CHAP 2 WEIGHTED RESIDUAL METHODS FOR 1D PROBLEMS

INTRODUCTION

- Direct stiffness method is limited for simple 1D problems
- FEM can be applied to many engineering problems that are governed by a differential equation
- Need systematic approaches to generate FE equations
  - Weighted residual method
  - Energy method
- Ordinary differential equation (second-order or fourth-order) can be solved using the weighted residual method, in particular using Galerkin method
- Principle of minimum potential energy can be used to derive finite element equations

2.1 EXACT VS. APPROXIMATE SOLUTION

EXACT VS. APPROXIMATE SOLUTION

- Exact solution
  - Boundary value problem: differential equation + boundary conditions
  - Displacements in a uniaxial bar subject to a distributed force $p(x)$

$$\frac{d^2u}{dx^2} + p(x) = 0, \ 0 \le x \le 1$$

$$\left.\begin{aligned} u(0) &= 0 \\ \frac{du}{dx}(1) &= 1 \end{aligned}\right\} \quad \text{Boundary conditions}$$

  - Essential BC: The solution value at a point is prescribed (displacement or kinematic BC)
  - Natural BC: The derivative is given at a point (stress BC)
  - **Exact solution *u*(*x*)**: twice differential function
  - In general, it is difficult to find the exact solution when the domain and/or boundary conditions are complicated
  - Sometimes the solution may not exists even if the problem is well defined

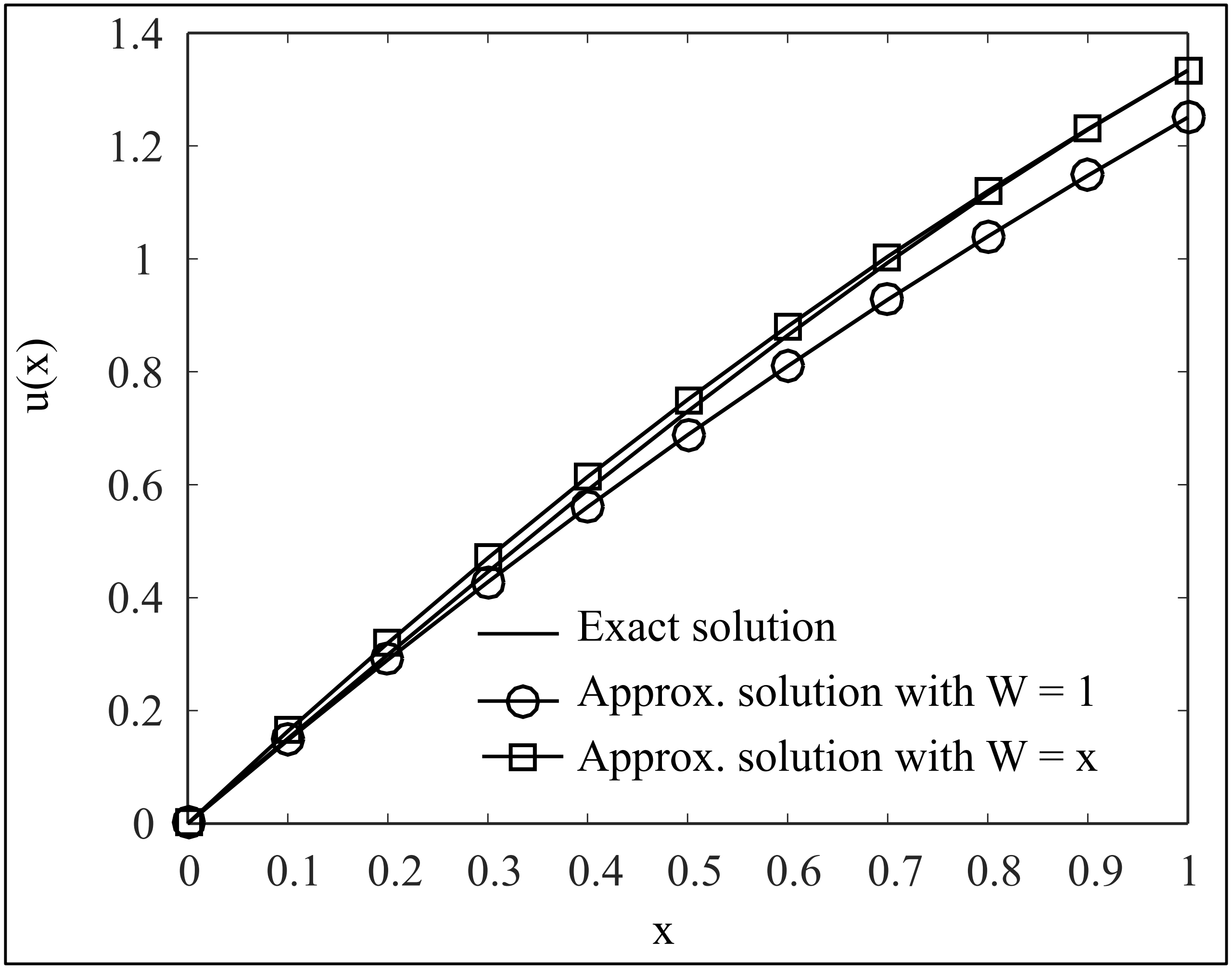
EXACT VS. APPROXIMATE SOLUTION *cont*.

- Approximate solution $\tilde{u}(x)$
  - It satisfies the essential BC, but not natural BC
  - The approximate solution may not satisfy the DE exactly
  - ***Residual***: $$\frac{d^2\tilde{u}}{dx^2} + p(x) = R(x)$$
  - Want to minimize the ***residual*** by multiplying with a weight ***W*** and integrate over the domain

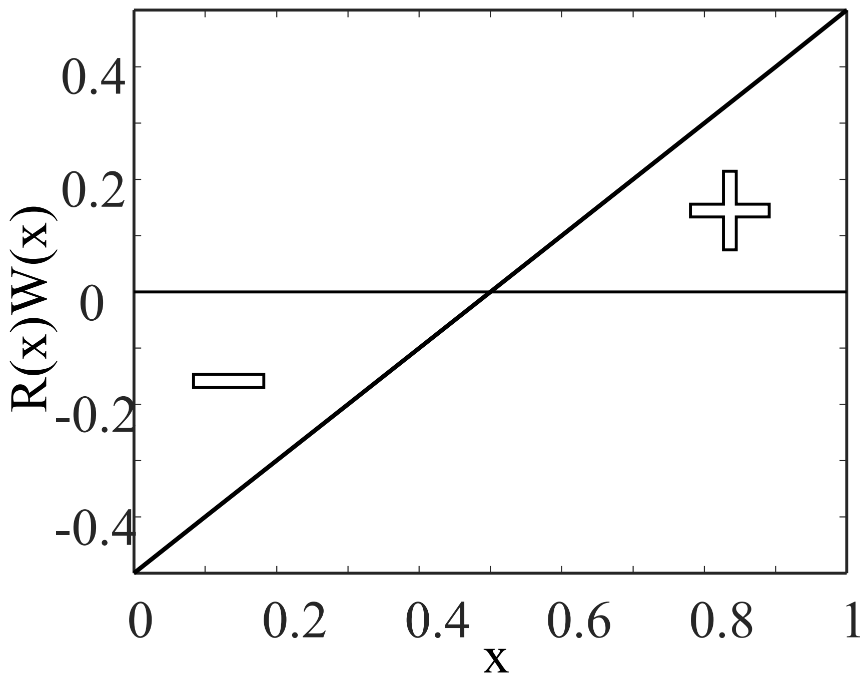
$$\int_0^1 R(x)W(x)dx = 0$$

Weight function

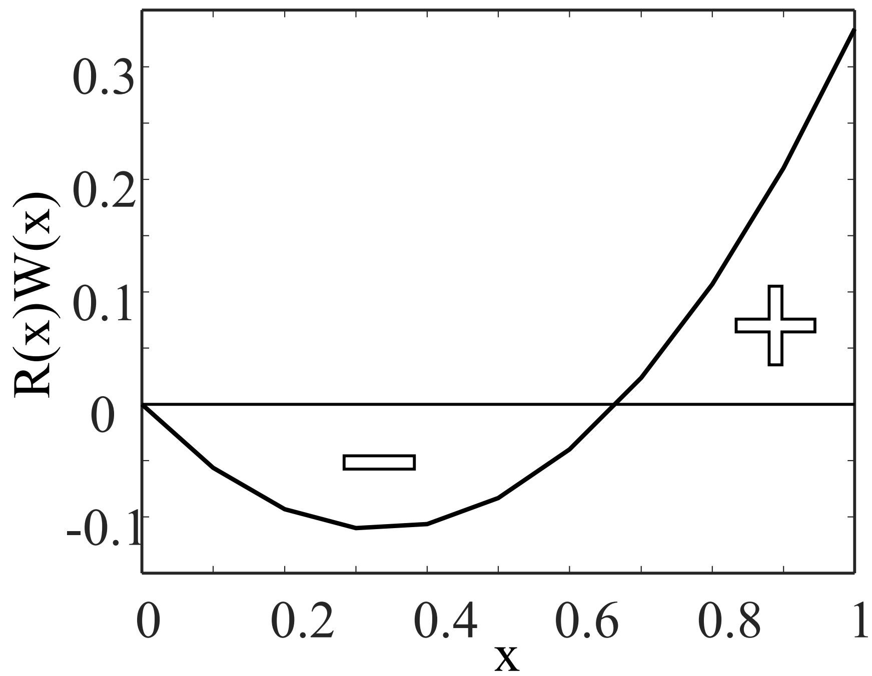
  - If it satisfies for any $W(x)$, then $R(x)$ will approaches zero, and the approximate solution will approach the exact solution
  - Depending on the choice of $W(x)$: least square error method, collocation method, Petrov-Galerkin method, and Galerkin method



1.4
1.2
1
0.8
0.6
0.4
0.2
0
u(x)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x
Exact solution
Approx. solution with W = 1
Approx. solution with W = x



(a) W = 1



(b) W = x

2.2 GALERKIN METHOD

GALERKIN METHOD

- Approximate solution is a linear combination of trial functions

$$\boxed{\tilde{u}(x) = \sum_{i=1}^{N} c_i \phi_i(x)}$$

Trial function

  - Accuracy depends on the choice of trial functions
  - The approximate solution must satisfy the essential BC

- Galerkin method
  - Use N trial functions for weight functions

$$\int_0^1 R(x)\phi_i(x)dx = 0, \quad i = 1,\ldots,N$$

$$\Rightarrow \int_0^1 \left(\frac{\mathrm{d}^2\tilde{u}}{\mathrm{d}x^2} + p(x) \right) \phi_i(x)\mathrm{d}x = 0, \quad i = 1,\ldots,N$$

$$\Rightarrow \int_0^1 \frac{\mathrm{d}^2\tilde{u}}{\mathrm{d}x^2} \phi_i(x)\mathrm{d}x = -\int_0^1 p(x)\phi_i(x)\mathrm{d}x, \quad i = 1,\ldots,N$$

GALERKIN METHOD *cont.*

- Galerkin method cont.
  - Integration-by-parts: reduce the order of differentiation in u(x)

$$\left.\frac{d\tilde{u}}{dx}\phi_i\right|_0^1 - \int_0^1 \frac{d\tilde{u}}{dx}\frac{d\phi_i}{dx}dx = -\int_0^1 p(x)\phi_i(x)dx, \quad i = 1,\ldots,N$$

