)}$
  - Equilibrium: $f_i^{(e)} + f_j^{(e)} = 0$ or $f_i^{(e)} = -f_j^{(e)}$: equal & opposite directions



$P^{(e)}$: Positive = tension

Negative = compression

SPRING ELEMENT *cont.*

- Spring Element (e)
  - Relation between internal nodal forces and displacements

$$f_i^{(e)} = k^{(e)}\left(u_i - u_j\right)$$

$$f_j^{(e)} = k^{(e)}\left(-u_i + u_j\right)$$

$$\begin{bmatrix} k^{(e)} & -k^{(e)} \\ -k^{(e)} & k^{(e)} \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} f_i^{(e)} \\ f_j^{(e)} \end{Bmatrix}$$

  - Matrix notation:

$$\left[\mathbf{k}^{(e)}\right] \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} f_i^{(e)} \\ f_j^{(e)} \end{Bmatrix}$$

$$[\mathbf{k}^{(e)}]\left\{\mathbf{q}^{(e)}\right\} = \left\{\mathbf{f}^{(e)}\right\}$$

$$\mathbf{k} \cdot \mathbf{q} = \mathbf{f}$$

  - **k**: stiffness matrix
  - **q**: vector of DOFs
  - **f**: vector of internal forces

SPRING ELEMENT *cont.*

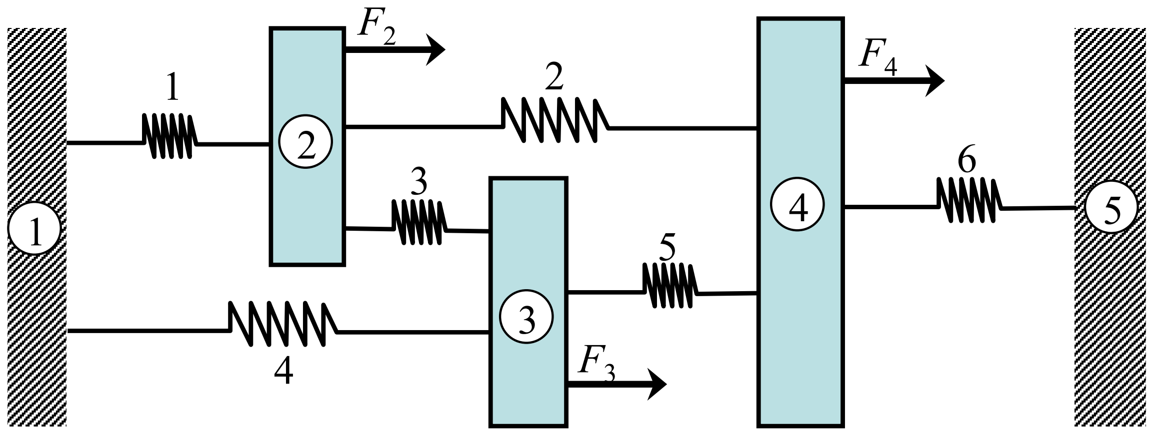
- Stiffness matrix
  - It is square as it relates to the same number of forces as the displacements.

$$[\mathbf{k}^{(e)}] = \begin{bmatrix} k^{(e)} & -k^{(e)} \\ -k^{(e)} & k^{(e)} \end{bmatrix}$$

  - It is symmetric.
  - It is singular, i.e., its determinant is equal to zero, so it cannot be inverted.
  - It is positive semi-definite
- Observation
  - For given nodal displacements, internal nodal forces can be calculated by $[\mathbf{k}^{(e)}]\{\mathbf{q}^{(e)}\} = \{\mathbf{f}^{(e)}\}$
  - For given internal nodal forces (equal and opposite directions), nodal displacements cannot be determined uniquely (rigid-body motion)

SYSTEM OF SPRINGS *cont.*

- Element equation and assembly



$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix} \Longrightarrow \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} f_2^{(2)} \\ f_4^{(2)} \end{Bmatrix} \Longrightarrow \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 + k_2 & 0 & -k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -k_2 & 0 & k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_2^{(2)} \\ 0 \\ f_4^{(2)} \\ 0 \end{Bmatrix}$$

SYSTEM OF SPRINGS *cont.*

$$\begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_2^{(3)} \\ f_3^{(3)} \end{Bmatrix} \Rightarrow \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_3 & -k_2 & 0 \\ 0 & -k_3 & k_3 & 0 & 0 \\ 0 & -k_2 & 0 & k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_2^{(2)} + f_2^{(3)} \\ f_3^{(3)} \\ f_4^{(2)} \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} k_4 & -k_4 \\ -k_4 & k_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1^{(4)} \\ f_3^{(4)} \end{Bmatrix} \Rightarrow \begin{bmatrix} k_1+k_4 & -k_1 & -k_4 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_3 & -k_2 & 0 \\ -k_4 & -k_3 & k_3+k_4 & 0 & 0 \\ 0 & -k_2 & 0 & k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} + f_1^{(4)} \\ f_2^{(1)} + f_2^{(2)} + f_2^{(3)} \\ f_3^{(3)} + f_3^{(4)} \\ f_4^{(2)} \\ 0 \end{Bmatrix}$$

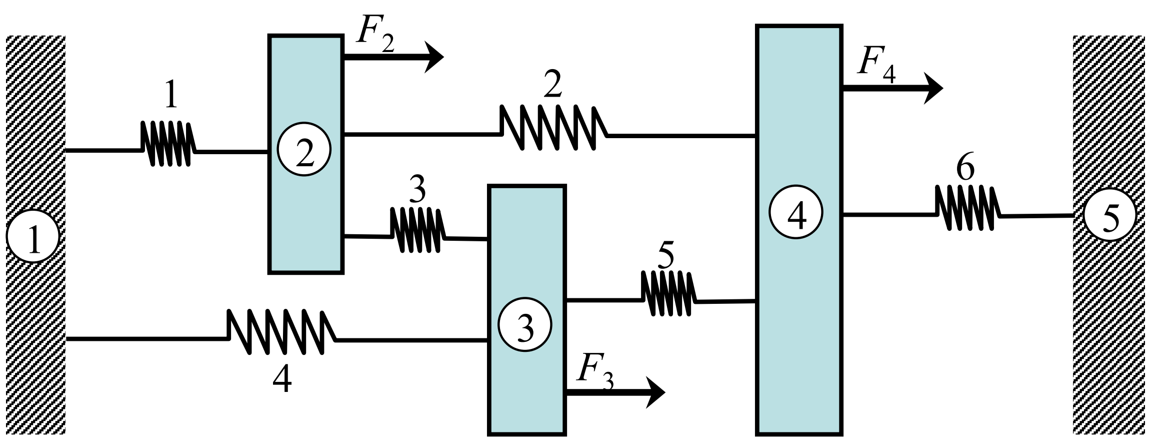
$$\begin{bmatrix} k_5 & -k_5 \\ -k_5 & k_5 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} f_3^{(5)} \\ f_4^{(5)} \end{Bmatrix}$$

$$\Rightarrow \begin{bmatrix} k_1+k_4 & -k_1 & -k_4 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_3 & -k_2 & 0 \\ -k_4 & -k_3 & k_3+k_4+k_5 & -k_5 & 0 \\ 0 & -k_2 & -k_5 & k_2+k_5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} + f_1^{(4)} \\ f_2^{(1)} + f_2^{(2)} + f_2^{(3)} \\ f_3^{(3)} + f_3^{(4)} + f_3^{(5)} \\ f_4^{(2)} + f_4^{(5)} \\ 0 \end{Bmatrix}$$

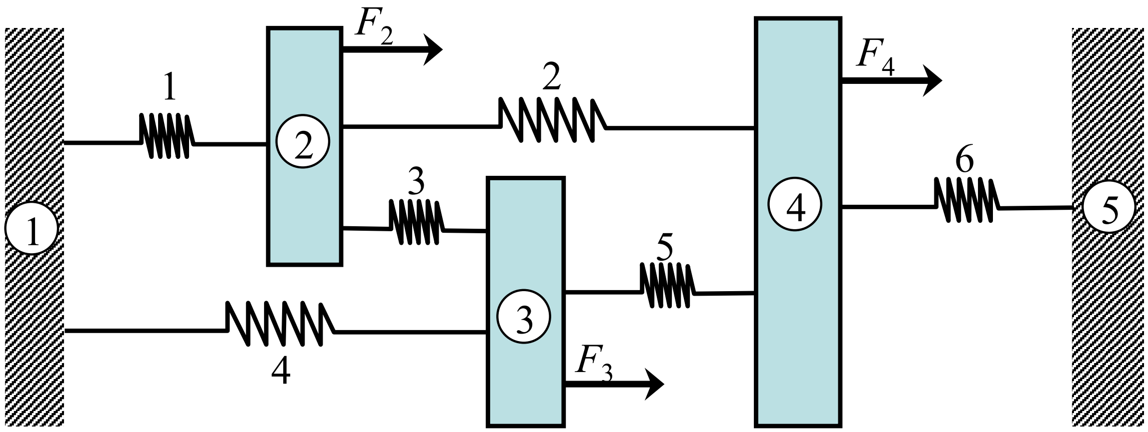
SYSTEM OF SPRINGS *cont.*

$$\begin{bmatrix} k_6 & -k_6 \\ -k_6 & k_6 \end{bmatrix} \begin{Bmatrix} u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} f_4^{(6)} \\ f_5^{(6)} \end{Bmatrix} \Longrightarrow$$

$$\begin{bmatrix} k_1 + k_4 & -k_1 & -k_4 & 0 & 0 \\ -k_1 & k_1 + k_2 + k_3 & -k_3 & -k_2 & 0 \\ -k_4 & -k_3 & k_3 + k_5 + k_4 & -k_5 & 0 \\ 0 & -k_2 & -k_5 & k_2 + k_5 + k_6 & -k_6 \\ 0 & 0 & 0 & -k_6 & +k_6 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} + f_1^{(4)} \\ f_2^{(1)} + f_2^{(2)} + f_2^{(3)} \\ f_3^{(3)} + f_3^{(4)} + f_3^{(5)} \\ f_4^{(2)} + f_4^{(5)} + f_4^{(6)} \\ f_5^{(6)} \end{Bmatrix}$$



SYSTEM OF SPRINGS *cont.*

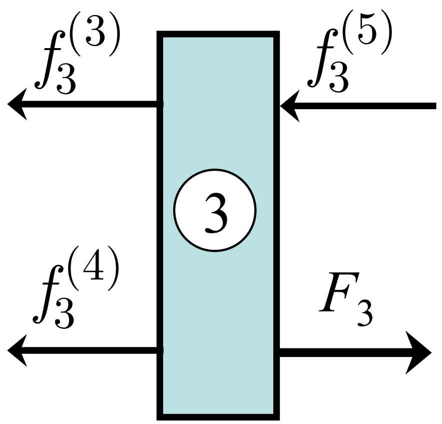


- Relation b/w internal forces and external force
- Force equilibrium

$$F_i - \sum_{e=1}^{i_e} f_i^{(e)} = 0$$

$$F_i = \sum_{e=1}^{i_e} f_i^{(e)}, \quad i = 1,\ldots ND$$

- At node 3



$$F_3 - f_3^{(3)} - f_3^{(4)} - f_3^{(5)} = 0$$

- At each node, the summation of **internal forces** is equal to the **applied, external force**

$$\begin{Bmatrix} f_1^{(1)} + f_1^{(4)} \\ f_2^{(1)} + f_2^{(2)} + f_2^{(3)} \\ f_3^{(3)} + f_3^{(4)} + f_3^{(5)} \\ f_4^{(2)} + f_4^{(5)} + f_4^{(6)} \\ f_5^{(6)} \end{Bmatrix} = \begin{Bmatrix} R_1 \\ F_2 \\ F_3 \\ F_4 \\ R_5 \end{Bmatrix}$$

SYSTEM OF SPRINGS *cont.*

- Assembled System of Matrix Equation:

$$\begin{bmatrix} k_1+k_4 & -k_1 & -k_4 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_3 & -k_2 & 0 \\ -k_4 & -k_3 & k_3+k_5+k_4 & -k_5 & 0 \\ 0 & -k_2 & -k_5 & k_2+k_5+k_6 & -k_6 \\ 0 & 0 & 0 & -k_6 & +k_6 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ F_2 \\ F_3 \\ F_4 \\ R_5 \end{Bmatrix}$$

$[\mathbf{K}_s]\{\mathbf{Q}_s\} = \{\mathbf{F}_s\}$ **[K$_S$]**: Structural stiffness matrix

- **[K$_s$]** is square, symmetric, singular and positive semi-definite.
- When displacement is known, force is unknown

$u_1 = u_5 = 0 \Longrightarrow$ R_1 and R_5 are unknown reaction forces

- When applied force is known, displacement is unknown

$\Longrightarrow$ Either displacement or force is known, not both

Alternative Way of Assembly

- From $F_i = \sum_{e=1}^{i_e} f_i^{(e)}, \quad i = 1,\dots ND$

$$
\begin{cases}
F_1 = f_1^{(1)} + f_4^{(1)} = k^{(1)}(u_1 - u_2) + k^{(4)}(u_1 - u_3) \\
F_2 = f_2^{(1)} + f_2^{(3)} + f_2^{(2)} = k^{(1)}(u_2 - u_1) + k^{(3)}(u_2 - u_3) + k^{(2)}(u_2 - u_4) \\
F_3 = f_3^{(3)} + f_3^{(4)} + f_3^{(5)} = k^{(3)}(u_3 - u_2) + k^{(4)}(u_3 - u_1) + k^{(5)}(u_3 - u_4) \\
F_4 = f_4^{(2)} + f_4^{(5)} + f_4^{(6)} = k^{(2)}(u_4 - u_2) + k^{(5)}(u_4 - u_3) + k^{(6)}(u_4 - u_5) \\
F_5 = f_5^{(6)} = k^{(6)}(u_5 - u_4)
\end{cases}
$$

$$\Downarrow$$

$$[\mathbf{K}_s]\begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_5 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ F_2 \\ \vdots \\ R_5 \end{Bmatrix}$$

Same structural matrix equation!!

SYSTEM OF SPRINGS *cont.*

- Imposing Boundary Conditions
  - Ignore the equations for which the RHS forces are unknown and strike out the corresponding rows in [$\mathbf{K}_s$].
  - Eliminate the columns in [$\mathbf{K}_s$] that multiply into zero values of displacements of the boundary nodes.

$$\begin{bmatrix} k_1+k_4 & -k_1 & -k_4 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_3 & -k_2 & 0 \\ -k_4 & -k_3 & k_3+k_5+k_4 & -k_5 & 0 \\ 0 & -k_2 & -k_5 & k_2+k_5+k_6 & -k_6 \\ 0 & 0 & 0 & -k_6 & +k_6 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ F_2 \\ F_3 \\ F_4 \\ R_5 \end{Bmatrix}$$

SYSTEM OF SPRINGS *cont.*

- Global Matrix Equation

$$\begin{bmatrix} k_1+k_2+k_3 & -k_3 & -k_2 \\ -k_3 & k_3+k_4+k_5 & -k_5 \\ -k_2 & -k_5 & k_2+k_5+k_6 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

$$[\mathbf{K}]\{\mathbf{Q}\} = \{\mathbf{F}\}$$

- Global Stiffness Matrix [**K**]
  – square, symmetric and positive definite and hence non-singular
- Solution

$$\{\mathbf{Q}\} = [\mathbf{K}]^{-1}\{\mathbf{F}\}$$

- Once nodal displacements are obtained, element forces can be calculated from

$$P^{(e)} = k^{(e)}\Delta^{(e)} = k^{(e)}\left(u_j - u_i\right)$$

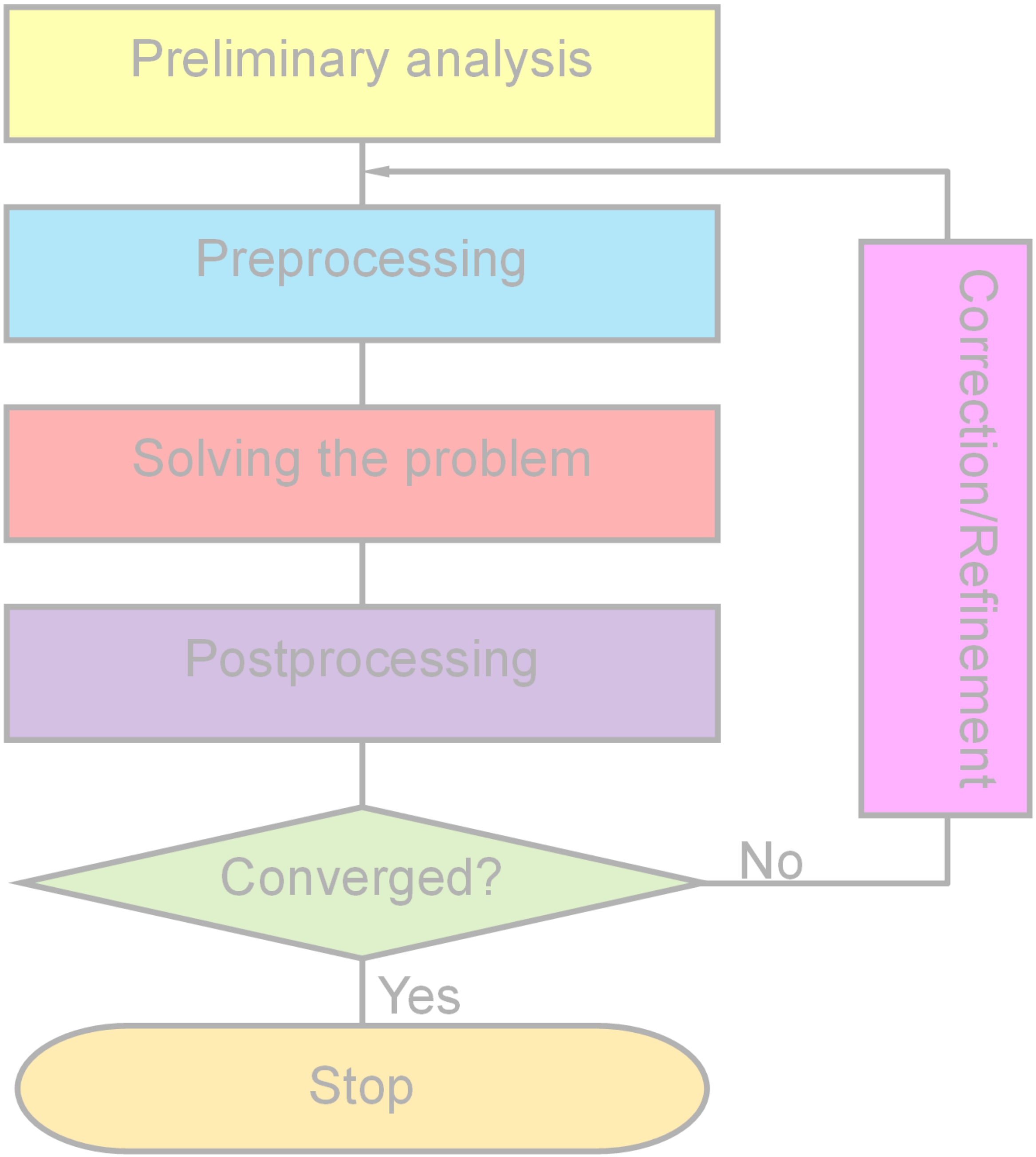
Finite Element Procedure

Element matrix Eq.

Assembly: $[\mathbf{K}_s]\{\mathbf{Q}_s\}=\{\mathbf{F}_s\}$

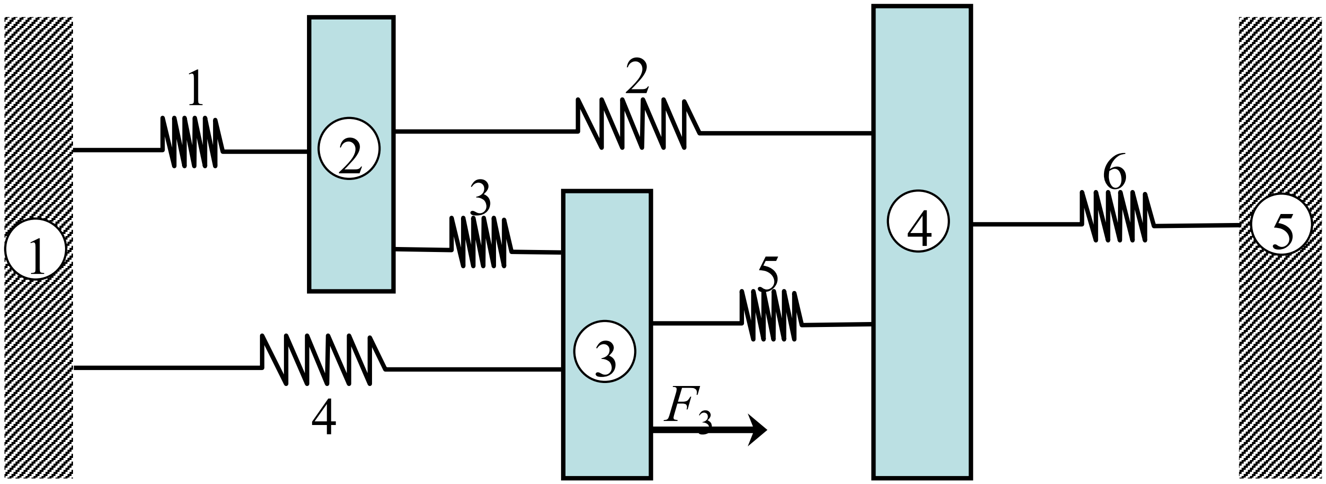
Apply BC: $[\mathbf{K}]\{\mathbf{Q}\}=\{\mathbf{F}\}$

Solving: $\{\mathbf{Q}\}=[\mathbf{K}]^{-1}\{\mathbf{F}\}$



Example 1.1

- Calculate element forces when F_3 = 1000N is applied. Use spring constants (N/mm): $k^{(1)} = 500$, $k^{(2)} = 400$, $k^{(3)} = 600$, $k^{(4)} = 200$, $k^{(5)} = 400$, and $k^{(6)} = 300$.



- Element equilibrium equations:

Example 1.1 *cont.*

- Nodal equilibrium equations:

- Assembly

Example 1.1 *cont*.

- Apply boundary conditions

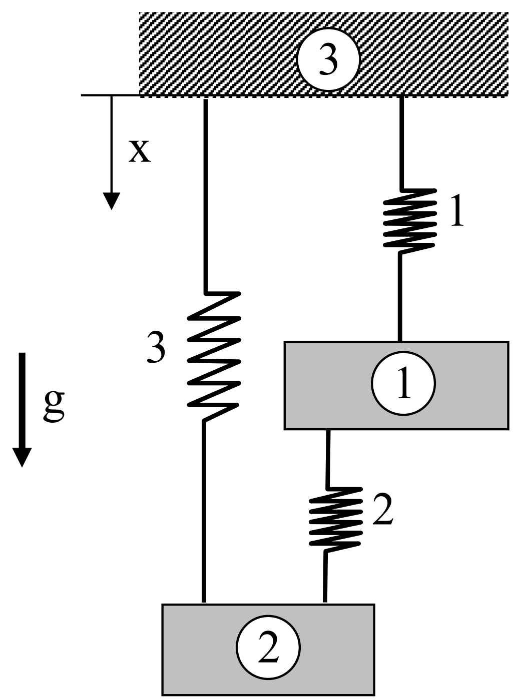
- Solving matrix equations:

$u_2 = 0.854$ mm, $u_3 = 1.55$ mm, and $u_4 = 0.875$ mm

- Element forces

Exercise

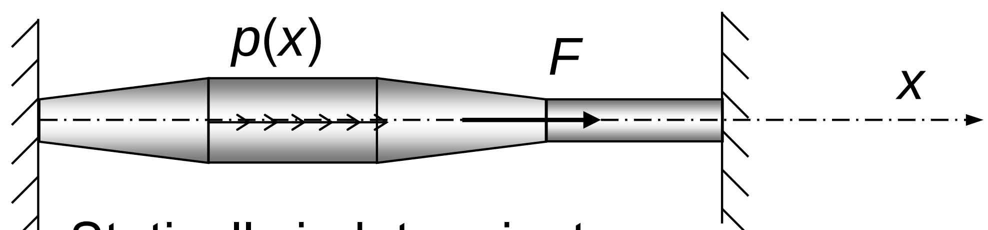
- Two rigid masses, 1 and 2, are connected by three springs as shown in the figure. When gravity is applied with g = 9.85 m/sec^2, using FE analysis, (a) find the displacements of the two masses (1 and 2), (2) find the element forces of all springs, and (3) the reaction force at the wall (Body 3). Assume the bodies can undergo only translation in the vertical direction. The spring constants (N/mm) are k_1 = 400, k_2 = 500, and k_3 = 500. The masses (kg) are m_1 = 20 and m_2 = 40.



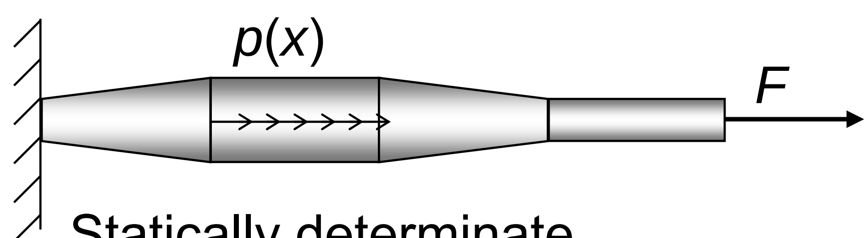
1.2 UNIAXIAL BAR ELEMENT

Uniaxial Bar

- A slender two–force member
  - Length is much larger than cross-sectional dimensions
- can have varying cross-sectional area and varying Young's modulus
- Both concentrated forces and distributed force can be applied
- Statically determinate system
  - Member forces can be calculated from force equilibrium
- Statically indeterminate system
  - Deformation must be used to calculate member forces
- In FE, no need to have a special treatment for statically indeterminate system



Statically indeterminate

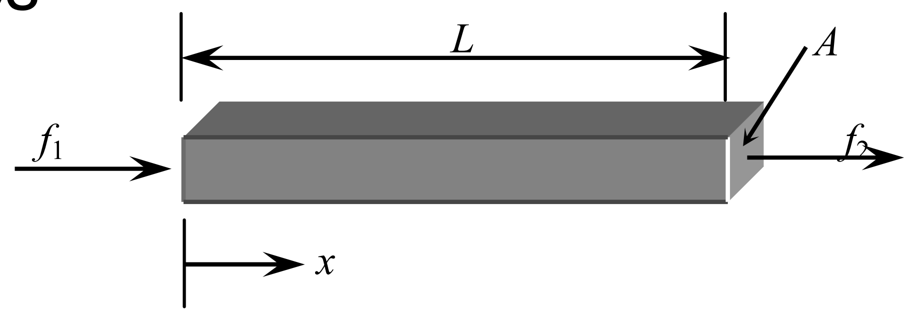


Statically determinate

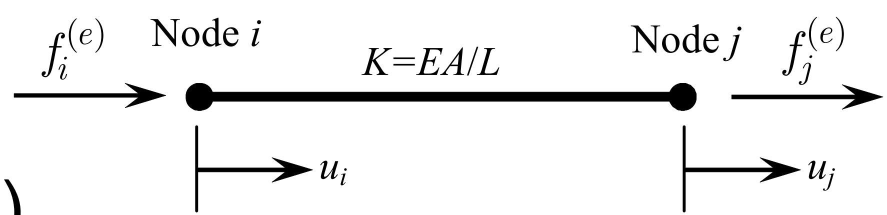
UNIAXIAL BAR

- Discretize the bar into a number of elements
  - Will discuss about determining element size later
  - It is assumed that each element has a constant axial rigidity, EA, throughout its length
- Elements are connected at nodes
  - More than one element can share a node
  - There will be nodes at points where the bar is supported
- Forces are applied only at the nodes
  - distributed load must be converted to the equivalent nodal forces
  - If displacement at a node is specified, then the reaction is unknown
  - If force at a node is specified, then the displacement is unknown
- Objective of FE analysis is to determine
  - (i) unknown DOF (u_i)
  - (ii) axial force resultant (P) in each element (element force)
  - (iii) support reactions

Direct Stiffness Method: 1D BAR ELEMENT



- 1D bar element with two nodes
- Positive displacement and force in positive x-direction
- Only constant cross-section
- First node: *i*
  second code: *j*



- Equilibrium (free-body diagram)

$$f_i^{(e)} + f_j^{(e)} = 0$$

- Force-displacement relation

$$f_j^{(e)} = \left(\frac{AE}{L}\right)^{(e)} (u_j - u_i)$$

$$f_i^{(e)} = -f_j^{(e)} = \left(\frac{AE}{L}\right)^{(e)} (u_i - u_j)$$

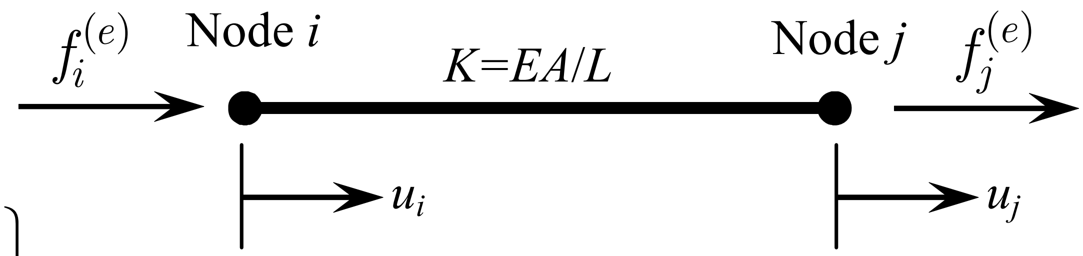
Elongation

$$\Delta^{(e)} = u_j - u_i$$

Similar to the spring element

$$k^{(e)} = \left(\frac{AE}{L}\right)^{(e)}$$

1D BAR ELEMENT *cont.*



– Matrix notation

$$\begin{Bmatrix} f_i^{(e)} \\ f_j^{(e)} \end{Bmatrix} = \left(\frac{AE}{L}\right)^{(e)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

$$\{\mathbf{f}^{(e)}\} = [\mathbf{k}^{(e)}]\{\mathbf{q}^{(e)}\}$$

– Either force or displacement (not both) must be given at each node.