  - Apply natural BC and rearrange

$$\int_0^1 \frac{d\phi_i}{dx}\frac{d\tilde{u}}{dx}dx = \int_0^1 p(x)\phi_i(x)dx + \frac{du}{dx}(1)\phi_i(1) - \frac{du}{dx}(0)\phi_i(0), \quad i = 1,\ldots,N$$

  - Same order of differentiation for both trial function and approx. solution
  - Substitute the approximate solution

$$\int_0^1 \frac{d\phi_i}{dx}\sum_{j=1}^{N} c_j \frac{d\phi_j}{dx}dx = \int_0^1 p(x)\phi_i(x)dx + \frac{du}{dx}(1)\phi_i(1) - \frac{du}{dx}(0)\phi_i(0), \quad i = 1,\ldots,N$$

GALERKIN METHOD *cont.*

- Galerkin method cont.
  - Write in matrix form

$$\sum_{j=1}^{N} K_{ij} c_j = F_i, \quad i = 1, \ldots, N \quad \Longrightarrow \quad \boxed{\underset{(N\times N)}{[\mathbf{K}]}\ \underset{(N\times 1)}{\{\mathbf{c}\}} = \underset{(N\times 1)}{\{\mathbf{F}\}}}$$

$$K_{ij} = \int_0^1 \frac{d\phi_i}{dx}\frac{d\phi_j}{dx} dx$$

$$F_i = \int_0^1 p(x)\phi_i(x) dx + \frac{du}{dx}(1)\phi_i(1) - \frac{du}{dx}(0)\phi_i(0)$$

  - Coefficient matrix is symmetric; $K_{ij} = K_{ji}$
  - N equations with N unknown coefficients

EXAMPLE1

- Differential equation

$$\frac{d^2u}{dx^2} + 1 = 0, 0 \le x \le 1$$

$$\left.\begin{aligned} u(0) &= 0 \\ \frac{du}{dx}(1) &= 1 \end{aligned}\right\} \text{ Boundary conditions}$$

Trial functions

$$\phi_1(x) = x \qquad \phi_1'(x) = 1$$

$$\phi_2(x) = x^2 \qquad \phi_2'(x) = 2x$$

- Approximate solution (satisfies the essential BC)

$$\tilde{u}(x) = \sum_{i=1}^{2} c_i \phi_i(x) = c_1 x + c_2 x^2$$

- Coefficient matrix and RHS vector

$$K_{11} = \int_0^1 (\phi_1')^2 \, dx = 1$$

$$K_{12} = K_{21} = \int_0^1 (\phi_1' \phi_2') dx = 1$$

$$K_{22} = \int_0^1 (\phi_2')^2 \, dx = \frac{4}{3}$$

$$F_1 = \int_0^1 \phi_1(x) dx + \phi_1(1) - \cancel{\frac{du}{dx}}(0)\phi_1(0) = \frac{3}{2}$$

$$F_2 = \int_0^1 \phi_2(x) dx + \phi_2(1) - \cancel{\frac{du}{dx}}(0)\phi_2(0) = \frac{4}{3}$$

EXAMPLE1 *cont.*

- Matrix equation

$$[\mathbf{K}] = \frac{1}{3}\begin{bmatrix} 3 & 3 \\ 3 & 4 \end{bmatrix} \qquad \{\mathbf{F}\} = \frac{1}{6}\begin{Bmatrix} 9 \\ 8 \end{Bmatrix} \qquad \Longrightarrow \qquad \{\mathbf{c}\} = [\mathbf{K}]^{-1}\{\mathbf{F}\} = \begin{Bmatrix} 2 \\ -\frac{1}{2} \end{Bmatrix}$$

- Approximate solution

$$\tilde{u}(x) = 2x - \frac{x^2}{2}$$

  – Approximate solution is also the exact solution because the linear combination of the trial functions can represent the exact solution

EXAMPLE2

- Differential equation

$$\frac{d^2u}{dx^2} + x = 0, \qquad 0 \le x \le 1$$

$$\left. \begin{aligned} & u(0) = 0 \\ & \frac{du}{dx}(1) = 1 \end{aligned} \right\} \text{ Boundary conditions}$$

Trial functions

$$\phi_1(x) = x \qquad \phi_1'(x) = 1$$

$$\phi_2(x) = x^2 \qquad \phi_2'(x) = 2x$$

- Coefficient matrix is same, force vector: $\{\mathbf{F}\} = \frac{1}{12}\begin{Bmatrix} 16 \\ 15 \end{Bmatrix}$

$$\{\mathbf{c}\} = [\mathbf{K}]^{-1}\{\mathbf{F}\} = \begin{Bmatrix} \frac{19}{12} \\ -\frac{1}{4} \end{Bmatrix} \Longrightarrow \tilde{u}(x) = \frac{19}{12}x - \frac{x^2}{4}$$

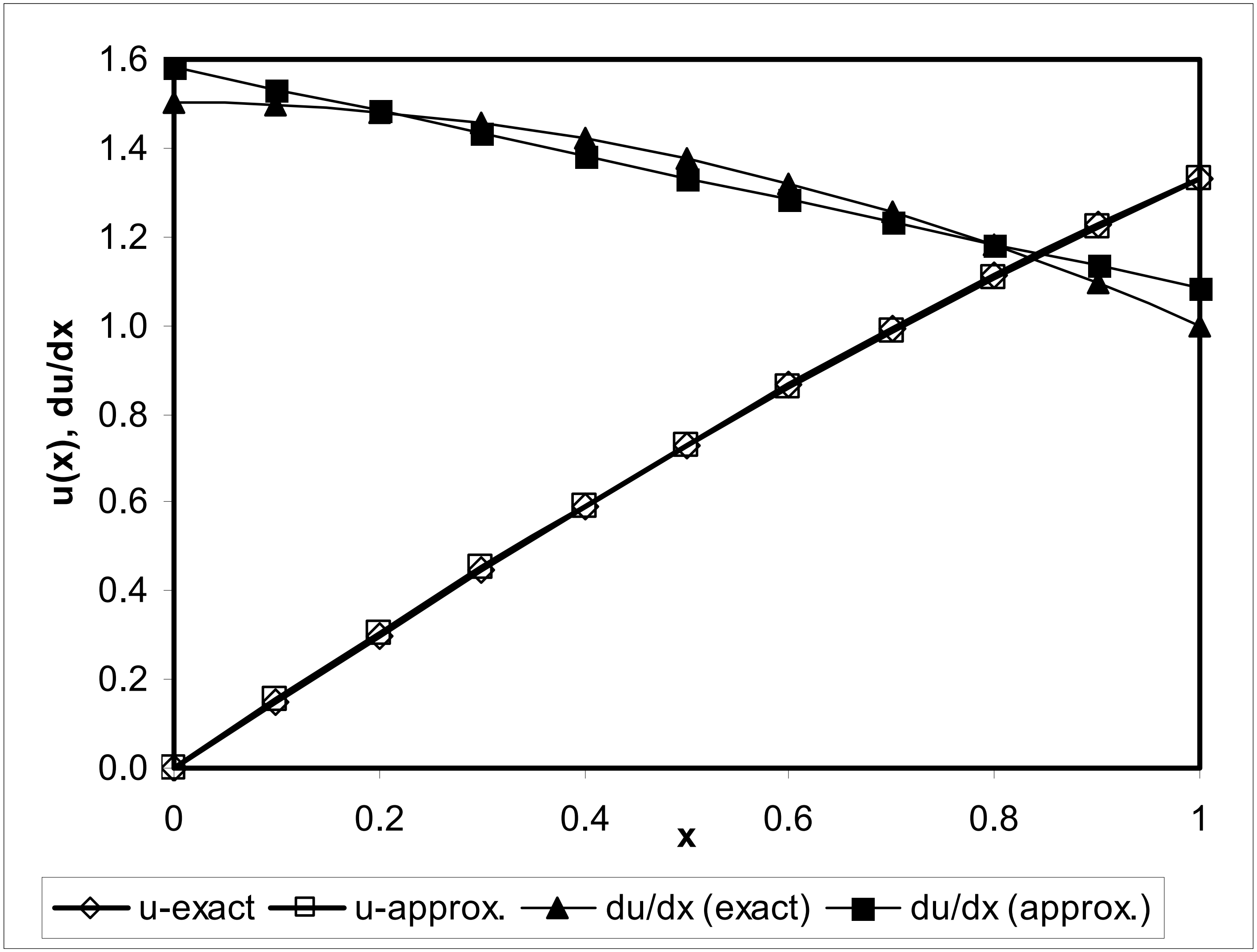
- Exact solution

$$u(x) = \frac{3}{2}x - \frac{x^3}{6}$$

  – The trial functions cannot express the exact solution; thus, approximate solution is different from the exact one

EXAMPLE2 *cont.*

- Approximation is good for $u(x)$, but not good for du/dx



2.3 HIGHER-ORDER EQUATIONS

HIGHER-ORDER DIFFERENTIAL EQUATIONS

- Fourth-order differential equation

$$\boxed{\frac{d^4w}{dx^4} - p(x) = 0, \quad 0 \le x \le L}$$

$$\left.\begin{aligned} w(0) &= 0 \\ \frac{dw}{dx}(0) &= 0 \end{aligned}\right\} \text{Essential BC}$$

$$\left.\begin{aligned} \frac{d^2w}{dx^2}(L) &= M \\ \frac{d^3w}{dx^3}(L) &= -V \end{aligned}\right\} \text{Natural BC}$$

  - Beam bending under pressure load

- Approximate solution

$$\tilde{w}(x) = \sum_{i=1}^{N} c_i \phi_i(x)$$

- Weighted residual equation (Galerkin method)

$$\int_0^L \left(\frac{d^4\tilde{w}}{dx^4} - p(x) \right) \phi_i(x) dx = 0, \quad i = 1, \ldots, N$$

  - In order to make the order of differentiation same, integration-by-parts must be done twice

HIGHER-ORDER DE *cont.*

- After integration-by-parts twice

$$\frac{d^3\tilde{w}}{dx^3}\phi_i\bigg|_0^L - \frac{d^2\tilde{w}}{dx^2}\frac{d\phi_i}{dx}\bigg|_0^L + \int_0^L \frac{d^2\tilde{w}}{dx^2}\frac{d^2\phi_i}{dx^2}dx = \int_0^L p(x)\phi_i(x)dx, \quad i = 1,\ldots,N$$

$$\Rightarrow \int_0^L \frac{d^2\tilde{w}}{dx^2}\frac{d^2\phi_i}{dx^2}dx = \int_0^L p(x)\phi_i(x)dx - \frac{d^3\tilde{w}}{dx^3}\phi_i\bigg|_0^L + \frac{d^2\tilde{w}}{dx^2}\frac{d\phi_i}{dx}\bigg|_0^L, \quad i = 1,\ldots,N$$

- Substitute approximate solution

$$\int_0^L \sum_{j=1}^{N} c_j \frac{d^2\phi_j}{dx^2}\frac{d^2\phi_i}{dx^2}dx = \int_0^L p(x)\phi_i(x)dx - \frac{d^3\tilde{w}}{dx^3}\phi_i\bigg|_0^L + \frac{d^2\tilde{w}}{dx^2}\frac{d\phi_i}{dx}\bigg|_0^L, \quad i = 1,\ldots,N$$

  - Do not substitute the approx. solution in the boundary terms

- Matrix form

$$\underset{N\times N}{[\mathbf{K}]}\,\underset{N\times 1}{\{\mathbf{c}\}} = \underset{N\times 1}{\{\mathbf{F}\}}$$

$$K_{ij} = \int_0^L \frac{d^2\phi_i}{dx^2}\frac{d^2\phi_j}{dx^2}dx$$

$$F_i = \int_0^L p(x)\phi_i(x)dx - \frac{d^3w}{dx^3}\phi_i\bigg|_0^L + \frac{d^2w}{dx^2}\frac{d\phi_i}{dx}\bigg|_0^L$$

EXMAPLE

- Fourth-order DE

$$\frac{d^4w}{dx^4} - 1 = 0, \quad 0 \le x \le 1$$

$$w(0) = 0 \quad \frac{dw}{dx}(0) = 0$$

$$\frac{d^2w}{dx^2}(1) = 2 \quad \frac{d^3w}{dx^3}(1) = -1$$

- Two trial functions

$$\phi_1 = x^2, \quad \phi_2 = x^3 \qquad \phi_1'' = 2, \quad \phi_2'' = 6x$$

- Coefficient matrix

$$K_{11} = \int_0^1 (\phi_1'')^2 \, dx = 4$$

$$K_{12} = K_{21} = \int_0^1 (\phi_1'' \phi_2'') \, dx = 6 \quad \Longrightarrow \quad [\mathbf{K}] = \begin{bmatrix} 4 & 6 \\ 6 & 12 \end{bmatrix}$$

$$K_{22} = \int_0^1 (\phi_2'')^2 \, dx = 12$$

EXAMPLE *cont.*

- RHS

$$F_1 = \int_0^1 x^2 dx + V\phi_1(1) + \cancel{\frac{d^3w(0)}{dx^3}\phi_1(0)} + M\phi_1'(1) - \cancel{\frac{d^2w(0)}{dx^2}\phi_1'(0)} = \frac{16}{3}$$

$$F_2 = \int_0^1 x^3 dx + V\phi_2(1) + \cancel{\frac{d^3w(0)}{dx^3}\phi_2(0)} + M\phi_2'(1) - \cancel{\frac{d^2w(0)}{dx^2}\phi_2'(0)} = \frac{29}{4}$$

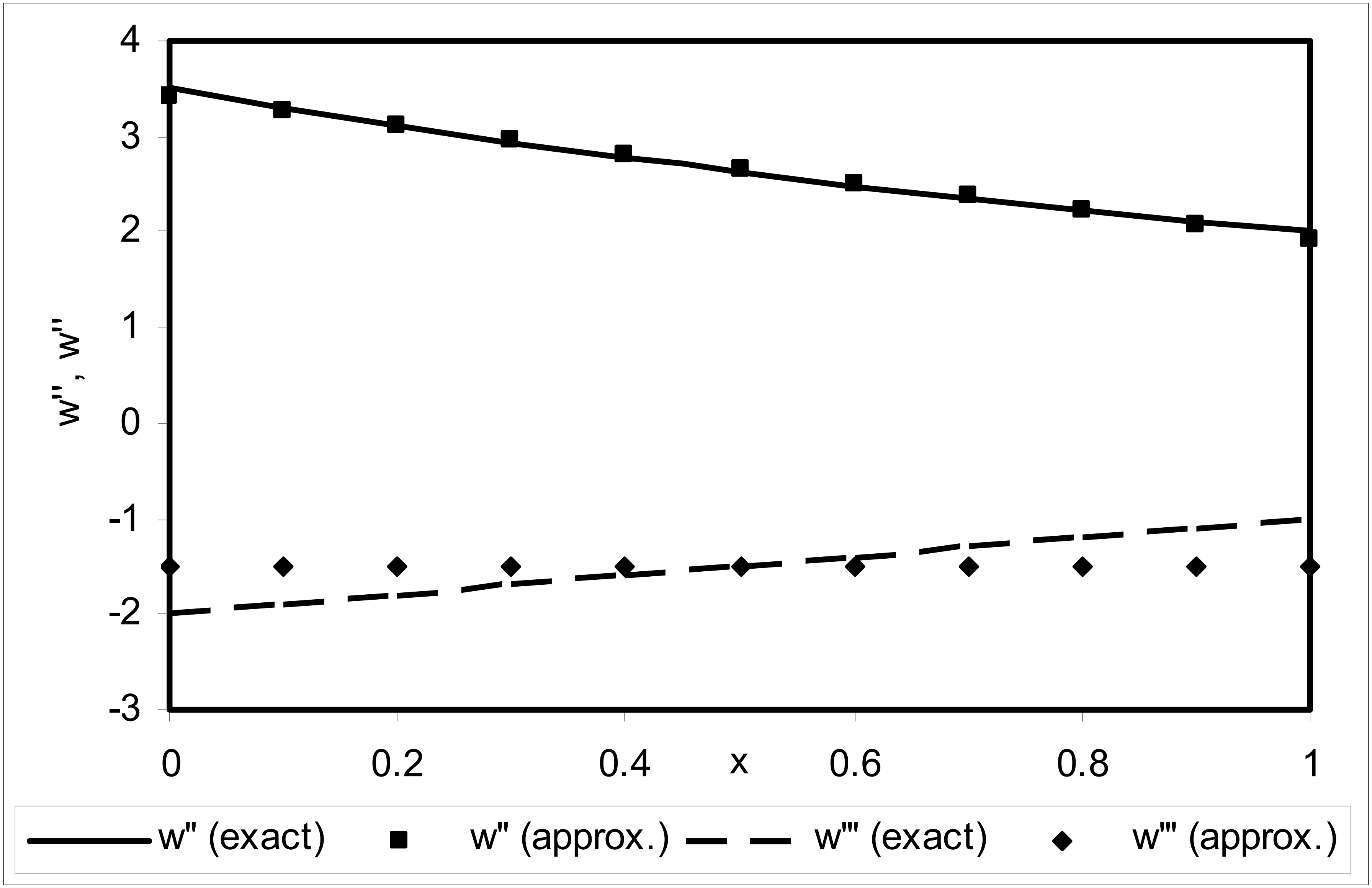
- Approximate solution

$$\{\mathbf{c}\} = [\mathbf{K}]^{-1}\{\mathbf{F}\} = \begin{Bmatrix} \frac{41}{24} \\ -\frac{1}{4} \end{Bmatrix} \quad \Longrightarrow \quad \tilde{w}(x) = \frac{41}{24}x^2 - \frac{1}{4}x^3$$

- Exact solution

$$w(x) = \frac{1}{24}x^4 - \frac{1}{3}x^3 + \frac{7}{4}x^2$$

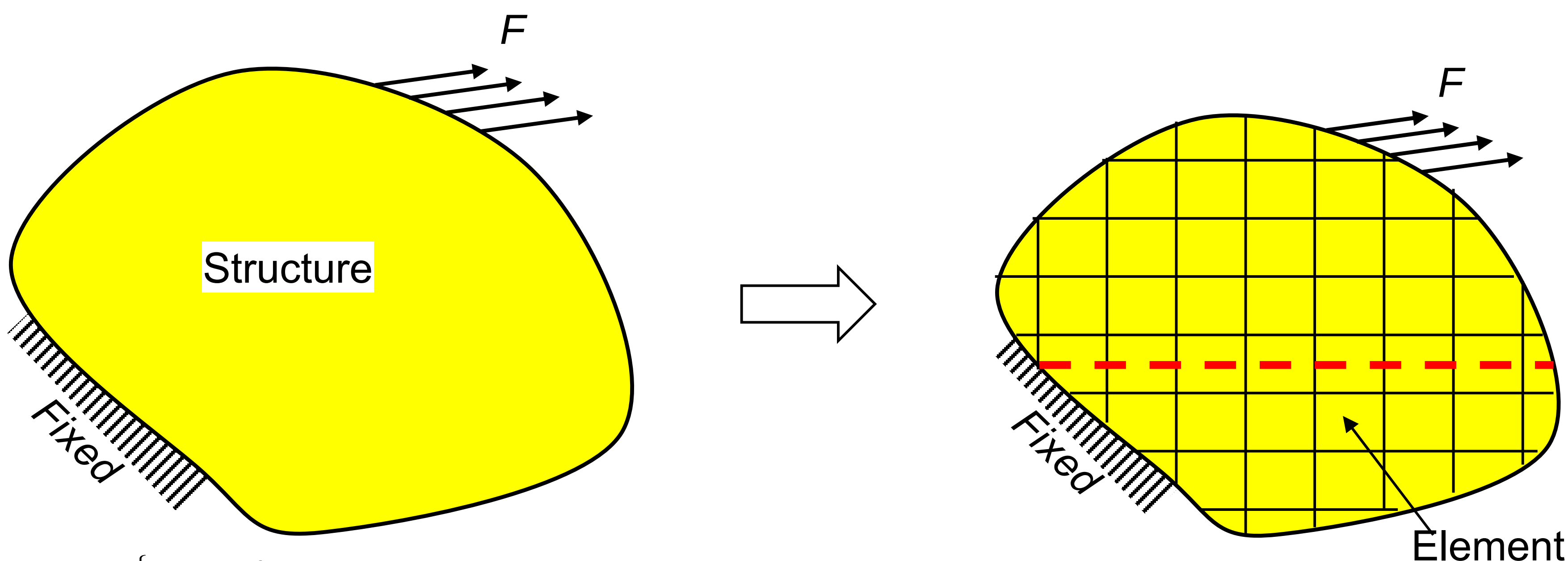
EXAMPLE *cont.*



2.4 FINITE ELEMENT APPROXIMATION

FINITE ELEMENT APPROXIMATION

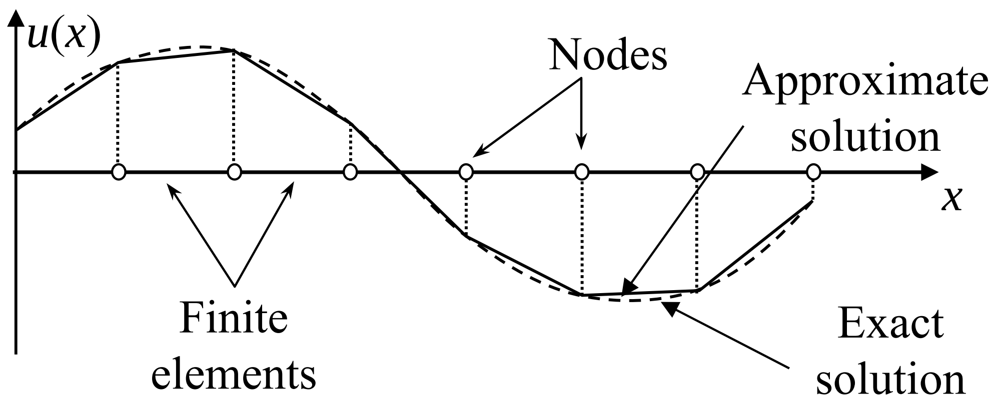
- Domain Discretization
  - Weighted residual method is still difficult to obtain the trial functions that satisfy the essential BC
  - FEM is to divide the entire domain into a set of simple sub-domains (finite element) and share nodes with adjacent elements