– Example: $u_i = 0$ and $f_j = 100$ N.

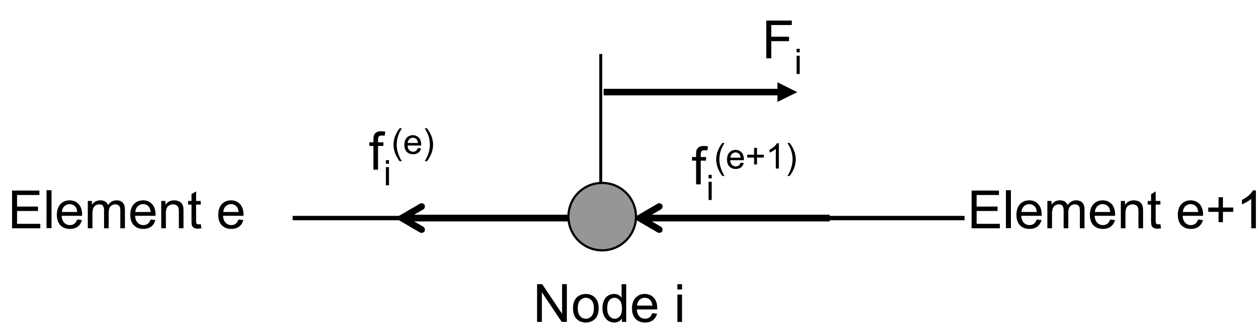
– What happens when f_i and f_j are given?

- Nodal equilibrium
  – Equilibrium of forces acting on Node i

$$F_i - f_i^{(e)} - f_i^{(e+1)} = 0 \quad \Longrightarrow \quad f_i^{(e)} + f_i^{(e+1)} = F_i$$

  – In general

$$F_i = \sum_{e=1}^{i_e} f_i^{(e)}$$



1D BAR ELEMENT *cont.*

- Assembly
  - Similar process as spring elements
  - Replace all internal nodal forces with **Externally Applied Nodal Force**
  - Obtain system of equations

$$[\mathbf{K}_s]\{\mathbf{Q}_s\} = \{\mathbf{F}_s\}$$

$[\mathbf{K}_s]$: Structural stiffness matrix

$\{\mathbf{Q}_s\}$ Vector of nodal DOFs

$\{\mathbf{F}_s\}$: Vector of applied forces

- Property of $[\mathbf{K}_s]$
  - Square, symmetric, positive semi-definite, singular, non-negative diagonal terms
- Applying boundary conditions
  - Remove rigid-body motion by fixing DOFs
  - Striking-the-rows and striking-the-columns (Refer to spring elements)

$$[\mathbf{K}]\{\mathbf{Q}\} = \{\mathbf{F}\}$$

$[\mathbf{K}]$: Global stiffness matrix

$\{\mathbf{Q}\}$ Vector of unknown nodal DOFs

$\{\mathbf{F}\}$: Vector of known applied forces

1D BAR ELEMENT *cont.*

- Applying boundary conditions cont.
  - [**K**] is square, symmetric, positive definite, non-singular, invertible, and positive diagonal terms
  - Can obtain unique {**Q**}
- Element forces
  - After solving nodal displacements, the element force can be calculated

$$P^{(e)} = \left(\frac{AE}{L}\right)^{(e)} \left(u_j - u_i\right) = f_j^{(e)} \qquad \begin{Bmatrix} -P^{(e)} \\ +P^{(e)} \end{Bmatrix} = \left(\frac{AE}{L}\right)^{(e)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

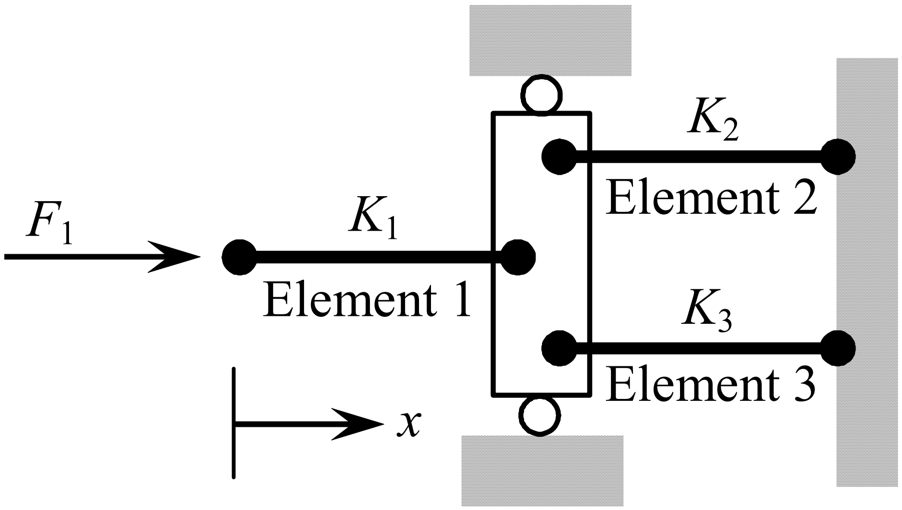
  - Element stress $\sigma^{(e)} = \dfrac{P^{(e)}}{A^{(e)}}$ $P^{(e)}$ and $\sigma^{(e)}$ have the same sign convention
- Reaction Forces
  - Use [$\mathbf{K}_s$]{$\mathbf{Q}_s$} = {$\mathbf{F}_s$}: the rows that have been deleted (strike-the-rows)
  - Or, use

$$F_i = \sum_{e=1}^{i_e} f_i^{(e)}$$

EXAMPLE

- 3 elements and 4 nodes
- At node 2:

$$F_2 = f_2^{(1)} + f_2^{(2)} + f_2^{(3)}$$

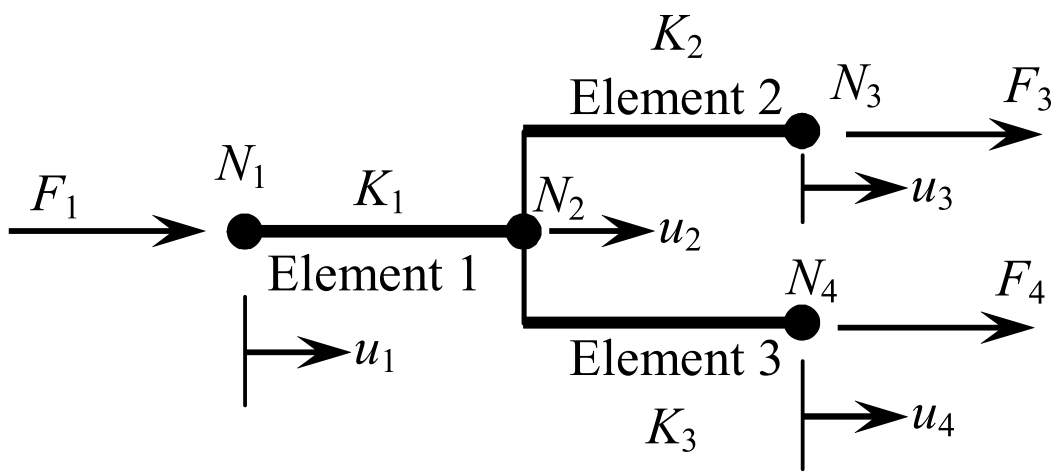


- Equation for each element:

$$\begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix} = \begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix} = \begin{bmatrix} K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} f_2^{(3)} \\ f_4^{(3)} \end{Bmatrix} = \begin{bmatrix} K_3 & -K_3 \\ -K_3 & K_3 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_4 \end{Bmatrix}$$



EXAMPLE *cont.*

- How can we combine different element equations? (Assembly)
  - First, prepare global matrix equation:

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

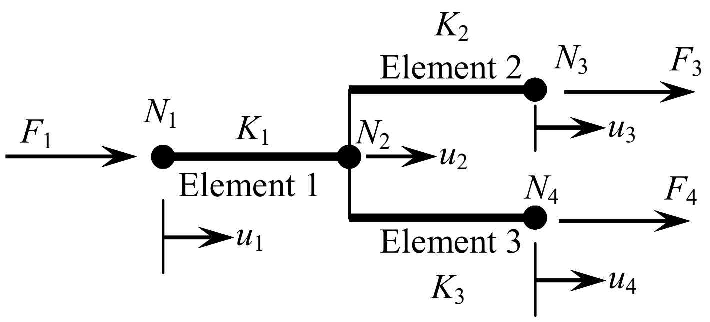
Displacement vector

Stiffness matrix

Applied force vector

  - Write the equation of element 1 in the corresponding location

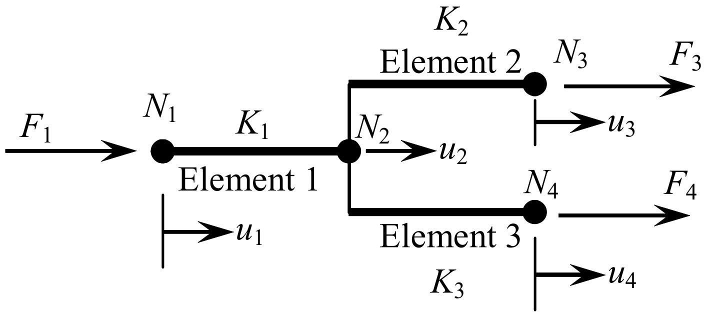
$$\begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} K_1 & -K_1 & 0 & 0 \\ -K_1 & K_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$



EXAMPLE *cont.*

– Write the equation of element 2:

$$\begin{Bmatrix} 0 \\ f_2^{(2)} \\ f_3^{(2)} \\ 0 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & K_2 & -K_2 & 0 \\ 0 & -K_2 & K_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$



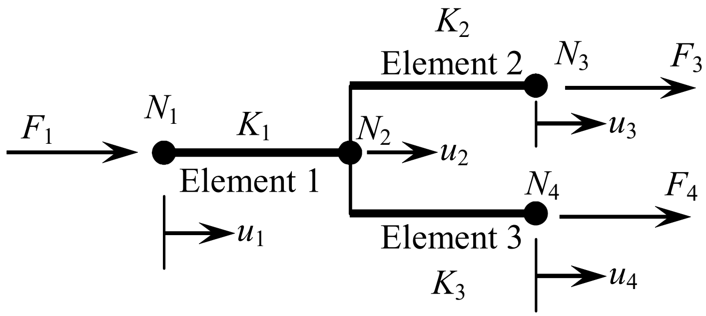
– Combine two equations of elements 1 and 2

$$\begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_2^{(2)} \\ f_3^{(2)} \\ 0 \end{Bmatrix} = \begin{bmatrix} K_1 & -K_1 & 0 & 0 \\ -K_1 & K_1 + K_2 & -K_2 & 0 \\ 0 & -K_2 & K_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

EXAMPLE *cont.*

– Write the equation of element 3

$$\begin{Bmatrix} 0 \\ f_2^{(3)} \\ 0 \\ f_4^{(3)} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & K_3 & 0 & -K_3 \\ 0 & 0 & 0 & 0 \\ 0 & -K_3 & 0 & K_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$



– Combine with other two elements

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_2^{(2)} + f_2^{(3)} \\ f_3^{(2)} \\ f_4^{(3)} \end{Bmatrix} = \underbrace{\begin{bmatrix} K_1 & -K_1 & 0 & 0 \\ -K_1 & (K_1 + K_2 + K_3) & -K_2 & -K_3 \\ 0 & -K_2 & K_2 & 0 \\ 0 & -K_3 & 0 & K_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}}_{\text{Structural Stiffness Matrix}}$$

EXAMPLE *cont.*

- Substitute boundary conditions and solve for the unknown displacements.
  - Let K_1 = 50 N/cm, K_2 = 30 N/cm, K_3 = 70 N/cm and f_1 = 40 N.