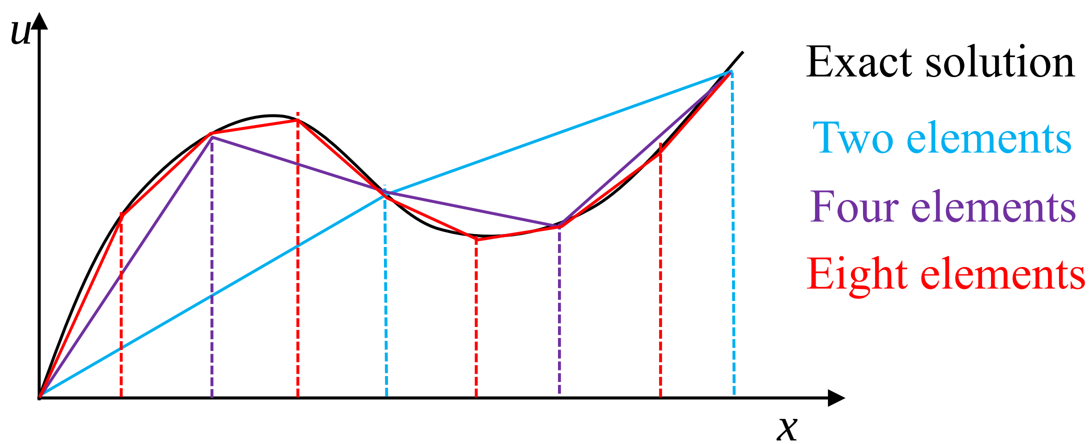
$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + b_x = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + b_y = 0 \end{cases}$$

FINITE ELEMENT APPROXIMATION

- Domain Discretization
  - Within a finite element, the solution is approximated in a simple polynomial form

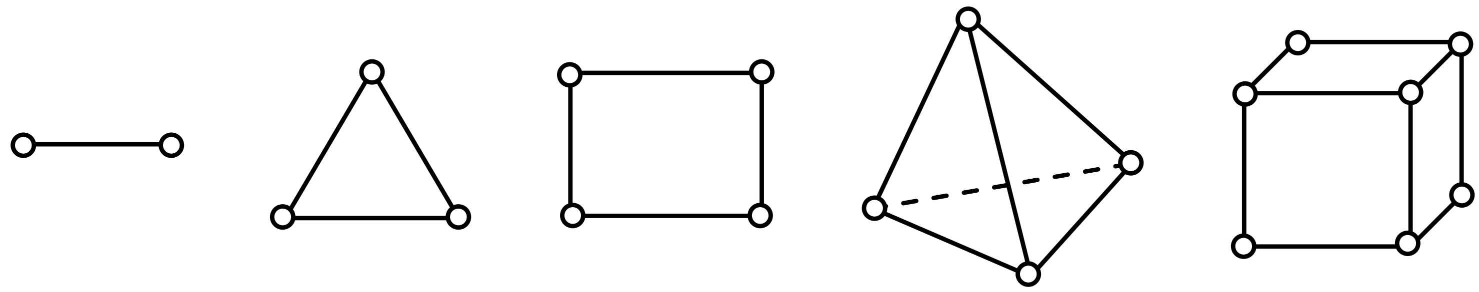


  - When more number of finite elements are used, the approximated piecewise linear solution may converge to the analytical solution



FINITE ELEMENT METHOD *cont.*

- Types of finite elements



1D 2D 3D

- Variational equation is imposed on each element.

$$\int_0^1 \square \, dx = \int_0^{0.1} \square \, dx + \int_{0.1}^{0.2} \square \, dx + \cdots + \int_{0.9}^{1} \square \, dx$$

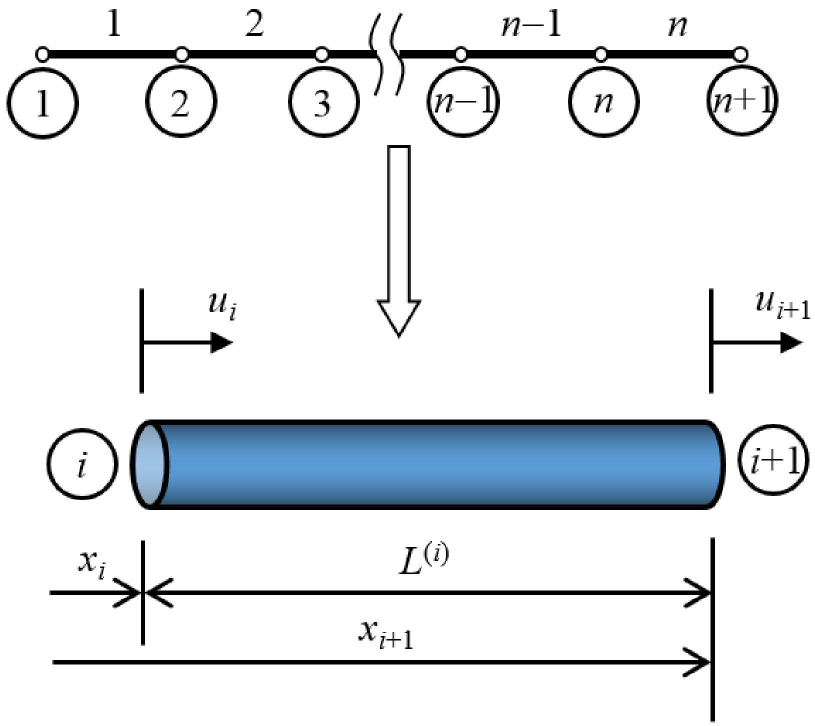
One element

Galerkin vs. Finite Element Method

Finite element method	**Galerkin method**
$\tilde{u}(x) = N_i(x)u_i + N_{i+1}(x)u_{i+1}$	$\tilde{u}(x) = \sum_{i=1}^{N} c_i \phi_i(x)$
Coefficients are nodal value	Coefficients don't have physical meaning
Defined only within an element	Defined in entire domain
Easy to satisfy BC	Difficult to find weight function to satisfy BC
Equilibrium in element by element	Equilibrium in entire domain
Need assembly	No need to assembly

TRIAL SOLUTION

– Solution within an element is approximated using simple polynomials.



– i-th element is composed of two nodes: x_i and x_{i+1}. Since two unknowns are involved, linear polynomial can be used:

$$\tilde{u}(x) = a_0 + a_1 x, \quad x_i \le x \le x_{i+1}$$

– The unknown coefficients, a_0 and a_1, will be expressed in terms of nodal solutions $u(x_i)$ and $u(x_{i+1})$.

TRIAL SOLUTION *cont.*

– Substitute two nodal values

$$\begin{cases} \tilde{u}(x_i) = u_i = a_0 + a_1 x_i \\ \tilde{u}(x_{i+1}) = u_{i+1} = a_0 + a_1 x_{i+1} \end{cases}$$

– Express a_0 and a_1 in terms of u_i and u_{i+1}. Then, the solution is approximated by

$$\tilde{u}(x) = \underbrace{\frac{x_{i+1} - x}{L^{(i)}}}_{N_i(x)} u_i + \underbrace{\frac{x - x_i}{L^{(i)}}}_{N_{i+1}(x)} u_{i+1}$$

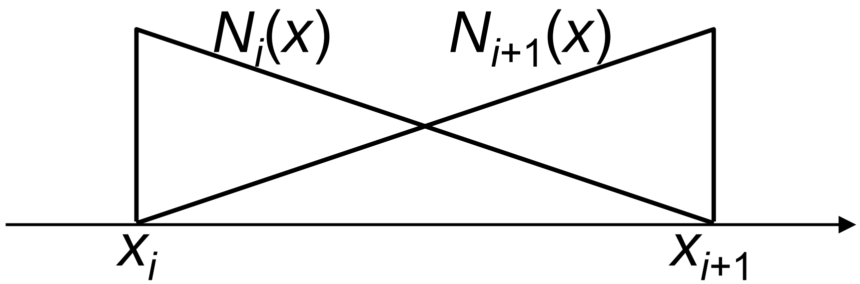
– Solution for i-th element:

$$\tilde{u}(x) = N_i(x) u_i + N_{i+1}(x) u_{i+1}, \quad x_i \le x \le x_{i+1}$$

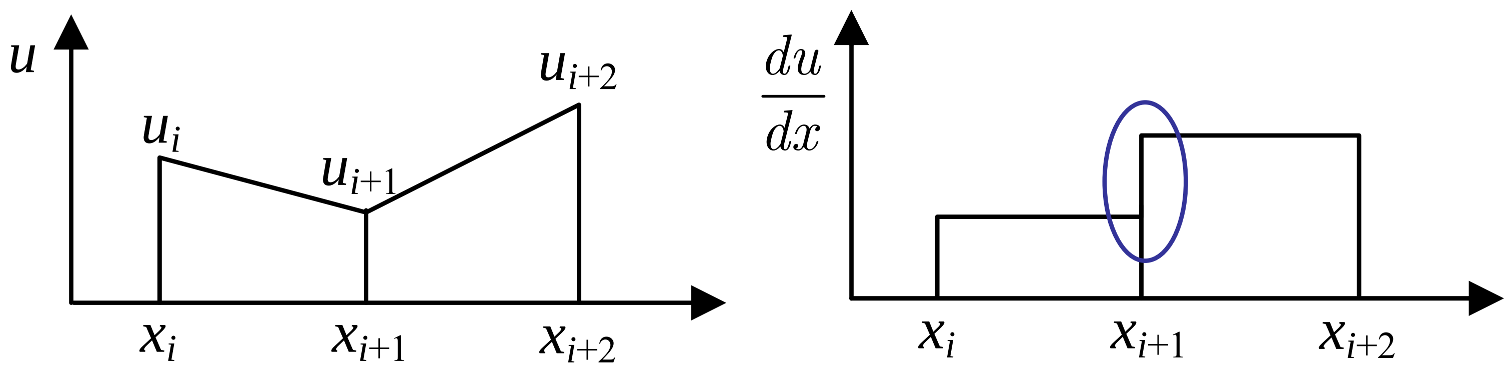
– $N_i(x)$ and $N_{i+1}(x)$: **Shape Function** or **Interpolation Function**

TRIAL SOLUTION *cont.*

- Observations
  - Solution $u(x)$ is interpolated using its nodal values u_i and u_{i+1}.
  - $N_i(x) = 1$ at node x_i, and $=0$ at node x_{i+1}.



  - The solution is approximated by piecewise linear polynomial and its gradient is constant within an element.



  - Stress and strain (derivative) are often averaged at the node.

GALERKIN METHOD

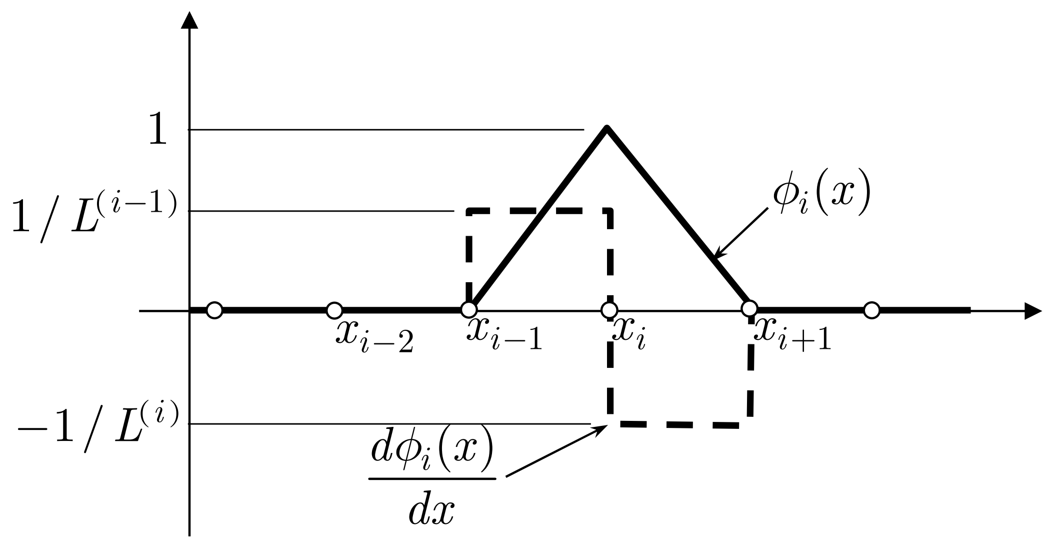
- Relation between interpolation functions and trial functions
  - 1D problem with linear interpolation

$$\tilde{u}(x) = \sum_{i=1}^{N_D} u_i \phi_i(x) \qquad \phi_i(x) = \begin{cases} 0, & 0 \le x \le x_{i-1} \\ N_i^{(i-1)}(x) = \dfrac{x - x_{i-1}}{L^{(i-1)}}, & x_{i-1} < x \le x_i \\ N_i^{(i)}(x) = \dfrac{x_{i+1} - x}{L^{(i)}}, & x_i < x \le x_{i+1} \\ 0, & x_{i+1} < x \le x_{N_D} \end{cases}$$

  - Difference: the interpolation function does not exist in the entire domain, but it exists only in elements connected to the node

- Derivative

$$\frac{d\phi_i(x)}{dx} = \begin{cases} 0, & 0 \le x \le x_{i-1} \\ \dfrac{1}{L^{(i-1)}}, & x_{i-1} < x \le x_i \\ -\dfrac{1}{L^{(i)}}, & x_i < x \le x_{i+1} \\ 0, & x_{i+1} < x \le x_{N_D} \end{cases}$$



EXAMPLE

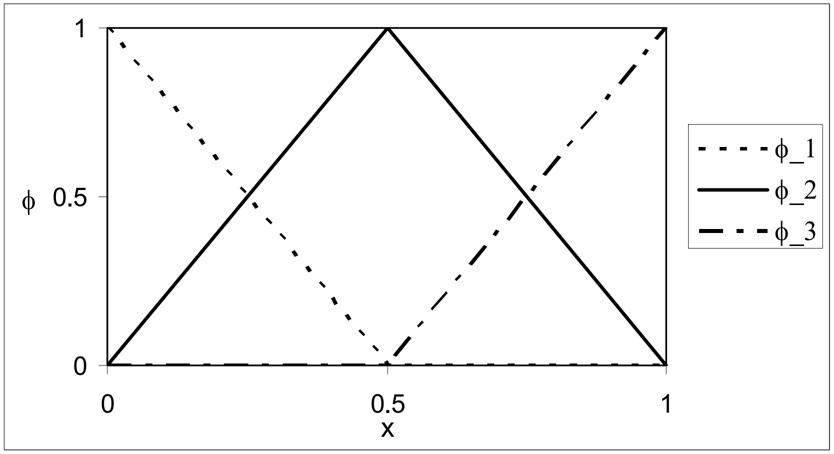
- Solve using two equal-length elements

$$\frac{d^2u}{dx^2}+1=0,0\le x\le 1 \qquad \left.\begin{aligned} u(0)&=0\\ \frac{du}{dx}(1)&=1\end{aligned}\right\}\ \text{Boundary conditions}$$

- Three nodes at x = 0, 0.5, 1.0; displ at nodes = u_1, u_2, u_3
- Approximate solution $\tilde{u}(x)=u_1\phi_1(x)+u_2\phi_2(x)+u_3\phi_3(x)$

$$\phi_1(x)=\begin{cases}1-2x, & 0\le x\le 0.5\\ 0, & 0.5<x\le 1\end{cases} \qquad \phi_2(x)=\begin{cases}2x, & 0\le x\le 0.5\\ 2-2x, & 0.5<x\le 1\end{cases}$$

$$\phi_3(x)=\begin{cases}0, & 0\le x\le 0.5\\ -1+2x, & 0.5<x\le 1\end{cases}$$



EXAMPLE *cont.*

- Derivatives of interpolation functions

$$\frac{\mathrm{d}\phi_1(x)}{\mathrm{d}x} = \begin{cases} -2, & 0 \le x \le 0.5 \\ 0, & 0.5 < x \le 1 \end{cases} \qquad \frac{\mathrm{d}\phi_2(x)}{\mathrm{d}x} = \begin{cases} 2, & 0 \le x \le 0.5 \\ -2, & 0.5 < x \le 1 \end{cases}$$

$$\frac{\mathrm{d}\phi_3(x)}{\mathrm{d}x} = \begin{cases} 0, & 0 \le x \le 0.5 \\ 2, & 0.5 < x \le 1 \end{cases}$$

- Coefficient matrix

$$K_{12} = \int_0^1 \frac{\mathrm{d}\phi_1}{\mathrm{d}x}\frac{\mathrm{d}\phi_2}{\mathrm{d}x}\mathrm{d}x = \int_0^{0.5} (-2)(2)\mathrm{d}x + \int_{0.5}^1 (0)(-2)\mathrm{d}x = -2$$

$$K_{22} = \int_0^1 \frac{\mathrm{d}\phi_2}{\mathrm{d}x}\frac{\mathrm{d}\phi_2}{\mathrm{d}x}dx = \int_0^{0.5} 4\mathrm{d}x + \int_{0.5}^1 4\mathrm{d}x = 4$$

- RHS

$$F_1 = \int_0^{0.5} 1\times(1-2x)\mathrm{d}x + \int_{0.5}^1 1\times(0)\mathrm{d}x + \cancel{\frac{du}{dx}(1)\phi_1(1)} - \frac{du}{dx}(0)\phi_1(0) = 0.25 - \frac{\mathrm{d}u}{\mathrm{d}x}(0)$$

$$F_2 = \int_0^{0.5} 2x\mathrm{d}x + \int_{0.5}^1 (2-2x)\mathrm{d}x + \cancel{\frac{\mathrm{d}u}{\mathrm{d}x}(1)\phi_2(1)} - \cancel{\frac{\mathrm{d}u}{\mathrm{d}x}(0)\phi_2(0)} = 0.5$$

EXAMPLE *cont.*

- Matrix equation

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0.5 \\ 1.25 \end{Bmatrix}$$

Consider it as unknown

- Striking the 1st row and striking the 1st column (BC)

$$\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0.5 \\ 1.25 \end{Bmatrix}$$

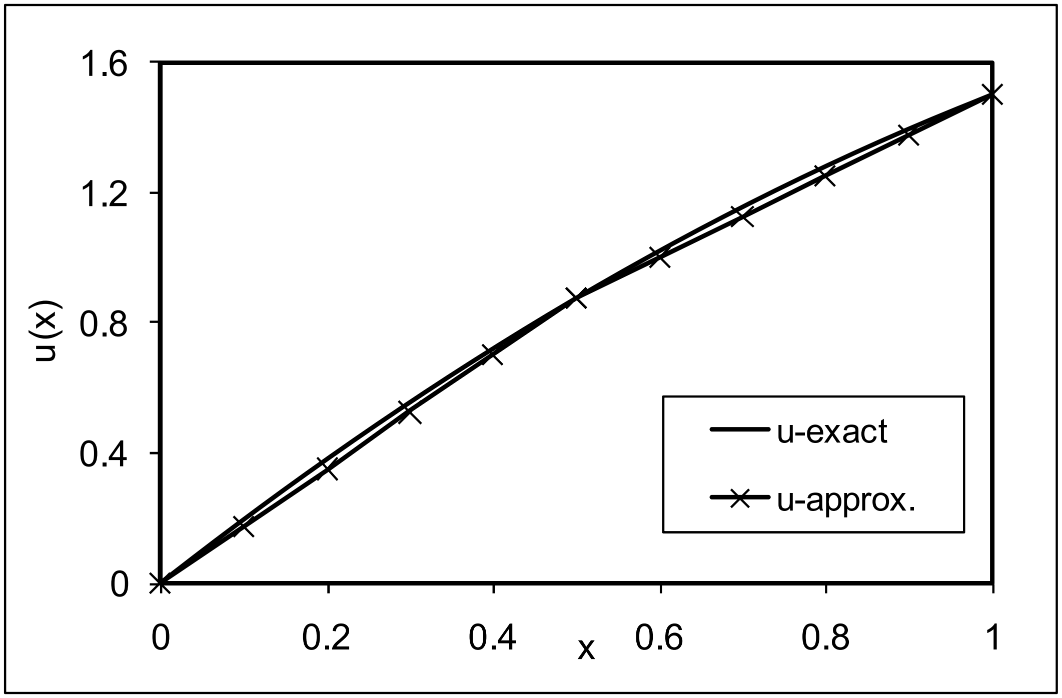
- Solve for $u_2 = 0.875$, $u_3 = 1.5$
- Approximate solution

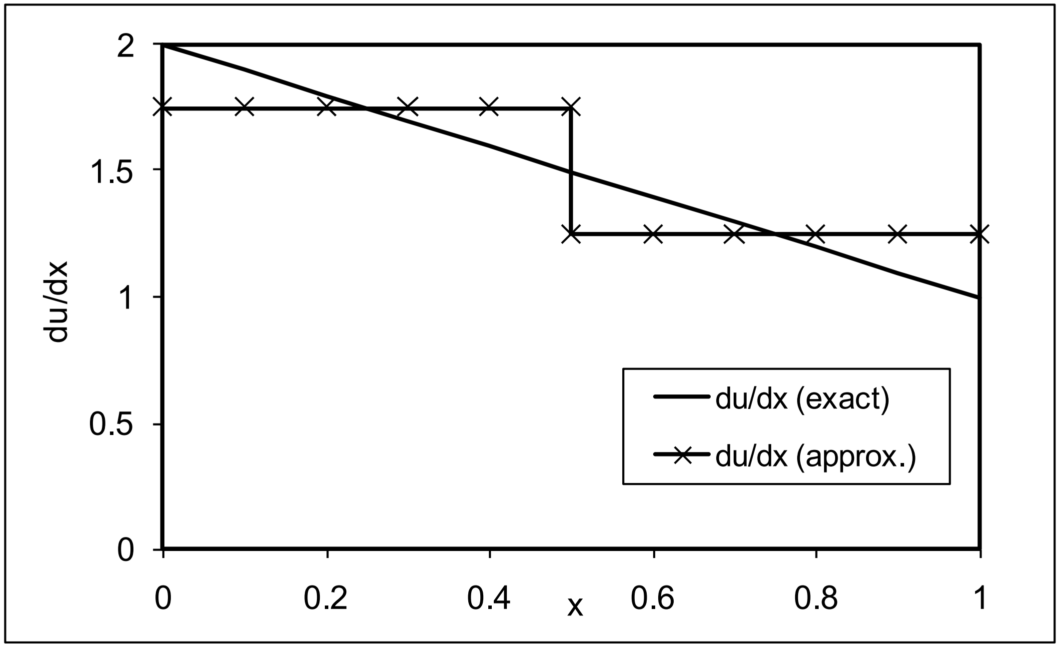
$$\tilde{u}(x) = \begin{cases} 1.75x, & 0 \le x \le 0.5 \\ 0.25 + 1.25x, & 0.5 \le x \le 1 \end{cases}$$