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{bmatrix} 50 & -50 & 0 & 0 \\ -50 & (50+30+70) & -30 & -70 \\ 0 & -30 & 30 & 0 \\ 0 & -70 & 0 & 70 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

  - Knowns: F_1, F_2, u_3, and u_4
  - Unknowns: F_3, F_4, u_1, and u_2

$$\begin{Bmatrix} 40 \\ 0 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{bmatrix} 50 & -50 & 0 & 0 \\ -50 & (50+30+70) & -30 & -70 \\ 0 & -30 & 30 & 0 \\ 0 & -70 & 0 & 70 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ 0 \\ 0 \end{Bmatrix}$$

EXAMPLE *cont.*

– Remove zero-displacement columns: u_3 and u_4.

$$\begin{Bmatrix} 40 \\ 0 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{bmatrix} 50 & -50 \\ -50 & 150 \\ 0 & -30 \\ 0 & -70 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

– Remove unknown force rows: F_3 and F_4.

$$\begin{Bmatrix} 40 \\ 0 \end{Bmatrix} = \begin{bmatrix} 50 & -50 \\ -50 & 150 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

– Now, the matrix should not be singular. Solve for u_1 and u_2.

$$u_1 = 1.2 \text{ cm}$$
$$u_2 = 0.4 \text{ cm}$$

– Using u_1 and u_2, Solve for F_3 and F_4.

$$F_3 = 0u_1 - 30u_2 = -12 \text{ N}$$
$$F_4 = 0u_1 - 70u_2 = -28 \text{ N}$$

EXAMPLE *cont.*

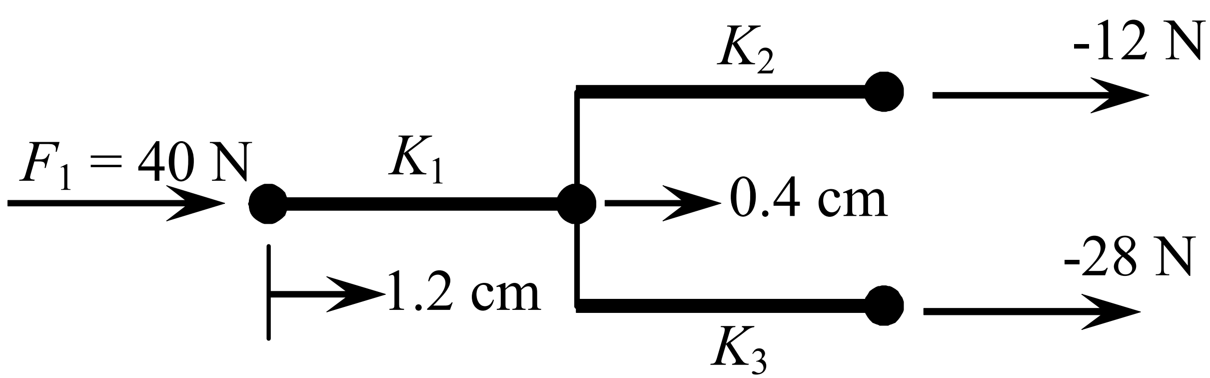
- Recover element data

Element force

$$\begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix} = \begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 50 & -50 \\ -50 & 50 \end{bmatrix} \begin{Bmatrix} 1.2 \\ 0.4 \end{Bmatrix} = \begin{Bmatrix} 40 \\ -40 \end{Bmatrix}$$

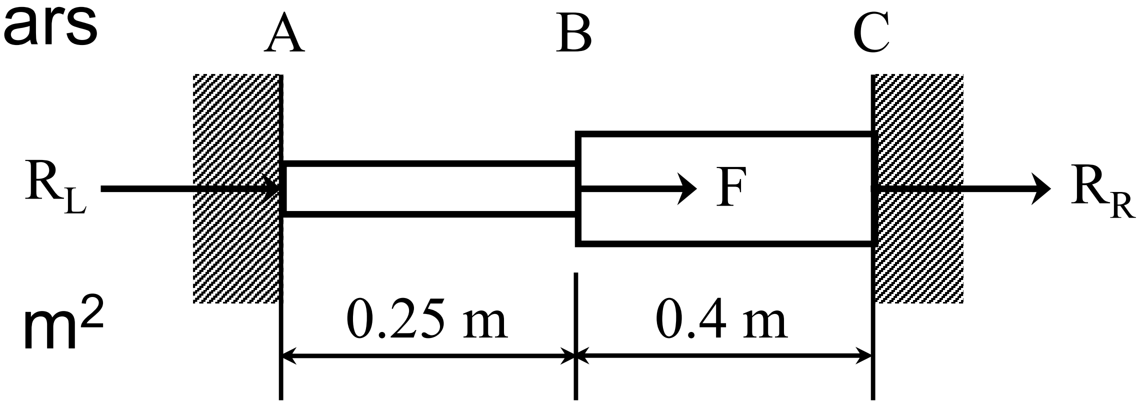
$$\begin{Bmatrix} f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix} = \begin{bmatrix} K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 30 & -30 \\ -30 & 30 \end{bmatrix} \begin{Bmatrix} 0.4 \\ 0.0 \end{Bmatrix} = \begin{Bmatrix} 12 \\ -12 \end{Bmatrix}$$

$$\begin{Bmatrix} f_2^{(3)} \\ f_4^{(3)} \end{Bmatrix} = \begin{bmatrix} K_3 & -K_3 \\ -K_3 & K_3 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_4 \end{Bmatrix} = \begin{bmatrix} 70 & -70 \\ -70 & 70 \end{bmatrix} \begin{Bmatrix} 0.4 \\ 0.0 \end{Bmatrix} = \begin{Bmatrix} 28 \\ -28 \end{Bmatrix}$$



EXAMPLE

- Statically indeterminate bars
- E = 100 GPa
- F = 10,000 N
- $A_1 = 10^{-4}$ m^2, $A_2 = 2\times10^{-4}$ m^2
- Element stiffness matrices:



$$[\mathbf{k}^{(1)}] = \frac{10^{11}\times10^{-4}}{0.25}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 10^7\begin{bmatrix}4 & -4\\ -4 & 4\end{bmatrix}\begin{matrix}u_1\\ u_2\end{matrix}$$

$$[\mathbf{k}^{(2)}] = \frac{10^{11}\times2\times10^{-4}}{0.4}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 10^7\begin{bmatrix}5 & -5\\ -5 & 5\end{bmatrix}\begin{matrix}u_2\\ u_3\end{matrix}$$

- Assembly

$$10^7\begin{bmatrix}4 & -4 & 0\\ -4 & 9 & -5\\ 0 & -5 & 5\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\\ u_3\end{Bmatrix} = \begin{Bmatrix}F_1\\ 10{,}000\\ F_3\end{Bmatrix}$$

EXAMPLE *cont.*

- Applying BC

$$10^7[9]\{u_2\} = \{10{,}000\} \Rightarrow u_2 = 1.11\times 10^{-4}\,\mathrm{m}$$

- Element forces or Element stresses

$$P = \frac{AE}{L}\left(u_j - u_i\right)$$

$$P^{(1)} = 4\times 10^7\left(u_2 - u_1\right) = 4{,}444\,\mathrm{N}$$

$$P^{(2)} = 5\times 10^7\left(u_3 - u_2\right) = -5{,}556\,\mathrm{N}$$

$$\sigma^{(1)} = \frac{P^{(1)}}{A^{(1)}} = 4.444\times 10^7\,\mathrm{Pa} = 44.44\,\mathrm{MPa}$$

$$\sigma^{(2)} = \frac{P^{(2)}}{A^{(2)}} = -2.778\times 10^7\,\mathrm{Pa} = -27.78\,\mathrm{MPa}$$

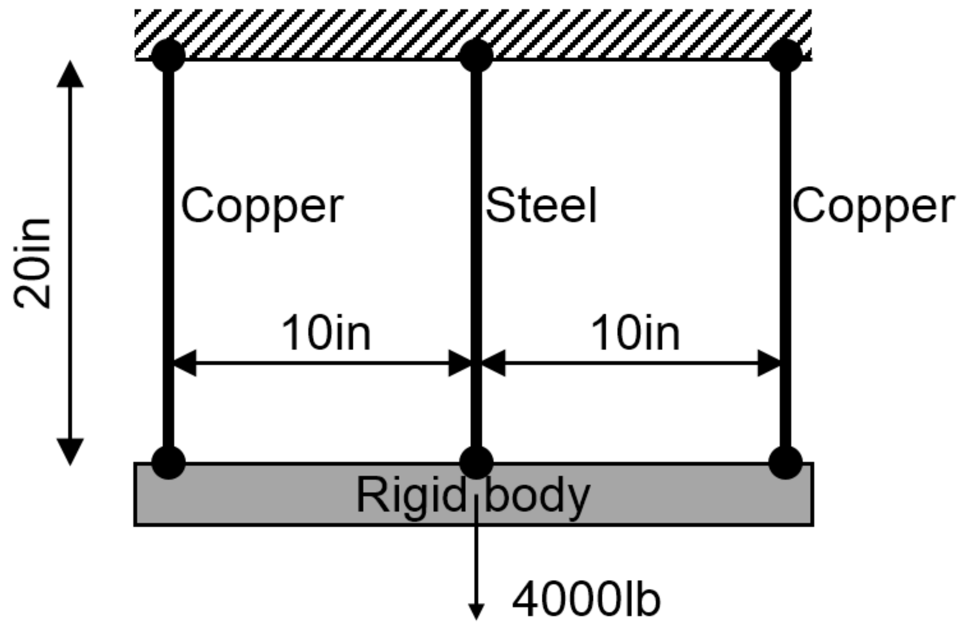
- Reaction forces

$$R_L = -P^{(1)} = -4{,}444\,\mathrm{N}$$

$$R_R = +P^{(2)} = -5{,}556\,\mathrm{N}$$

Exercise

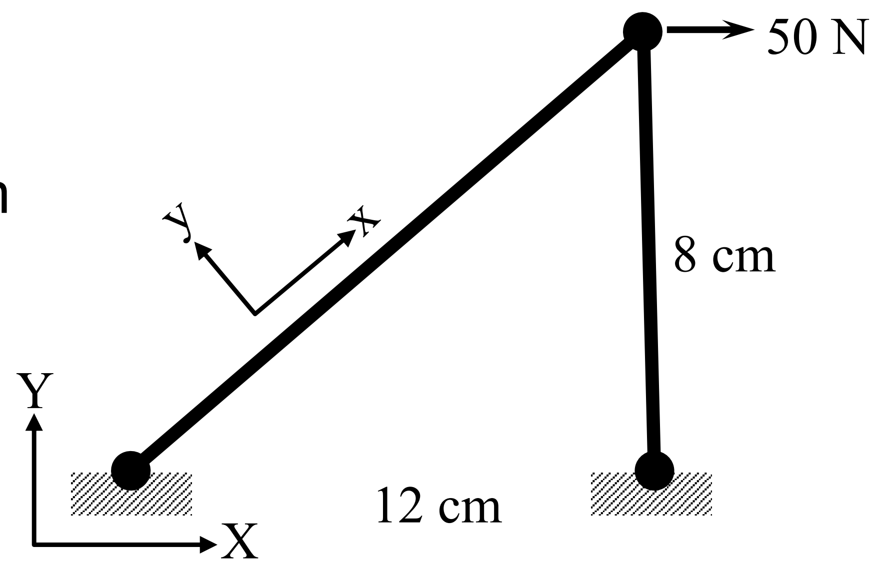
- Two copper wires and a steel wire are connected by a rigid body as shown in the figure. The three wires are all in equal length of 20in and in the same cross-sectional area of A = $0.1 in^2$. When a load Q = 4000lb is applied, find the stresses in the copper and steel wires using three bar elements. Note that the load Q is applied at the center of the rigid body. For the copper wire, use the elastic modulus $E_c = 1.6 \times 10^7$psi. For the steel wire, use the elastic modulus $E_s = 3.0 \times 10^7$psi.



1.3 PLANE TRUSS ELEMENT

PLANE TRUSS ELEMENT

- What is the difference between 1D and 2D finite elements?
  - 2D element can move x- and y-direction (2 DOFs per node).
  - However, the stiffness can be applied only axial direction.
- Local Coordinate System
  - 1D FE formulation can be used if a body-fixed local coordinate system is constructed along the length of the element
  - The global coordinate system (*X* and *Y* axes) is chosen to represent the entire structure
  - The local coordinate system (*x* and *y* axes) is selected to align the *x*-axis along the length of the element



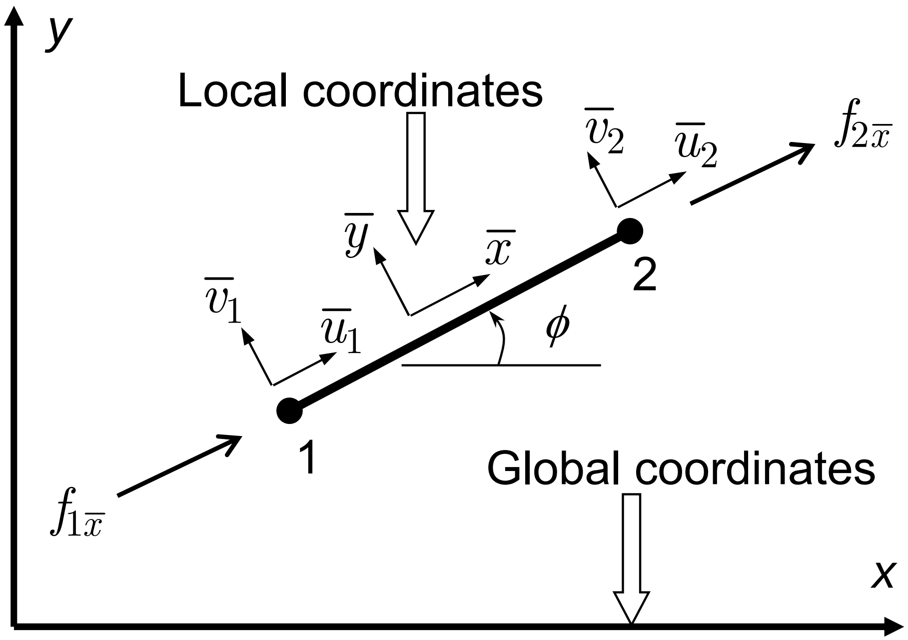
$$\begin{Bmatrix} f_{1\bar{x}} \\ f_{2\bar{x}} \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix}$$

PLANE TRUSS ELEMENT *cont.*

- Element Equation (Local Coordinate System)
  - Axial direction is the local x-axis.
  - 2D element equation

$$\begin{Bmatrix} f_{1\bar{x}} \\ f_{1\bar{y}} \\ f_{2\bar{x}} \\ f_{2\bar{y}} \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \end{Bmatrix}$$

$$\{\bar{\mathbf{f}}\} = [\bar{\mathbf{k}}]\{\bar{\mathbf{q}}\}$$



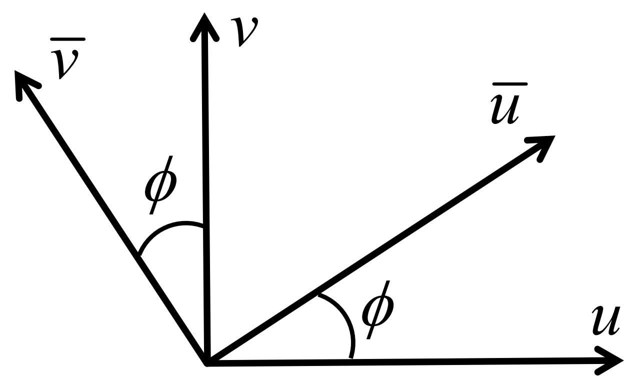
  - $[\bar{\mathbf{k}}]$ is square, symmetric, positive semi-definite, and non-negative diagonal components.
- How to connect to the neighboring elements?
  - Cannot connect to other elements because LCS is different
  - Use coordinate transformation

COORDINATE TRANSFORMATION

- Transform to the global coord. and assemble

$$\boxed{\begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}}$$

$$\boxed{\begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix}}$$



- Transformation matrix

$$\underbrace{\begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \end{Bmatrix}}_{\text{local}} = \begin{bmatrix} \cos\phi & \sin\phi & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi \\ 0 & 0 & -\sin\phi & \cos\phi \end{bmatrix} \underbrace{\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}}_{\text{global}}$$

$$\boxed{\{\bar{\mathbf{q}}\} = [\mathbf{T}]\{\mathbf{q}\}}$$

Transformation matrix

COORDINATE TRANSFORMATION *cont.*

- The same transformation for force vector

$$\underbrace{\begin{Bmatrix} f_{1\bar{x}} \\ f_{1\bar{y}} \\ f_{2\bar{x}} \\ f_{2\bar{y}} \end{Bmatrix}}_{\text{local}} = \begin{bmatrix} \cos\phi & \sin\phi & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi \\ 0 & 0 & -\sin\phi & \cos\phi \end{bmatrix} \underbrace{\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix}}_{\text{global}}$$

$$\boxed{\{\bar{\mathbf{f}}\} = [\mathbf{T}]\{\mathbf{f}\}}$$

- Property of transformation matrix

$$[\mathbf{T}]^{-1} = [\mathbf{T}]^T \qquad \{\bar{\mathbf{f}}\} = [\mathbf{T}]\{\mathbf{f}\} \quad \Longrightarrow \quad \{\mathbf{f}\} = [\mathbf{T}]^T\{\bar{\mathbf{f}}\}$$

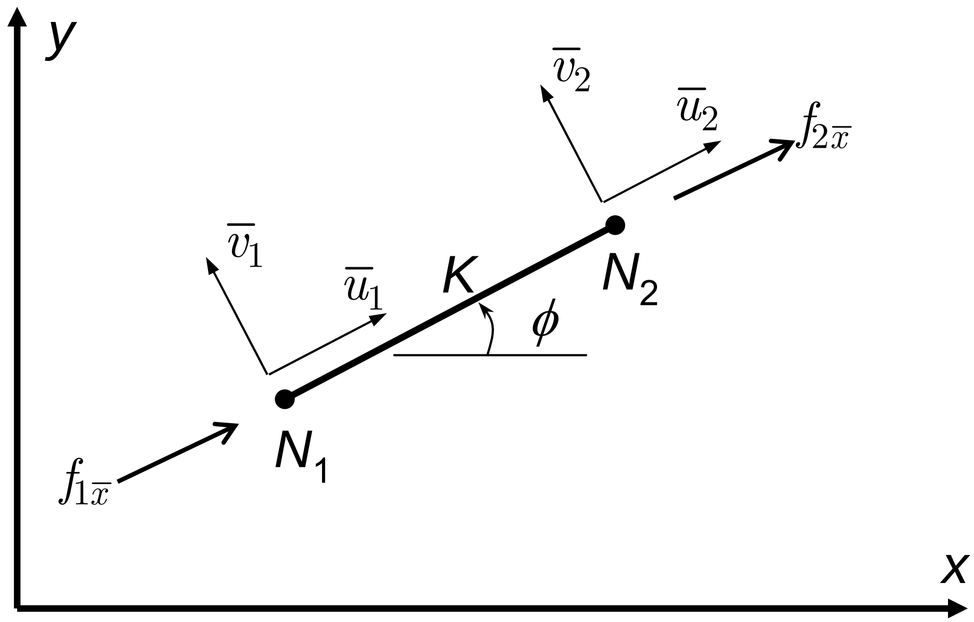
[**T**] is an orthogonal matrix

ELEMENT STIFFNESS IN GLOBAL COORD.

- Element 1

$$\begin{Bmatrix} f_{1\bar{x}} \\ f_{1\bar{y}} \\ f_{2\bar{x}} \\ f_{2\bar{y}} \end{Bmatrix} = \underbrace{\boxed{\frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}}_{\text{element stiffness matrix}} \begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \end{Bmatrix}$$

$$\{\bar{\mathbf{f}}\} = [\bar{\mathbf{k}}]\{\bar{\mathbf{q}}\}$$



- Transform to the global coordinates

$$[\mathbf{T}]\{\mathbf{f}\} = [\bar{\mathbf{k}}][\mathbf{T}]\{\mathbf{q}\} \Longrightarrow \underbrace{\{\mathbf{f}\}}_{\text{global}} = [\mathbf{T}]^{-1}[\bar{\mathbf{k}}][\mathbf{T}]\underbrace{\{\mathbf{q}\}}_{\text{global}}$$

$$\Downarrow$$

$$\boxed{[\mathbf{k}] = [\mathbf{T}]^{-1}[\bar{\mathbf{k}}][\mathbf{T}]} \Longleftarrow \{\mathbf{f}\} = [\mathbf{k}]\{\mathbf{q}\}$$

ELEMENT STIFFNESS IN GLOBAL COORD. *cont.*

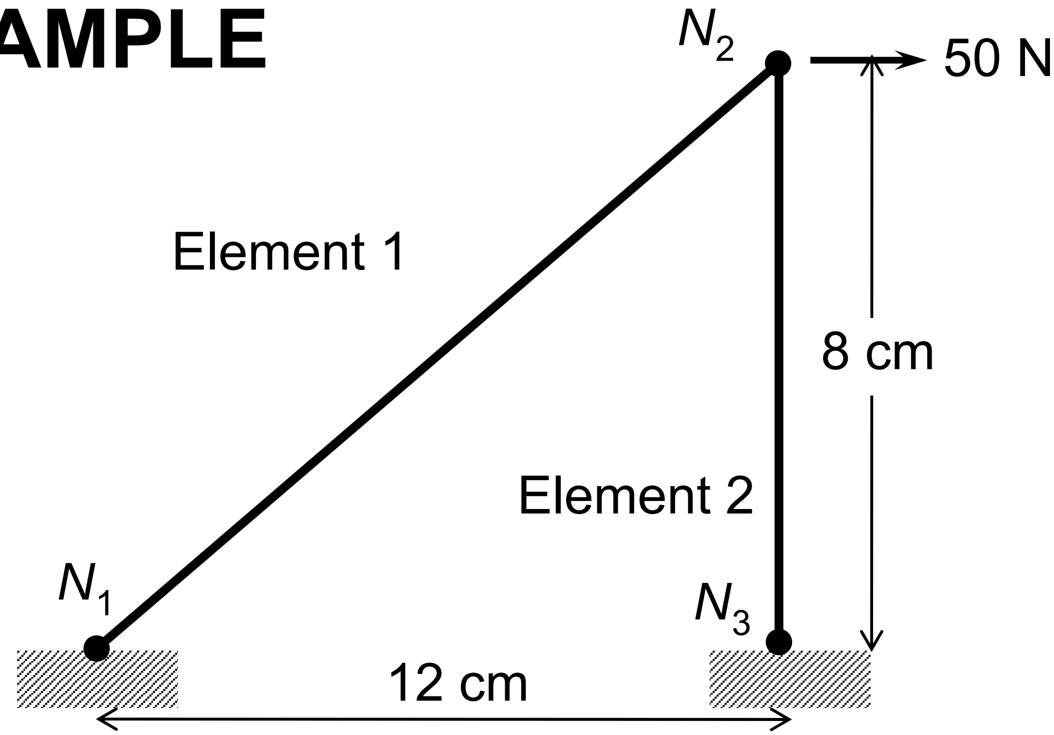
- Element stiffness matrix in global coordinates

$$[\mathbf{k}] = [\mathbf{T}]^T [\bar{\mathbf{k}}][\mathbf{T}]$$

$$[\mathbf{k}] = \frac{EA}{L}\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi & -\cos^2\phi & -\cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi & -\cos\phi\sin\phi & -\sin^2\phi \\ -\cos^2\phi & -\cos\phi\sin\phi & \cos^2\phi & \cos\phi\sin\phi \\ -\cos\phi\sin\phi & -\sin^2\phi & \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

  - Depends on Young's modulus (E), cross-sectional area (A), length (L), and angle of rotation (ϕ)
  - Axial rigidity = EA
  - Square, symmetric, positive semi-definite, singular, and non-negative diagonal terms

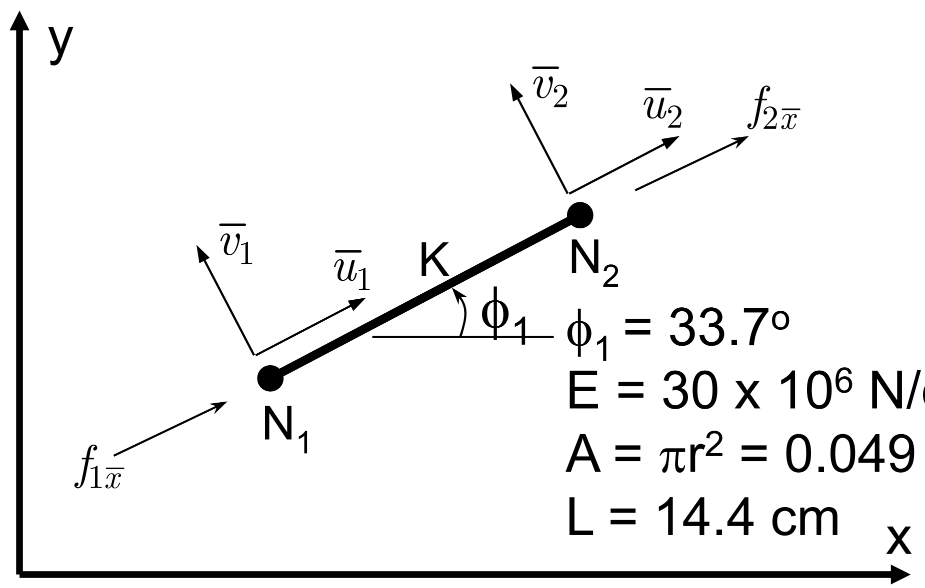
EXAMPLE



- Two-bar truss
  - Diameter = 0.25 cm
  - E = 30×10^6 N/cm^2
- Element 1
  - Connectivity: $N_1 \rightarrow N_2$
  - In local coordinate

$$\{\bar{\mathbf{f}}^{(1)}\} = [\bar{\mathbf{k}}^{(1)}]\{\bar{\mathbf{q}}^{(1)}\}$$

$$\begin{Bmatrix} f_{1\bar{x}} \\ f_{2\bar{x}} \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix}$$



EXAMPLE *cont.*

- Element 1 cont.
  - Element equation in the global coordinates

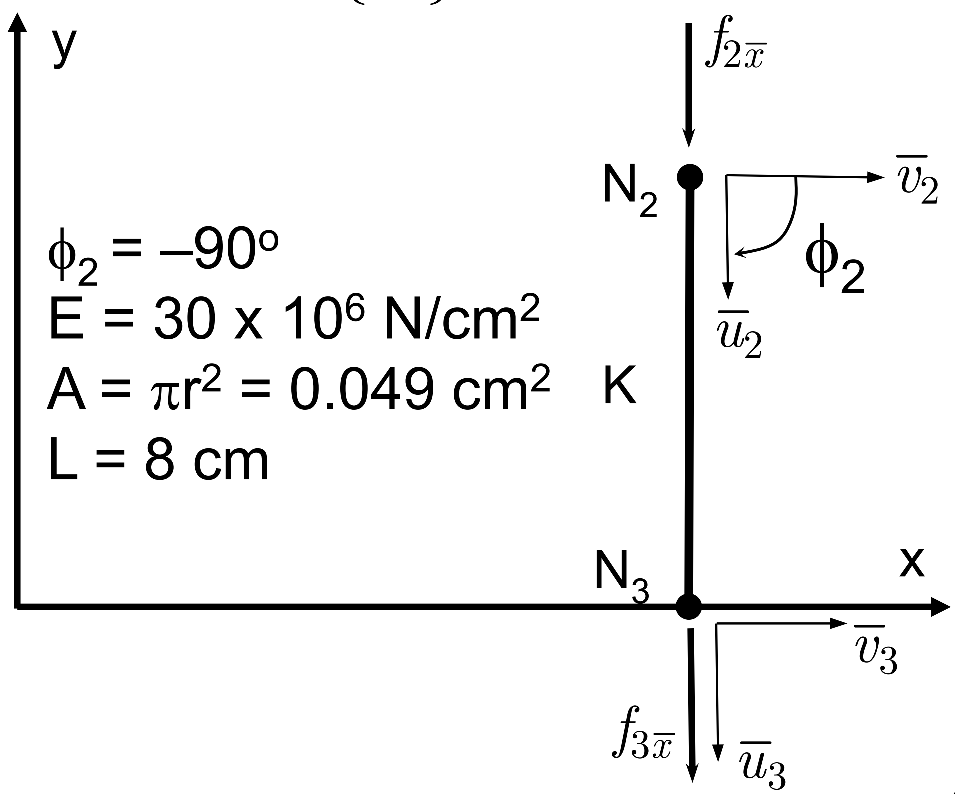
$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{1y}^{(1)} \\ f_{2x}^{(1)} \\ f_{2y}^{(1)} \end{Bmatrix} = 102150 \begin{bmatrix} 0.692 & 0.462 & -0.692 & -0.462 \\ 0.462 & 0.308 & -0.462 & -0.308 \\ -0.692 & -0.462 & 0.692 & 0.462 \\ -0.462 & -0.308 & 0.462 & 0.308 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$\{\mathbf{f}^{(1)}\} = [\mathbf{k}^{(1)}]\{\mathbf{q}^{(1)}\}$$

- Element 2
  - Connectivity: $N_2 \rightarrow N_3$

$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{2y}^{(2)} \\ f_{3x}^{(2)} \\ f_{3y}^{(2)} \end{Bmatrix} = 184125 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$\phi_2 = -90^o$
$E = 30 \times 10^6$ N/cm^2
$A = \pi r^2 = 0.049$ cm^2
$L = 8$ cm



EXAMPLE *cont.*

- Assembly
  - After transforming to the global coordinates

Element 1

$$
\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix} = \begin{bmatrix} 70687 & 47193 & -70687 & -47193 & 0 & 0 \\ 47193 & 31462 & -47193 & -31462 & 0 & 0 \\ -70687 & -47193 & 70687 & 47193 & 0 & 0 \\ -47193 & -31462 & 47193 & 215587 & 0 & -184125 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -184125 & 0 & 184125 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}
$$

Element 2

- Boundary Conditions
  - Nodes 1 and 3 are fixed.
  - Node 2 has known applied forces: $F_{2x} = 50$ N, $F_{2y} = 0$ N

EXAMPLE *cont.*

- Boundary conditions (striking-the-columns)

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ 50 \\ 0 \\ F_{3x} \\ F_{3y} \end{Bmatrix} = \begin{bmatrix} 70687 & 47193 & -70687 & -47193 & 0 & 0 \\ 47193 & 31462 & -47193 & -31462 & 0 & 0 \\ -70687 & -47193 & 70687 & 47193 & 0 & 0 \\ -47193 & -31462 & 47193 & 215587 & 0 & -184125 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -184125 & 0 & 184125 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ 0 \\ 0 \end{Bmatrix}$$

  - Striking-the-rows

$$\begin{Bmatrix} 50 \\ 0 \end{Bmatrix} = \begin{bmatrix} 70687 & 47193 \\ 47193 & 215587 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix}$$

- Solve the global matrix equation

$$u_2 = 8.28 \times 10^{-4} \text{ cm}$$

$$v_2 = -1.81 \times 10^{-4} \text{ cm}$$

EXAMPLE *cont.*

- Support reactions

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{3x} \\ F_{3y} \end{Bmatrix} = \begin{bmatrix} -70687 & -47193 \\ -47193 & -31462 \\ 0 & 0 \\ 0 & -184125 \end{bmatrix} \begin{Bmatrix} 8.28\times 10^{-4} \\ -1.81\times 10^{-4} \end{Bmatrix} = \begin{Bmatrix} -50 \\ -33.39 \\ 0 \\ 33.39 \end{Bmatrix} N$$

  - The reaction force is parallel to the element length (two-force member)

- Element force and stress (Element 1)
  - Need to transform to the element local coordinates

$$\begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix} = \begin{bmatrix} .832 & .555 & 0 & 0 \\ 0 & 0 & .832 & .555 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 5.89\times 10^{-4} \end{Bmatrix}$$

EXAMPLE *cont.*

- Element force and stress (Element 1) *cont*.
  - Element force can only be calculated using local element equation

$$\begin{Bmatrix} f_{1\bar{x}} \\ f_{2\bar{x}} \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 5.89\times10^{-4} \end{Bmatrix} = \begin{Bmatrix} -60.2 \\ 60.2 \end{Bmatrix} N$$

  - There is no force components in the local y-direction
  - In x-direction, two forces are equal and opposite
  - The force in the second node is equal to the element force
  - Normal stress = 60.2 / 0.049 = 1228 N/cm^2.