  – Piecewise linear solution

EXAMPLE *cont.*

- Solution comparison
- Approx. solution has about 8% error
- Derivative shows a large discrepancy
- Approx. derivative is constant as the solution is piecewise linear





2.5 FORMAL PROCEDURE

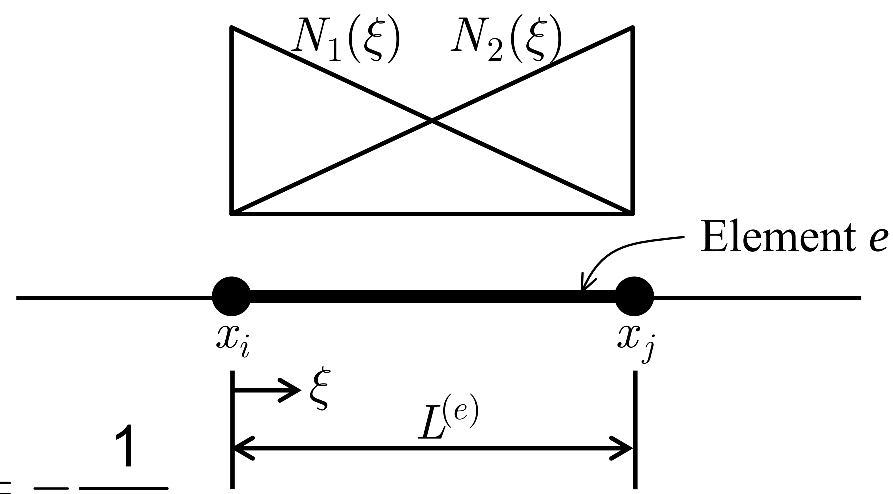
FORMAL PROCEDURE

- Galerkin method is still not general enough for computer code
- Apply Galerkin method to one element (e) at a time
- Introduce a local coordinate

$$x = x_i(1-\xi) + x_j\xi \qquad \xi = \frac{x - x_i}{x_j - x_i} = \frac{x - x_i}{L^{(e)}}$$

- Approximate solution within the element

$$\tilde{u}(x) = u_i N_1(x) + u_j N_2(x)$$



$$N_1(\xi) = (1-\xi)$$

$$N_2(\xi) = \xi$$

$$N_1(x) = \left(1 - \frac{x - x_i}{L^{(e)}}\right) \qquad \frac{dN_1}{dx} = \frac{dN_1}{d\xi}\frac{d\xi}{dx} = -\frac{1}{L^{(e)}}$$

$$N_2(x) = \frac{x - x_i}{L^{(e)}} \qquad \frac{dN_2}{dx} = \frac{dN_2}{d\xi}\frac{d\xi}{dx} = +\frac{1}{L^{(e)}}$$

FORMAL PROCEDURE *cont.*

- Interpolation property

$$N_1(x_i) = 1, \quad N_1(x_j) = 0 \qquad \tilde{u}(x_i) = u_i$$
$$N_2(x_i) = 0, \quad N_2(x_j) = 1 \qquad \tilde{u}(x_j) = u_j$$

- Derivative of approx. solution

$$\frac{d\tilde{u}}{dx} = u_i \frac{dN_1}{dx} + u_j \frac{dN_2}{dx}$$

$$\frac{d\tilde{u}}{dx} = \left\lfloor \frac{dN_1}{dx} \quad \frac{dN_2}{dx} \right\rfloor \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{L^{(e)}} \left\lfloor \frac{dN_1}{d\xi} \quad \frac{dN_2}{d\xi} \right\rfloor \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

- Apply Galerkin method in the element level

$$\int_{x_i}^{x_j} \frac{dN_i}{dx} \frac{d\tilde{u}}{dx} dx = \int_{x_i}^{x_j} p(x) N_i(x) dx + \frac{du}{dx}(x_j) N_i(x_j) - \frac{du}{dx}(x_i) N_i(x_i), \quad i = 1, 2$$

FORMAL PROCEDURE *cont.*

- Change variable from x to ξ

$$\frac{1}{L^{(e)}}\int_0^1 \frac{dN_i}{d\xi}\left[\frac{dN_1}{d\xi} \quad \frac{dN_2}{d\xi}\right] d\xi \bullet \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = L^{(e)}\int_0^1 p(x)N_i(\xi)d\xi$$

$$+\frac{du}{dx}(x_j)N_i(1) - \frac{du}{dx}(x_i)N_i(0), \quad i = 1,2$$

  - Do not use approximate solution for boundary terms

- Element-level matrix equation

$$[\mathbf{k}^{(e)}]\{\mathbf{q}^{(e)}\} = \{\mathbf{f}^{(e)}\} + \begin{Bmatrix} -\dfrac{du}{dx}(x_i) \\ +\dfrac{du}{dx}(x_j) \end{Bmatrix} \qquad \{\mathbf{f}^{(e)}\} = L^{(e)}\int_0^1 p(x)\begin{Bmatrix} N_1(\xi) \\ N_2(\xi) \end{Bmatrix} d\xi$$

$$\underset{2\times 2}{\left[\mathbf{k}^{(e)}\right]} = \frac{1}{L^{(e)}}\int_0^1 \begin{bmatrix} \left(\dfrac{dN_1}{d\xi}\right)^2 & \dfrac{dN_1}{d\xi}\dfrac{dN_2}{d\xi} \\ \dfrac{dN_2}{d\xi}\dfrac{dN_1}{d\xi} & \left(\dfrac{dN_2}{d\xi}\right)^2 \end{bmatrix} d\xi = \frac{1}{L^{(e)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

FORMAL PROCEDURE *cont.*

- Need to derive the element-level equation for all elements
- Consider Elements 1 and 2 (connected at Node 2)

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}^{(1)} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}^{(1)} + \begin{Bmatrix} -\dfrac{du}{dx}(x_1) \\ +\dfrac{du}{dx}(x_2) \end{Bmatrix}$$

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}^{(2)} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_2 \\ f_3 \end{Bmatrix}^{(2)} + \begin{Bmatrix} -\dfrac{du}{dx}(x_2) \\ +\dfrac{du}{dx}(x_3) \end{Bmatrix}$$

- Assembly

$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} \\ 0 & k_{21}^{(2)} & k_{22}^{(2)} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix} + \begin{Bmatrix} -\dfrac{du}{dx}(x_1) \\ 0 \\ \dfrac{du}{dx}(x_3) \end{Bmatrix}$$

Vanished unknown term

FORMAL PROCEDURE *cont.*

- Assembly of N_E elements ($N_D = N_E + 1$)

$$\underset{(N_D\times N_D)}{\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & \cdots & 0 \\ k_{21}^{(1)} & k_{22}^{(1)}+k_{11}^{(2)} & k_{12}^{(2)} & \cdots & 0 \\ 0 & k_{221}^{(2)} & k_{22}^{(2)}+k_{11}^{(2)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & k_{21}^{(N_E)} & k_{22}^{(N_E)} \end{bmatrix}} \underset{(N_D\times 1)}{\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_N \end{Bmatrix}} = \underset{(N_D\times 1)}{\begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)}+f_2^{(2)} \\ f_3^{(2)}+f_3^{(3)} \\ \vdots \\ f_N^{(N_E)} \end{Bmatrix}} + \underset{(N_D\times 1)}{\begin{Bmatrix} -\dfrac{du}{dx}(x_1) \\ 0 \\ 0 \\ \vdots \\ +\dfrac{du}{dx}(x_N) \end{Bmatrix}}$$

$$\boxed{[\mathbf{K}_s]\{\mathbf{Q}_s\} = \{\mathbf{F}_s\}}$$

- Coefficient matrix $[\mathbf{K}_s]$ is singular; it will become non-singular after applying boundary conditions

EXAMPLE

- Use three equal-length elements

$$\frac{d^2u}{dx^2} + x = 0, \qquad 0 \le x \le 1 \qquad\qquad u(0) = 0,\ u(1) = 0$$

- All elements have the same coefficient matrix

$$\left[\mathbf{k}^{(e)}\right]_{2\times 2} = \frac{1}{L^{(e)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}, \quad (e = 1,2,3)$$

- Change variable of $p(x) = x$ to $p(\xi)$: $\qquad p(\xi) = x_i(1-\xi) + x_j\xi$
- RHS

$$\{\mathbf{f}^{(e)}\} = L^{(e)}\int_0^1 p(x)\begin{Bmatrix} N_1(\xi) \\ N_2(\xi) \end{Bmatrix} d\xi = L^{(e)}\int_0^1 \left[x_i(1-\xi) + x_j\xi\right]\begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix} d\xi$$

$$= L^{(e)}\begin{Bmatrix} \dfrac{x_i}{3} + \dfrac{x_j}{6} \\ \dfrac{x_i}{6} + \dfrac{x_j}{3} \end{Bmatrix}, \qquad (e = 1,2,3)$$

EXAMPLE *cont.*

- RHS cont. $\begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix} = \frac{1}{54}\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}, \quad \begin{Bmatrix} f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix} = \frac{1}{54}\begin{Bmatrix} 4 \\ 5 \end{Bmatrix}, \quad \begin{Bmatrix} f_3^{(3)} \\ f_4^{(3)} \end{Bmatrix} = \frac{1}{54}\begin{Bmatrix} 7 \\ 8 \end{Bmatrix}$
- Assembly

$$\begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 3+3 & -3 & 0 \\ 0 & -3 & 3+3 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{54} - \frac{du}{dx}(0) \\ \frac{2}{54} + \frac{4}{54} \\ \frac{7}{54} + \frac{5}{54} \\ \frac{8}{54} + \frac{du}{dx}(1) \end{Bmatrix}$$

Element 1

Element 2

Element 3

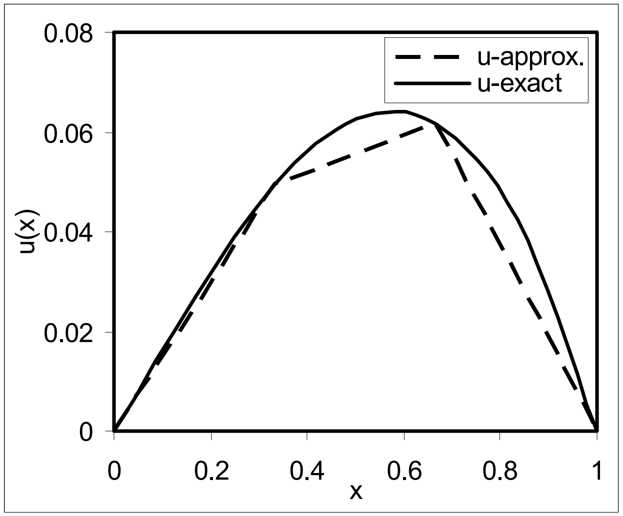
- Apply boundary conditions
  - Deleting 1st and 4th rows and columns

$$\begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \frac{1}{9}\begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \Longrightarrow \begin{aligned} u_2 &= \tfrac{4}{81} \\ u_3 &= \tfrac{5}{81} \end{aligned}$$