OTHER WAY OF ELEMENT FORCE CALCULATION

- Element force for plane truss

$$P^{(e)} = \left(\frac{AE}{L}\right)^{(e)} \Delta^{(e)} = \left(\frac{AE}{L}\right)^{(e)} \left(\bar{u}_j - \bar{u}_i\right)$$

  – Write in terms of global displacements

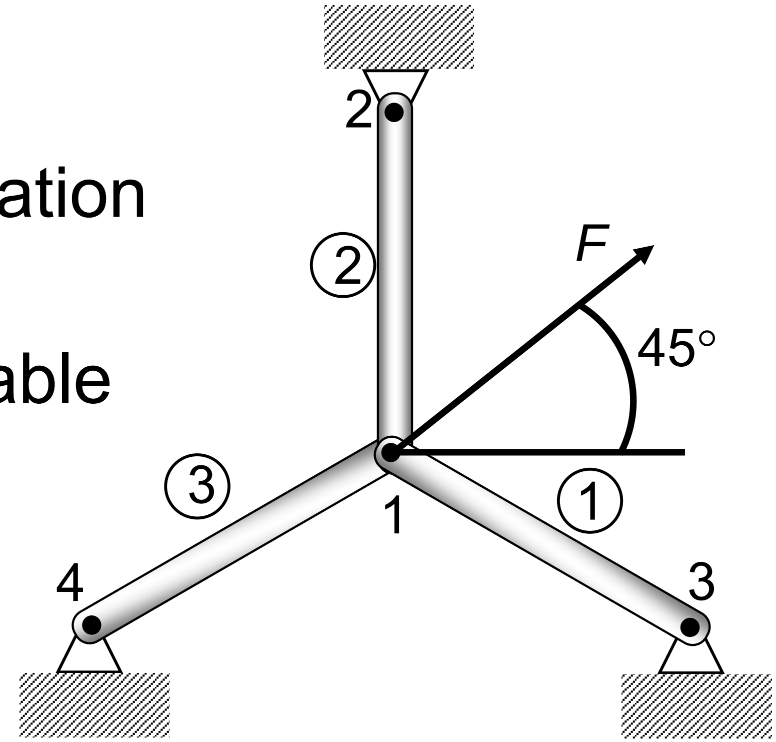
$$P^{(e)} = \left(\frac{AE}{L}\right)^{(e)} \left(\left(lu_j + mv_j\right) - \left(lu_i + mv_i\right)\right)$$

$$= \left(\frac{AE}{L}\right)^{(e)} \left(l\left(u_j - u_i\right) + m\left(v_j - v_i\right)\right)$$

$$l = \cos\phi$$

$$m = \sin\phi$$

EXAMPLE



- Directly assembling global matrix equation (applying BC in the element level)
- Element property & direction cosine table

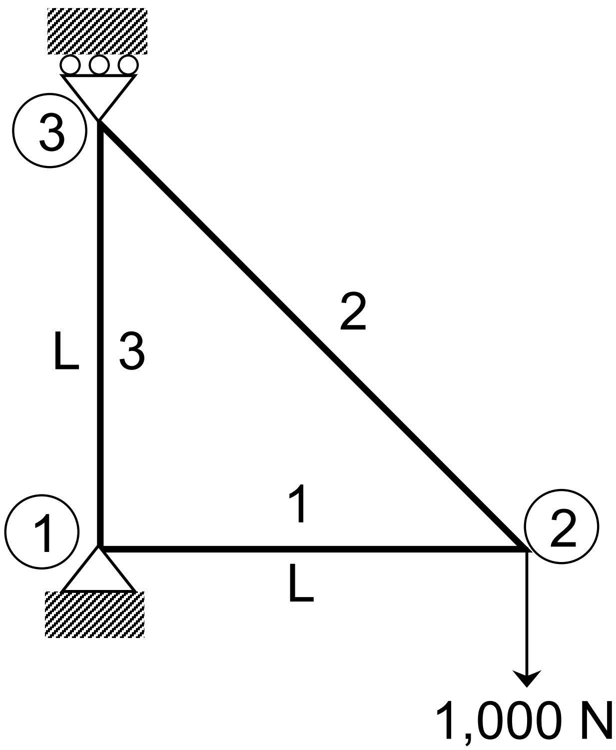
Elem	AE/L	i -> j	ϕ	$l = \cos\phi$	$m = \sin\phi$
1	206×10^5	1 -> 3	-30	$\sqrt{3}/2$	$-1/2$
2	206×10^5	1 -> 2	90	0	1
3	206×10^5	1 -> 4	210	$-\sqrt{3}/2$	$-1/2$

- Since u_3 and v_3 will be deleted after assembly, it is not necessary to keep them

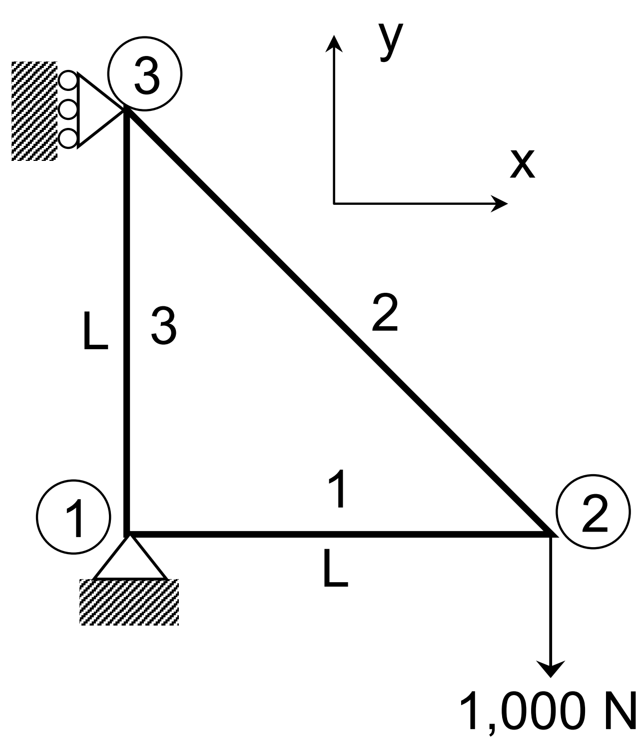
$$
[\mathbf{k}^{(1)}] = \left(\frac{EA}{L}\right)^{(1)}
\begin{array}{c}
\begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\
\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}
\end{array}
\begin{array}{c} u_1 \\ v_1 \\ u_3 \\ v_3 \end{array}
\quad \Longrightarrow \quad
[\mathbf{k}^{(1)}] = \left(\frac{EA}{L}\right)^{(1)}
\begin{array}{c}
\begin{array}{cc} u_1 & v_1 \end{array} \\
\begin{bmatrix} l^2 & lm \\ lm & m^2 \end{bmatrix}
\end{array}
\begin{array}{c} u_1 \\ v_1 \end{array}
$$

Exercise

- Using FE method, solve the vertical displacement at Node 2. All three truss elements have EA = 10^6N and L = 1m. For boundary conditions, use a pin joint at Node 1 and $v_3 = 0$ for (a) and $u_3 = 0$ for (b). Discuss about finite element results.



(a)



(b)

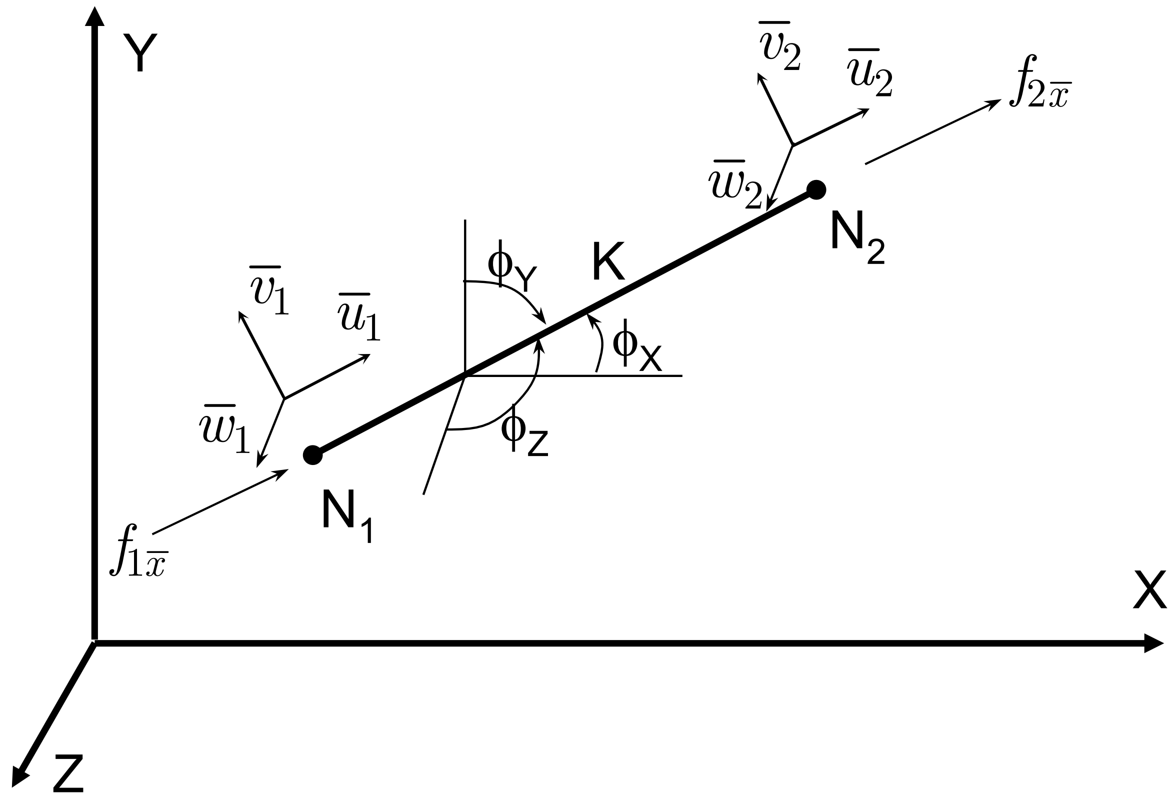
1.4 SPACE TRUSS

SPACE TRUSS ELEMENT

- A similar extension using coordinate transformation

- 3DOF per node
  - u, v, and w
  - f_x, f_y, and f_z

- Element stiffness matrix is 6x6



- FE equation in the local coord.

$$\begin{Bmatrix} f_{i\bar{x}} \\ f_{j\bar{x}} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \bar{u}_i \\ \bar{u}_j \end{Bmatrix} \qquad \{\bar{\mathbf{f}}\} = [\bar{\mathbf{k}}]\{\bar{\mathbf{q}}\}$$

SPACE TRUSS ELEMENT cont.

- Relation between local and global displacements
  - Each node has 3 DOFs (u_i, v_i, w_i)

$$\begin{Bmatrix} \bar{u}_i \\ \bar{u}_j \end{Bmatrix} = \begin{bmatrix} l & m & n & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & n \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ w_i \\ u_j \\ v_j \\ w_j \end{Bmatrix}$$

$$\underset{(2\times 1)}{\{\bar{\mathbf{q}}\}} = \underset{(2\times 6)}{[\mathbf{T}]} \cdot \underset{(6\times 1)}{\{\mathbf{q}\}}$$

  - Direction cosines

$$l = \cos\phi_x = \frac{x_j - x_i}{L}, \quad m = \cos\phi_y = \frac{y_j - y_i}{L}, \quad n = \cos\phi_z = \frac{z_j - z_i}{L}$$

$$L = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2}$$

SPACE TRUSS ELEMENT cont.