EXAMPLE *cont.*

- Approximate solution

$$\tilde{u}(x) = \begin{cases} \dfrac{4}{27}x, & 0 \le x \le \dfrac{1}{3} \\ \dfrac{4}{81} + \dfrac{1}{27}\left(x - \dfrac{1}{3}\right), & \dfrac{1}{3} \le x \le \dfrac{2}{3} \\ \dfrac{5}{81} - \dfrac{5}{27}\left(x - \dfrac{2}{3}\right), & \dfrac{2}{3} \le x \le 1 \end{cases}$$



- Exact solution

$$u(x) = \frac{1}{6}x\left(1 - x^2\right)$$

  - Three element solutions are poor
  - Need more elements

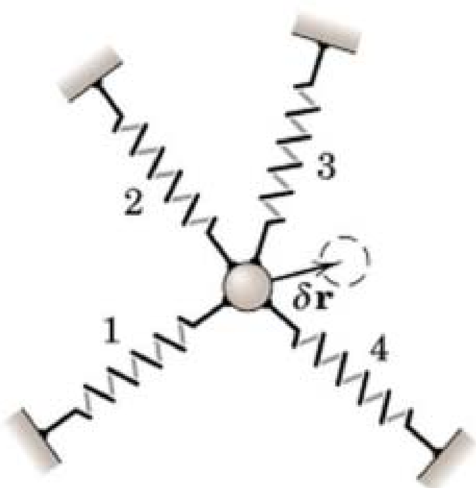
2.6 ENERGY METHODS

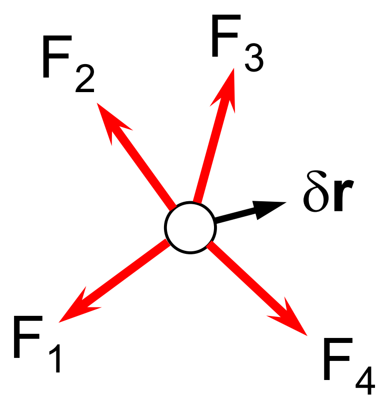
Virtual Displacement

- Virtual displacement is not experienced but only assumed to exist so that various possible equilibrium positions may be compared to determine the correct one
- Mass m and springs are in equilibrium at the current position
- Then, a small arbitrary perturbation, $\delta\mathbf{r}$, can be assumed
  - Since $\delta\mathbf{r}$ is so small, the member forces are assumed unchanged
- The work done by virtual displacement is

$$\delta W = \mathbf{F}_1 \cdot \delta\mathbf{r} + \mathbf{F}_2 \cdot \delta\mathbf{r} + \mathbf{F}_3 \cdot \delta\mathbf{r} + \mathbf{F}_4 \cdot \delta\mathbf{r} = (\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4) \cdot \delta\mathbf{r}$$

- If the current position is in force equilibrium, $\boxed{\delta W = 0}$





Principle of Virtual Work

- Powerful alternative method to obtain FE equations
- Principle of virtual work for a particle
  - for a particle in equilibrium the virtual work is identically equal to zero
  - Virtual work: work done by the (real) external forces through the virtual displacements
  - Virtual displacement: small arbitrary (imaginary, not real) displacement that is consistent with the kinematic constraints of the particle
- Force equilibrium

$$\sum F_x = 0, \sum F_y = 0, \sum F_z = 0$$

- Virtual displacements: δu, δv, and δw
- Virtual work

$$\delta W = \delta u \sum F_x + \delta v \sum F_y + \delta w \sum F_z = 0$$

- If the virtual work is zero for arbitrary virtual displacements, then the particle is in equilibrium under the applied forces

PVW of 1DOF System

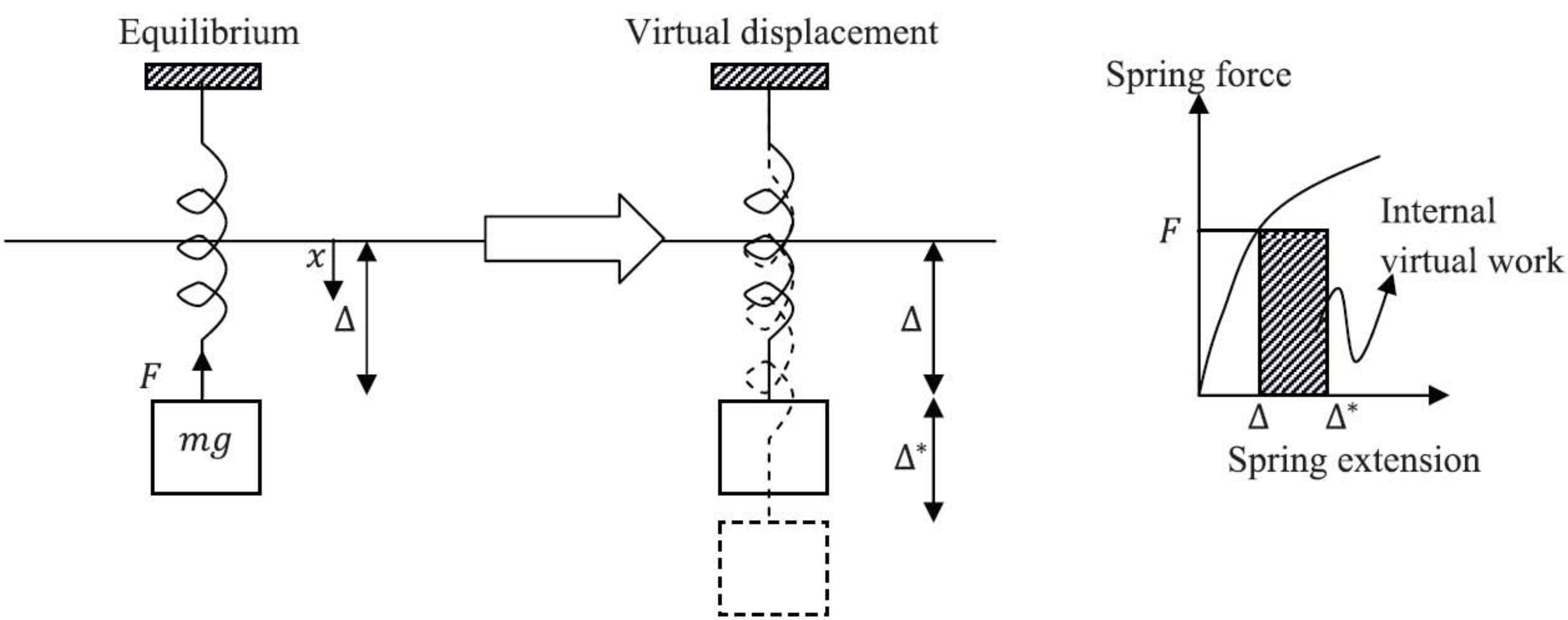
- Static equilibrium of a mass-sprint system
  - At equilibrium, the spring force, F, is equal to applied load, mg

$$F(\Delta) - mg = 0$$

  - If the position is perturbed by Δ^*

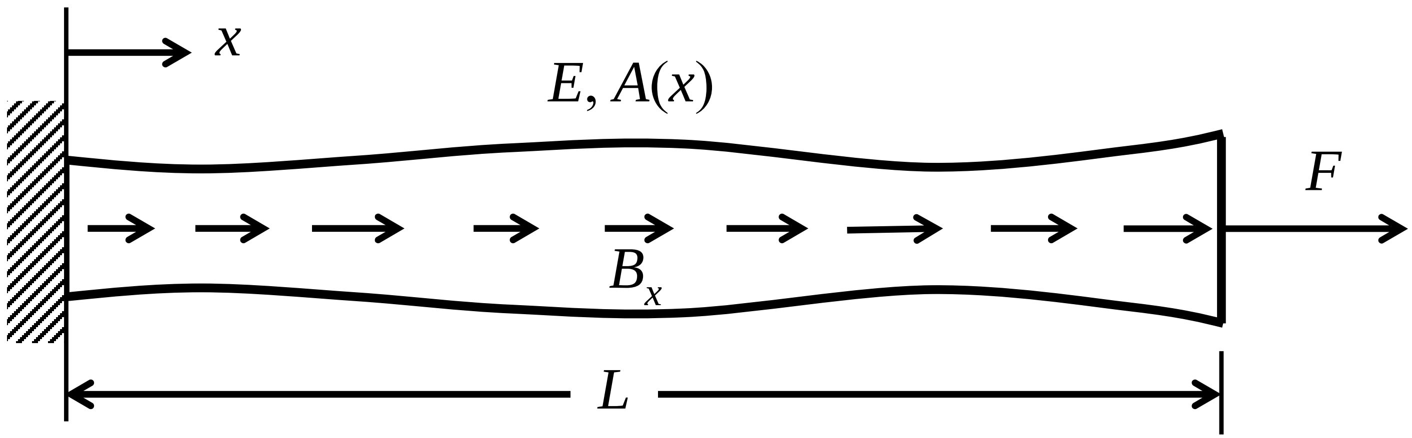
$$F(\Delta) \cdot \Delta^* = mg \cdot \Delta^*$$

  - External work done by the external forces mg during the application of a small virtual displacement is equal to the internal work done by the spring force during the application of that small virtual displacement



PRINCIPLE OF VIRTUAL WORK

- Deformable body (uniaxial bar under body force and tip force)



- Equilibrium equation: $\dfrac{d\sigma_x}{dx} + B_x = 0$ ⟵ This is force equilibrium
- PVW

$$\int_0^L \int_A \left(\frac{d\sigma_x}{dx} + B_x \right) \delta u(x) \, dA \, dx = 0$$

- Integrate over the area, axial force $P(x) = A\sigma(x)$

$$\int_0^L \left(\frac{dP}{dx} + b_x \right) \delta u(x) \, dx = 0$$

PVW *cont.*

- Integration by parts

$$P\delta u\Big|_0^L - \int_0^L P\frac{\mathrm{d}(\delta u)}{\mathrm{d}x}\mathrm{d}x + \int_0^L b_x \delta u(x)\mathrm{d}x = 0$$

  - At $x = 0$, $u(0) = 0$. Thus, $\delta u(0) = 0$
  - the virtual displacement should be consistent with the displacement constraints of the body
  - At $x = L$, $P(L) = F$
- Virtual strain $\delta\varepsilon(x) = \dfrac{\mathrm{d}(\delta u)}{\mathrm{d}x}$
- PVW:

$$\underbrace{F\delta u(L) + \int_0^L b_x \delta u(x)\mathrm{d}x}_{\delta W_e} = \underbrace{\int_0^L P\delta\varepsilon(x)\mathrm{d}x}_{-\delta W_i}$$

$$\boxed{\delta W_e + \delta W_i = 0}$$

PVW *cont*.