- Relation between local and global force vectors

$$\begin{Bmatrix} f_{ix} \\ f_{iy} \\ f_{iz} \\ f_{jx} \\ f_{jy} \\ f_{jz} \end{Bmatrix} = \begin{bmatrix} l & 0 \\ m & 0 \\ n & 0 \\ 0 & l \\ 0 & m \\ 0 & n \end{bmatrix} \begin{Bmatrix} f_{i\bar{x}} \\ f_{j\bar{x}} \end{Bmatrix} \qquad \{\mathbf{f}\} = [\mathbf{T}]^T \{\bar{\mathbf{f}}\}$$

- Stiffness matrix

$$\{\bar{\mathbf{f}}\} = [\bar{\mathbf{k}}]\{\bar{\mathbf{q}}\} \Rightarrow [\mathrm{T}]^T\{\bar{\mathbf{f}}\} = [\mathrm{T}]^T[\bar{\mathbf{k}}][\mathrm{T}]\{\mathbf{q}\} \Rightarrow \{\mathbf{f}\} = [\mathbf{k}]\{\mathbf{q}\}$$

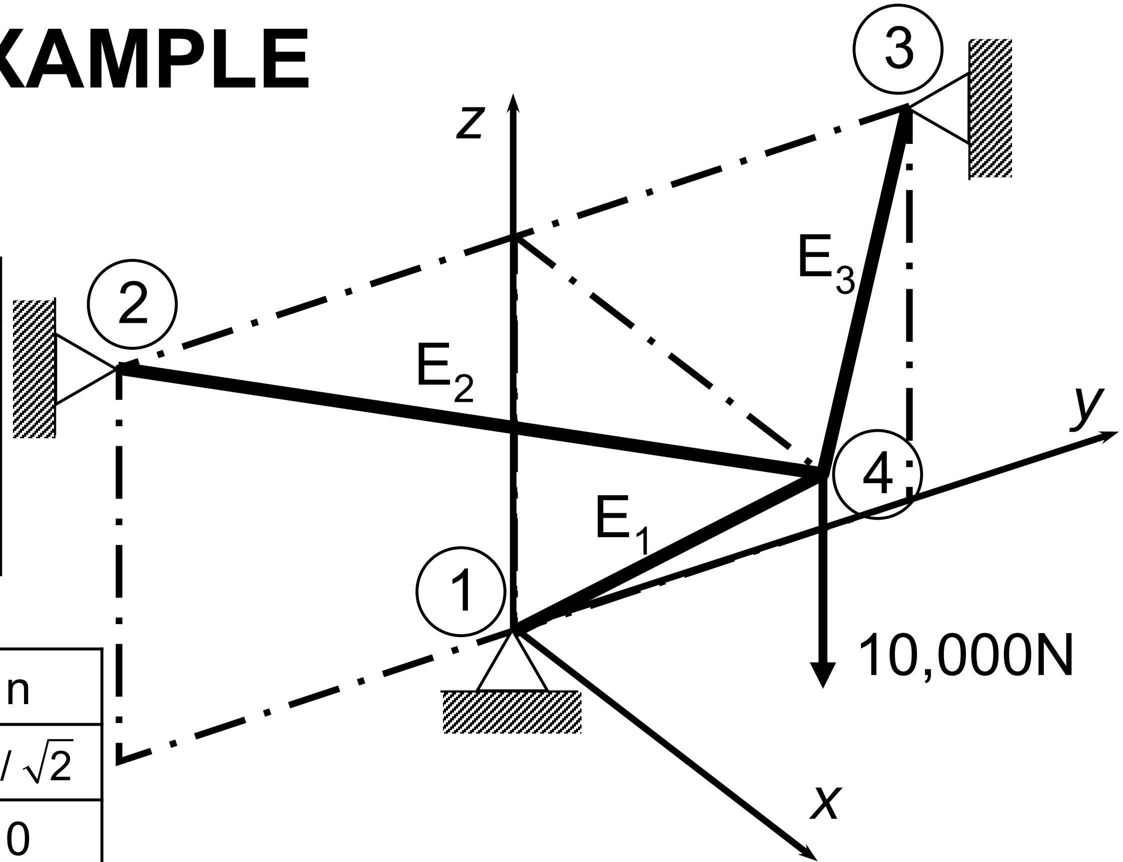
$$[\mathbf{k}] = [\mathbf{T}]^T[\bar{\mathbf{k}}][\mathbf{T}]$$

$$[\mathbf{k}] = \frac{EA}{L} \begin{array}{c} \begin{bmatrix} l^2 & lm & ln & -l^2 & -lm & -ln \\ & m^2 & mn & -lm & -m^2 & -mn \\ & & n^2 & -ln & -mn & -n^2 \\ & & & l^2 & lm & ln \\ & \text{sym} & & & m^2 & mn \\ & & & & & n^2 \end{bmatrix} \end{array} \begin{array}{c} u_i \\ v_i \\ w_i \\ u_j \\ v_j \\ w_j \end{array}$$

EXAMPLE

- Space truss

Node	x	y	z
1	0	0	0
2	0	–1	1
3	0	1	1
4	1	0	1



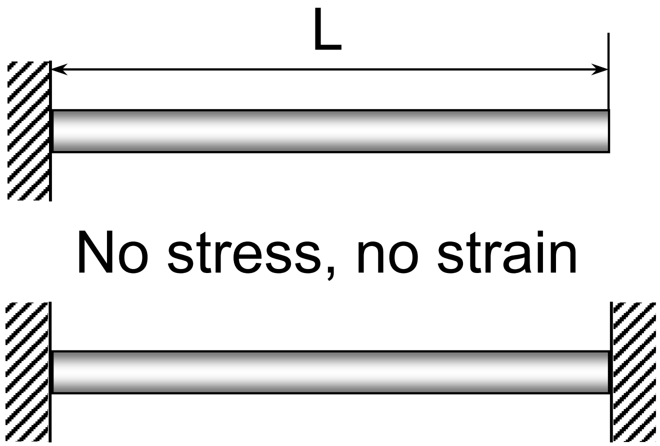
Elem	EA/L	i -> j	l	m	n
1	$35\sqrt{2}\times10^5$	1 -> 4	$1/\sqrt{2}$	0	$1/\sqrt{2}$
2	$35\sqrt{2}\times10^5$	2 -> 4	$1/\sqrt{2}$	$1/\sqrt{2}$	0
3	$35\sqrt{2}\times10^5$	3 -> 4	$1/\sqrt{2}$	$-1/\sqrt{2}$	0

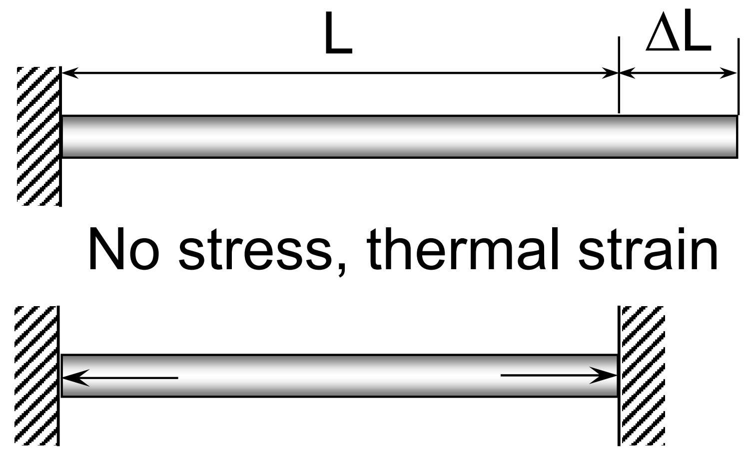
$$[\mathbf{k}] = \frac{EA}{L}\begin{array}{c}\left[\begin{array}{cccccc} l^2 & lm & ln & -l^2 & -lm & -ln \\ & m^2 & mn & -lm & -m^2 & -mn \\ & & n^2 & -ln & -mn & -n^2 \\ & & & l^2 & lm & ln \\ & \text{sym} & & & m^2 & mn \\ & & & & & n^2 \end{array}\right]\end{array}\begin{array}{c} u_i \\ v_i \\ w_i \\ u_j \\ v_j \\ w_j \end{array}$$

1.5 THERMAL STRESSES

THERMAL STRESSES

- Temperature change causes thermal strain





Thermal stress, no strain

(a) at $T = T_{ref}$

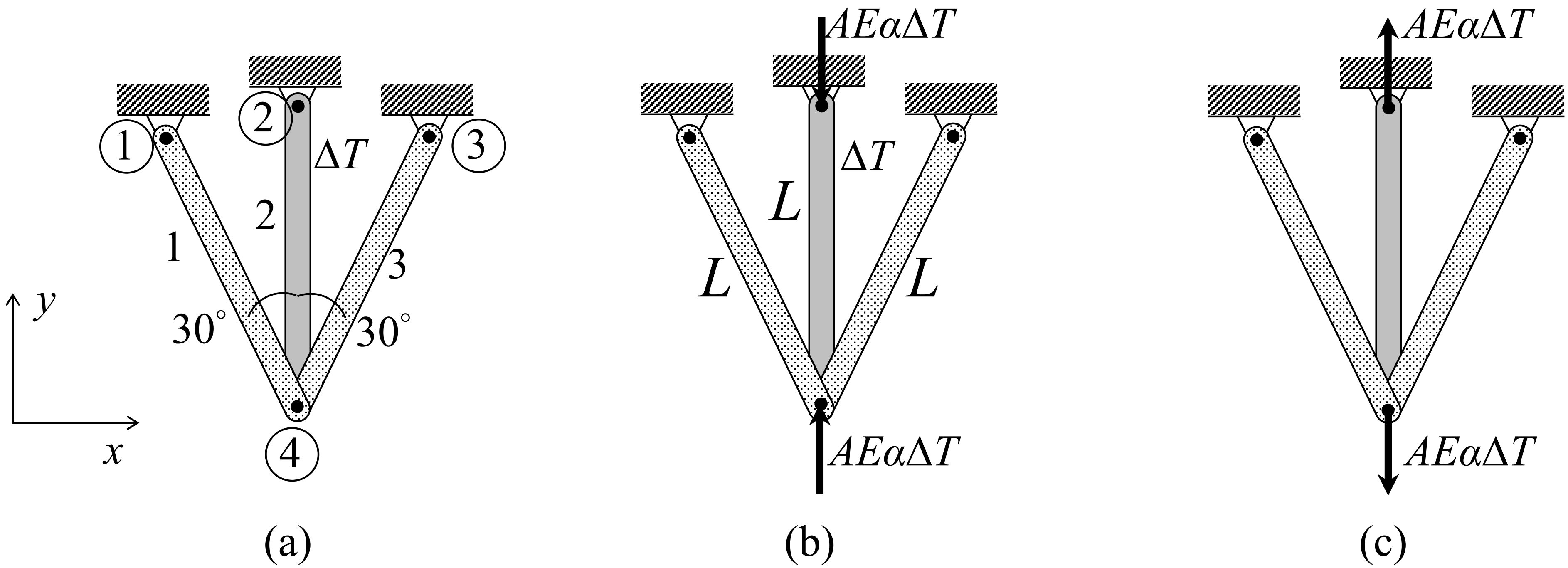
(b) at $T = T_{ref} + \Delta T$

- Constraints cause thermal stresses
- Thermo-elastic stress-strain relationship

$$\sigma = E(\varepsilon - \underbrace{\alpha}_{\text{Coefficient of thermal expansion (CTE)}}\Delta T)$$

$$\underbrace{\varepsilon}_{\text{Total strain (FE solution)}} = \underbrace{\frac{\sigma}{E}}_{\text{Mechanical strain (Generate stress))}} + \underbrace{\alpha\Delta T}_{\text{Thermal strain}}$$

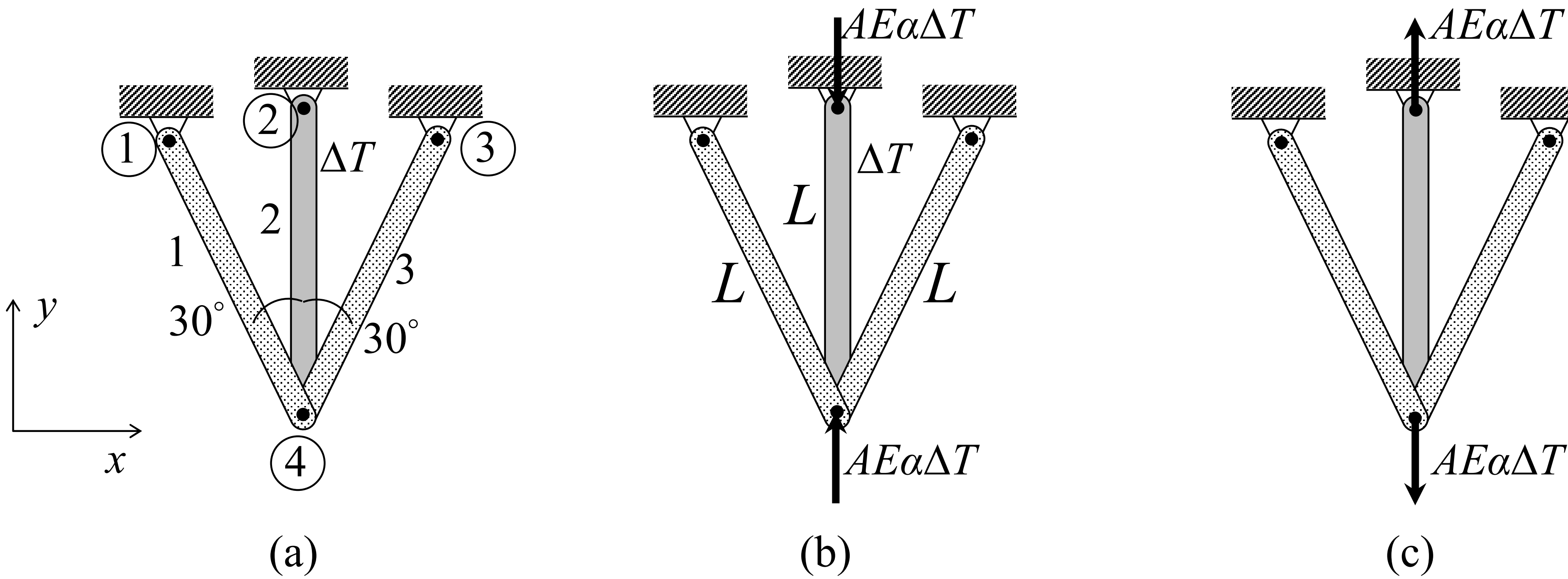
Method of Superposition



- Problem I
  - Initially 3 truss elements have no stresses (a)
  - Temperature in Elem 2 increases by ΔT
  - Assume 3 elements are disconnected and Elem 2 expands
  - Apply force $-AE\alpha\Delta T$ to Elem 2 to remove elongation (b)

$$P^{(1)} = P^{(3)} = 0,\ P^{(2)} = -AE\alpha\Delta T$$

Method of Superposition cont.



- Problem II
  - Now all 3 truss elements are connected
  - Forces applied in Problem I is reversed, no thermal effect (c)
  - Fictitious thermal force

$$P^{(2)} = AE\alpha\Delta T$$

  - Use standard FEA to calculate truss problem
- Final solution = Solution of Problem I + Solution of Problem II

Example

- Solve the method of superposition example with $AE = 10^7$ N, $L = 1$ m, $\alpha = 10^{-5}/°C$, $\Delta T = 100°C$
- Problem I:

$$u_4 = v_4 = 0$$

$$P^{(1)} = 0,\ P^{(2)} = -AE\alpha\Delta T = -10{,}000\,\text{N},\ P^{(3)} = 0$$

- Problem II:
  - Connectivity table

Elem.	LN1 (i)	LN2 (j)	AE/L (N/m)	$AE\alpha\Delta T$ (N)	ϕ (degree)	$l=\cos\phi$	$m=\sin\phi$
1	1	4	10^7	0	–60	1/2	$-\sqrt{3}/2$
2	2	4	10^7	10,000	–90	0	–1
3	3	4	10^7	0	240	–1/2	$-\sqrt{3}/2$

Example cont.

- Element stiffness (after removing boundary DOFs)

$$[\mathbf{k}^{(1)}] = \frac{10^7}{4}\begin{bmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{bmatrix}\begin{matrix} u_4 \\ v_4 \end{matrix} \qquad [\mathbf{k}^{(2)}] = 10^7\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{matrix} u_4 \\ v_4 \end{matrix}$$

$$[\mathbf{k}^{(3)}] = \frac{10^7}{4}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{bmatrix}\begin{matrix} u_4 \\ v_4 \end{matrix}$$

- Assembly and solution

$$\frac{10^7}{4}\begin{bmatrix} 2 & 0 \\ 0 & 10 \end{bmatrix}\begin{Bmatrix} u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10{,}000 \end{Bmatrix} \Rightarrow \begin{aligned} u_4 &= 0 \\ v_4 &= -0.4\times 10^{-3}\,\mathrm{m} \end{aligned}$$

- Element forces

$$P^{(1)} = 3{,}464\,\mathrm{N}$$

$$P^{(2)} = 4{,}000\,\mathrm{N}$$

$$P^{(3)} = 3{,}464\,\mathrm{N}$$

Example cont.

- Final solution = Solution of Problem I + Solution of Problem II

Variable	Problem I	Problem II	Final Solution
u_4	0	0	0
v_4	0	-0.4×10^{-3} m	-0.4×10^{-3} m
$P^{(1)}$	$-AE\alpha\Delta T^{(1)}=0$	3,464 N	3,464 N
$P^{(2)}$	$-AE\alpha\Delta T^{(2)}= -10{,}000$ N	4,000 N	–6,000 N
$P^{(3)}$	$-AE\alpha\Delta T^{(3)}=0$	3,464 N	3,464 N

Thermal Stress Using FEA

- Force-displacement relation

$$P = AE\left(\frac{\Delta L}{L} - \alpha\Delta T\right) = AE\frac{\Delta L}{L} - AE\alpha\Delta T$$

- Finite element equation (1D)

$$\{\bar{\mathbf{f}}^{(e)}\} = [\bar{\mathbf{k}}^{(e)}]\{\bar{\mathbf{q}}^{(e)}\} - \{\bar{\mathbf{f}}_T^{(e)}\}$$

Thermal force vector

$$\{\bar{\mathbf{f}}_T^{(e)}\} = AE\alpha\Delta T\begin{Bmatrix}-1\\+1\end{Bmatrix}\begin{matrix}\bar{u}_i\\\bar{u}_j\end{matrix}$$

- For plane truss, transform to the global coord.

$$\{\mathbf{f}\} = [\mathbf{k}]\{\mathbf{q}\} - \{\mathbf{f}_T\}$$

$$\Downarrow$$

$$[\mathbf{k}]\{\mathbf{q}\} = \{\mathbf{f}\} + \{\mathbf{f}_T\}$$

$$\Downarrow$$

$$[\mathbf{K}_s]\{\mathbf{Q}_s\} = \{\mathbf{F}_s\} + \{\mathbf{F}_{Ts}\}$$

$$\{\mathbf{f}_T\} = AE\alpha\Delta T\begin{Bmatrix}-l\\-m\\+l\\+m\end{Bmatrix}\begin{matrix}u_i\\v_i\\u_j\\v_j\end{matrix}$$

Example

- Three truss example using FEA

$$\{\mathbf{f}_T^{(1)}\} = AE\alpha\Delta T^{(1)} \begin{Bmatrix} 1/2 \\ -\sqrt{3}/2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \begin{matrix} u_4 \\ v_4 \end{matrix}$$

$$\{\mathbf{f}_T^{(2)}\} = AE\alpha\Delta T^{(2)} \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10{,}000 \end{Bmatrix} \begin{matrix} u_4 \\ v_4 \end{matrix} \quad \Longrightarrow \quad \{\mathbf{F}_T\} = \begin{Bmatrix} 0 \\ -10{,}000 \end{Bmatrix}$$

$$\{\mathbf{f}_T^{(3)}\} = AE\alpha\Delta T^{(3)} \begin{Bmatrix} 1/2 \\ -\sqrt{3}/2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \begin{matrix} u_4 \\ v_4 \end{matrix}$$

- Global matrix equation

$$[\mathbf{K}]\{\mathbf{Q}\} = \cancel{\{\mathbf{F}\}} + \{\mathbf{F}_T\} \quad \Longrightarrow \quad \frac{10^7}{4} \begin{bmatrix} 2 & 0 \\ 0 & 10 \end{bmatrix} \begin{Bmatrix} u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10{,}000 \end{Bmatrix}$$

$$\boxed{\begin{aligned} u_4 &= 0 \\ v_4 &= -0.4 \times 10^{-3}\ \text{m} \end{aligned}}$$

Example cont.

- Element forces

$$P = AE\left(\frac{\Delta L}{L} - \alpha \Delta T\right)$$

$$= \frac{AE}{L}\left[l\left(u_j - u_i\right) + m\left(v_j - v_i\right)\right] - AE\alpha\Delta T$$

$$\boxed{\begin{aligned} P^{(1)} &= 3{,}464\,N \\ P^{(2)} &= -6{,}000\,N \\ P^{(3)} &= 3{,}464\,N \end{aligned}}$$

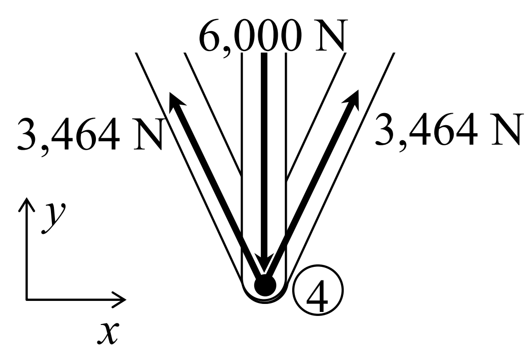
- Force equilibrium check

$$\sum F_x = -P^{(1)}\sin 30 + P^{(2)}\sin 30 = 0$$

$$\sum F_y = P^{(1)}\cos 30 - P^{(2)} + P^{(3)}\cos 30$$

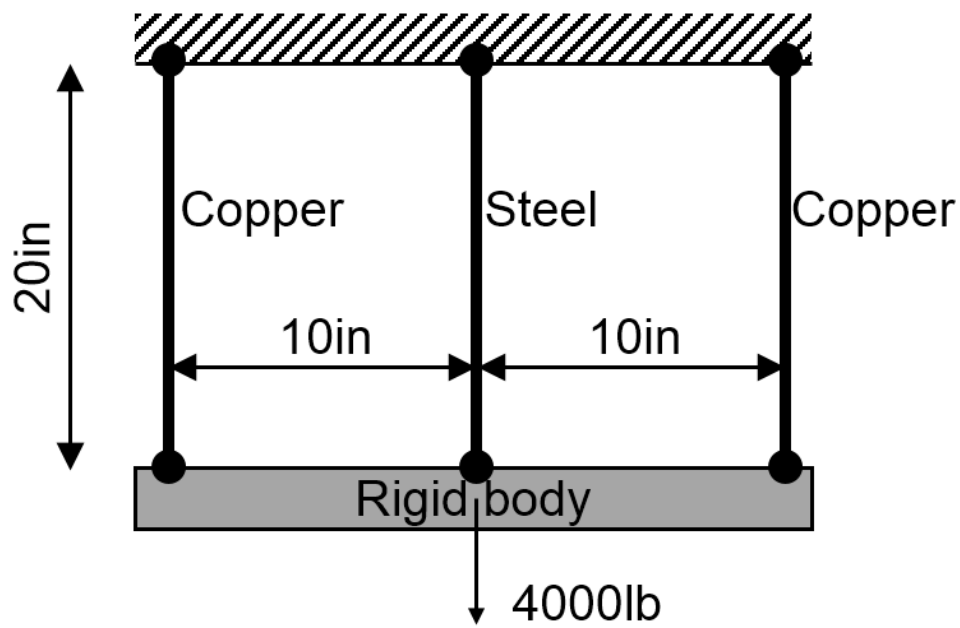
$$= 3464 \times \frac{\sqrt{3}}{2} - 6000 + 3464 \times \frac{\sqrt{3}}{2}$$

$$= 0$$



Ecercise

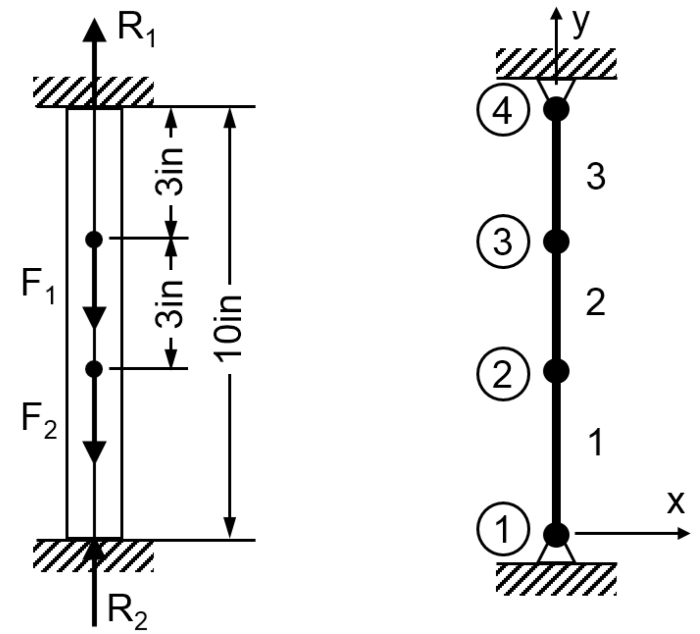
- For the three-truss homework problem in Section 1.2, when the temperature is increased by 10^{o}F, find the stresses in the copper and steel wires. For the copper wire, use the elastic modulus E_c = 1.6x10^{7}psi and the thermal expansion coefficient α_c = 9.2x10^{-6} in/in^{o}F. For the steel wire, use the elastic modulus E_s = 3.0x10^{7}psi and the thermal expansion coefficient α_s = 7.0x10^{-6} in/in^{o}F.



1.6 FE MODELING PRACTICE

Reaction Force of Statically Indeterminate Bar

- A vertical prismatic bar is fixed at both ends. When two downward forces, F_1 = 1000lb and F_2 = 500lb, are applied, calculate the reaction forces R_1 and R_2 at both ends. Use E = 30×10^6psi and A = $0.1in^2$
- Statically indeterminate system
- Remove bottom constraint and apply R_2



(a) Clamped bar (b) Finite element model

$$R_1 + R_2 = 1500\,\text{lb}$$

$$\delta = \frac{(F_1 + F_2 - R_2)\times 3}{EA} + \frac{(F_2 - R_2)\times 3}{EA} - \frac{R_2 \times 4}{EA} = 0$$

$\Rightarrow$ R_2 = 600lb, R_1 = 900 lb

Reaction Force of Statically Indeterminate Bar

- FE method does not require special treatment for statically indeterminate system
- Assembled matrix equation:

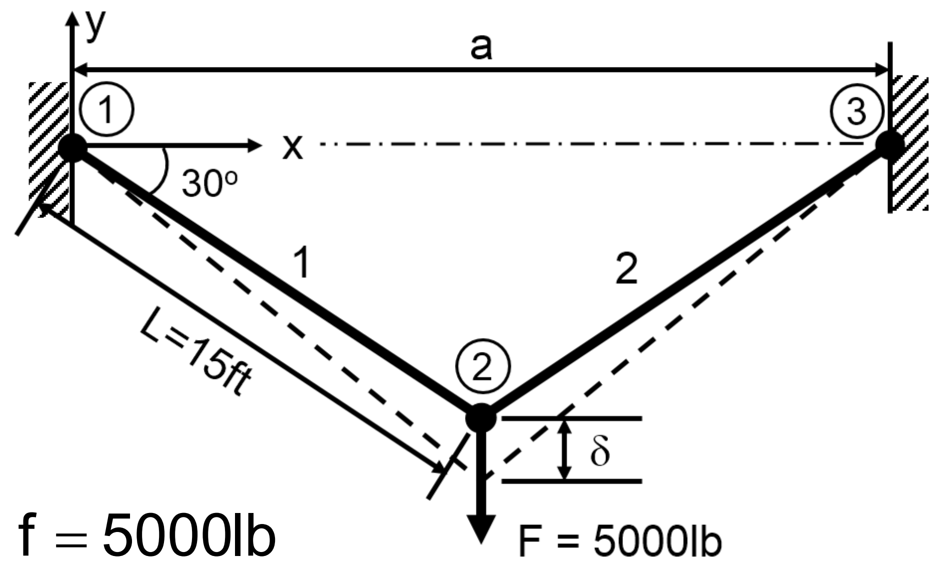
$$10^6 \begin{bmatrix} 0.75 & -0.75 & 0 & 0 \\ -0.75 & 1.75 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} R_2 \\ -F_2 \\ -F_1 \\ R_1 \end{Bmatrix}$$

- BC: Strike out 1st and 4th rows and columns

$$10^6 \begin{bmatrix} 1.75 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} -500 \\ -1000 \end{Bmatrix}$$

$$\Rightarrow \quad \begin{aligned} u_2 &= -8 \times 10^{-4} \\ u_3 &= -9 \times 10^{-4} \end{aligned} \quad \Rightarrow \quad \begin{aligned} R_2 &= 10^6 \times (0.75 \times u_1 - 0.75 \times u_2) = 600\,\text{lb} \\ R_1 &= 10^6 \times (-1 \times u_3 + 1 \times u_4) = 900\,\text{lb} \end{aligned}$$

Deflection of Two-Bar Truss



- Two steel bars with L = 15ft, A = 0.5in^2 and load F = 5000lb. Determine σ and δ. Use E = 30x10^{6}psi
- Two-force member:

$$2f\sin 30 = 5000 \quad \Rightarrow \quad f = 5000\text{lb}$$

- Stress in both members

$$\sigma^{(1)} = \sigma^{(2)} = \frac{f}{A} = 10{,}000\text{psi}$$

- Member elongation

$$\delta' = \frac{fL}{EA} = \frac{5000 \times 15 \times 12}{3 \times 10^7 \times 0.5} = 0.06\text{in}$$

- New angle

$$\theta = \cos^{-1}\left(\frac{180 \times \cos(30)}{180.06}\right) = 30.033^\circ$$

$$\delta = 180.06 \times \sin(30.033) - 180 \times \sin(30) = 0.12\text{in}$$

Deflection of Two-Bar Truss

- Connectivity table

Element	LN1 (i)	LN2 (j)	AE/L (N/m)	f (degrees)	l=cosf	m=sinf
1	1	2	83333	–30	$\sqrt{3}/2$	–1/2
2	3	2	83333	–150	$-\sqrt{3}/2$	–1/2

- Element equations (after applying BC)

$$83333\begin{bmatrix} 3/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 1/4 \end{bmatrix}\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} f_{x2}^{(1)} \\ f_{y2}^{(1)} \end{Bmatrix} \qquad 83333\begin{bmatrix} 3/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 1/4 \end{bmatrix}\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} f_{x2}^{(2)} \\ f_{y2}^{(2)} \end{Bmatrix}$$

- Assembly and solutions

$$83333\begin{bmatrix} 3/2 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -5000 \end{Bmatrix} \Longrightarrow \begin{matrix} u_2 = 0 \\ v_2 = -0.12 \end{matrix} \qquad \sigma^{(1)} = \frac{P^{(1)}}{A} = 10{,}000\text{psi}$$

$$P^{(1)} = \left(\frac{AE}{L}\right)^{(1)}\Big(l\big(u_j - u_i\big) + m\big(v_j - v_i\big)\Big) = 83333\left(-\frac{1}{2}(-0.12 - 0)\right) = 5000\text{lb}$$