- in equilibrium, the sum of external and internal virtual work is zero for every virtual displacement field
- 3D PVW has the same form with different expressions
- With distributed forces and concentrated forces

$$\delta W_e = \int_S \left(t_x \delta u + t_y \delta v + t_z \delta w \right) \mathrm{d}S + \sum_i \left(F_{xi} \delta u_i + F_{yi} \delta v_i + F_{zi} \delta w_i \right)$$

- Internal virtual work

$$\delta W_i = -\int_V \left(\sigma_x \delta \varepsilon_x + \sigma_y \delta \varepsilon_y + ... + \tau_{xy} \delta \gamma_{xy} \right) \mathrm{d}V$$

Variation of a Function

- Virtual displacements in the previous section can be considered as a variation of real displacements
- Perturbation of displ $u(x)$ by arbitrary virtual displ $\delta u(x)$

$$u_\tau(x) = u(x) + \tau\delta u(x)$$

- Variation of displacement

$$\left.\frac{\mathrm{d}u_\tau(x)}{\mathrm{d}\tau}\right|_{\tau=0} = \delta u(x) \longleftarrow \text{Displacement variation}$$

- Variation of a function $f(u)$

$$\delta f = \left.\frac{\mathrm{d}f(u_\tau)}{\mathrm{d}\tau}\right|_{\tau=0} = \frac{\mathrm{d}f}{\mathrm{d}u}\delta u$$

- The order of variation & differentiation can be interchangeable

$$\delta\varepsilon_x = \delta\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right) = \frac{\mathrm{d}(\delta u)}{\mathrm{d}x}$$

Variation of a Function

- Displacement Variation
  - Need to satisfy homogeneous essential BC
  - Both $u_\tau(x)$ and u(x) are solution

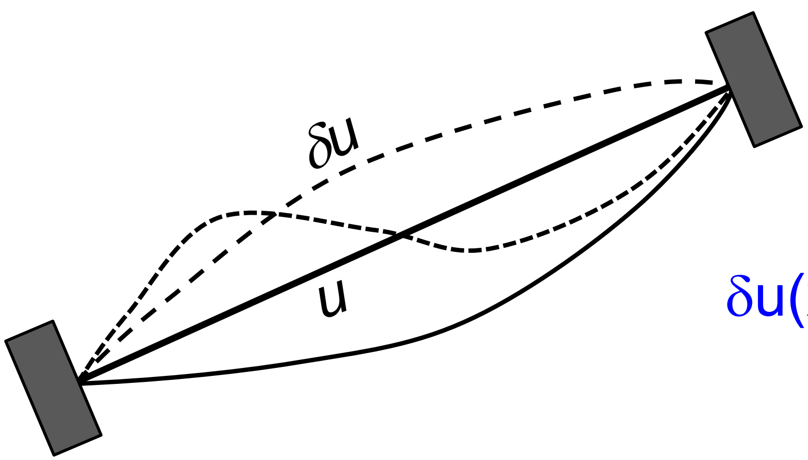
$$u_\tau(x) = u(x) + \tau\delta u(x)$$

  - Let assume that u(1) = 1 (essential BC)
  - Then

$$u_\tau(1) = u(1) + \tau\delta u(1) = 1$$

  - Therefore,

$$\delta u(1) = 0 \qquad \text{Homogeneous (i.e., = 0) BC}$$



δu(x) can be treated as u(x)

PRINCIPLE OF MINIMUM POTENTIAL ENERGY

- Strain energy density of 1D body

$$U_0 = \frac{1}{2}\sigma_x \varepsilon_x = \frac{1}{2}E\varepsilon_x^2$$

- Variation in the strain energy density by $\delta u(x)$

$$\delta U_0 = E\varepsilon_x \delta\varepsilon_x = \sigma_x \delta\varepsilon_x$$

- Variation of strain energy

$$\delta U = \int_0^L \int_A \delta U_0 dA\mathrm{d}x = \int_0^L \int_A \sigma_x \delta\varepsilon_x \mathrm{d}A\mathrm{d}x = \int_0^L P\delta\varepsilon_x \mathrm{d}x$$

$$\Rightarrow \boxed{\delta U = -\delta W_i}$$

PMPE *cont.*

- Potential energy of external forces
  - Force F is applied at x = L with corresponding virtual displ $\delta u(L)$
  - Work done by the force = $F\delta u(L)$
  - The potential is reduced by the amount of work

$$\delta V = -F\delta u(L) \Longrightarrow \delta V = -\delta\left(Fu(L)\right)$$

F is constant
virtual displacement

  - With distributed forces and concentrated force

$$V = -Fu(L) - \int_0^L b_x u(x)\mathrm{d}x \Longrightarrow \boxed{\delta V = -\delta W_e}$$

- PVW $\delta U + \delta V = 0$ or $\delta\left(U + V\right) = 0$

  - Define total potential energy $\Pi = U + V$

$$\Longrightarrow \boxed{\delta\Pi = 0}$$

PMPE *cont.*

- Principle of minimum potential energy

  Of all displacement configurations of a solid consistent with its displacement (kinematic) constraints, the actual one that satisfies the equilibrium equations is given by the minimum value of total potential energy

- Variation is similar to differentiation

$$\boxed{\delta\Pi = \frac{\mathrm{d}\Pi}{\mathrm{d}u}\delta u = 0}$$

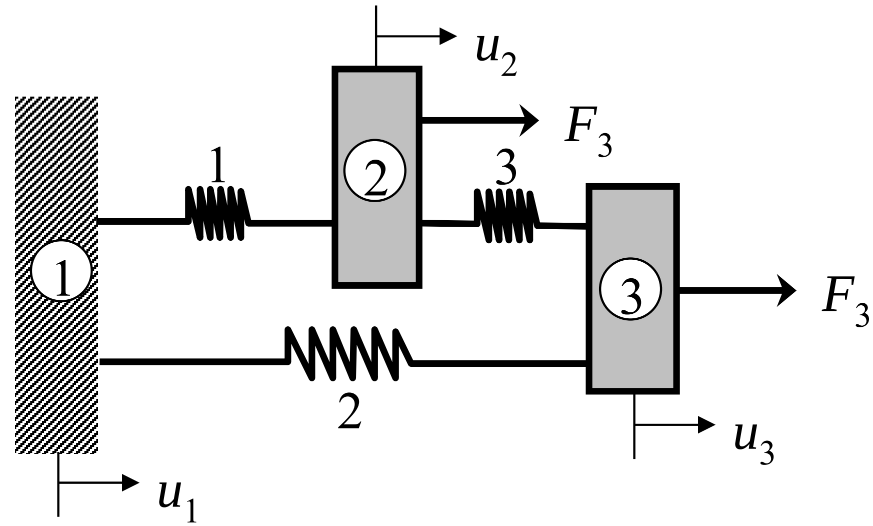
Variation = 0 $\Longrightarrow$ Derivative = 0

- In general, $u(x) = \sum_{i=1}^{N} c_i \phi_i(x) \quad \Rightarrow \quad \Pi(u) = \Pi(c_i)$

$$\delta\Pi = 0 \quad \Rightarrow \quad \frac{\mathrm{d}\Pi}{\mathrm{d}c_i} = 0, \; i = 1, \ldots, N$$

EXAMPLE: PMPE TO DISCRETE SYSTEMS

- Express U and V in terms of displacements, and then differential Π w.r.t displacements
- $k^{(1)} = 100$ N/mm, $k^{(2)} = 200$ N/mm
  $k^{(3)} = 150$ N/mm, $F_2 = 1{,}000$ N
  $F_3 = 500$ N



- Strain energy of elements (springs)

$$U^{(1)} = \frac{1}{2}k^{(1)}(u_2 - u_1)^2 \quad \Longrightarrow \quad U^{(1)} = \frac{1}{2}\underset{(1\times 2)}{\lfloor u_1 \quad u_2 \rfloor}\underset{(2\times 2)}{\begin{bmatrix} k^{(1)} & -k^{(1)} \\ -k^{(1)} & k^{(1)} \end{bmatrix}}\underset{(2\times 1)}{\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}}$$

$$U^{(2)} = \frac{1}{2}\lfloor u_1 \quad u_3 \rfloor\begin{bmatrix} k^{(2)} & -k^{(2)} \\ -k^{(2)} & k^{(2)} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_3 \end{Bmatrix}$$

$$U^{(3)} = \frac{1}{2}\lfloor u_2 \quad u_3 \rfloor\begin{bmatrix} k^{(3)} & -k^{(3)} \\ -k^{(3)} & k^{(3)} \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

EXAMPLE *cont.*

- Strain energy of the system $U = \sum_{e=1}^{3} U^{(e)}$

$$U = \frac{1}{2}\lfloor u_1 \quad u_2 \quad u_3 \rfloor \begin{bmatrix} k^{(1)} + k^{(2)} & -k^{(1)} & -k^{(2)} \\ -k^{(1)} & k^{(1)} + k^{(3)} & -k^{(3)} \\ -k^{(2)} & -k^{(3)} & k^{(2)} + k^{(3)} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\boxed{U = \frac{1}{2}\{\mathbf{Q}\}^{\mathrm{T}}[\mathbf{K}]\{\mathbf{Q}\}}$$

$$\{\mathbf{Q}\} = \{u_1, u_2, u_3\}^{\mathrm{T}}$$

- Potential energy of applied forces

$$V = -(F_1u_1 + F_2u_2 + F_3u_3) = -\lfloor u_1 \quad u_2 \quad u_3 \rfloor \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = -\{\mathbf{Q}\}^{\mathrm{T}}\{\mathbf{F}\}$$

- Total potential energy

$$\Pi = U + V = \frac{1}{2}\{\mathbf{Q}\}^{\mathrm{T}}[\mathbf{K}]\{\mathbf{Q}\} - \{\mathbf{Q}\}^{\mathrm{T}}\{\mathbf{F}\}$$

EXAMPLE *cont.*

- Total potential energy is minimized with respect to the DOFs

$$\frac{\partial \Pi}{\partial u_1} = 0, \quad \frac{\partial \Pi}{\partial u_2} = 0, \quad \frac{\partial \Pi}{\partial u_3} = 0 \quad \text{or,} \quad \frac{\partial \Pi}{\partial \{\mathbf{Q}\}} = 0$$

$$\Rightarrow \quad [\mathbf{K}]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \quad \Rightarrow \quad \boxed{[\mathbf{K}]\{\mathbf{Q}\} = \{\mathbf{F}\}}$$

Finite element equations

- Global FE equations

$$\begin{bmatrix} 300 & -100 & -200 \\ -100 & 250 & -150 \\ -200 & -150 & 350 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 1{,}000 \\ 500 \end{Bmatrix} \quad \Rightarrow \quad \begin{aligned} u_2 &= 6.538\text{mm} \\ u_3 &= 4.231\text{mm} \\ F_1 &= -1{,}500\text{N} \end{aligned}$$

- Forces in the springs $P^{(e)} = k^{(e)}(u_j - u_i)$

$$P^{(1)} = k^{(1)}(u_2 - u_1) = 654\text{N} \qquad P^{(2)} = k^{(2)}(u_3 - u_1) = 846\text{N}$$

$$P^{(3)} = k^{(3)}(u_3 - u_2) = -346\text{N}$$

RAYLEIGH-RITZ METHOD

- PMPE is easy to apply to discrete systems (exact solution)
  - Unknown DOFs are finite
- Continuous system (DOFs are infinite)
  - Use Rayleigh-Ritz method to approximate a continuous system as a discrete system with finite number of DOFs
  - Approximate the displacements by a function containing finite number of coefficients
- Total potential energy is evaluated in terms of the unknown coefficients
- Apply PMPE to determine the coefficients that minimizes the total potential energy
- Solution thus obtained may not be exact
  - It is the best solution from among the family of solutions that can be obtained from the assumed displacement functions

RAYLEIGH-RITZ METHOD *cont.*

- Assumed displacement (must satisfy the essential BC)

$$u(x) = c_1 f_1(x) + \cdots + c_n f_n(x)$$

- Determine the strain energy: U
- Find the potential of external forces: V
- Total potential energy in terms of unknown coefficients

$$\Pi(c_1, c_2, \ldots c_n) = U + V$$

- PMPE to determine coefficients

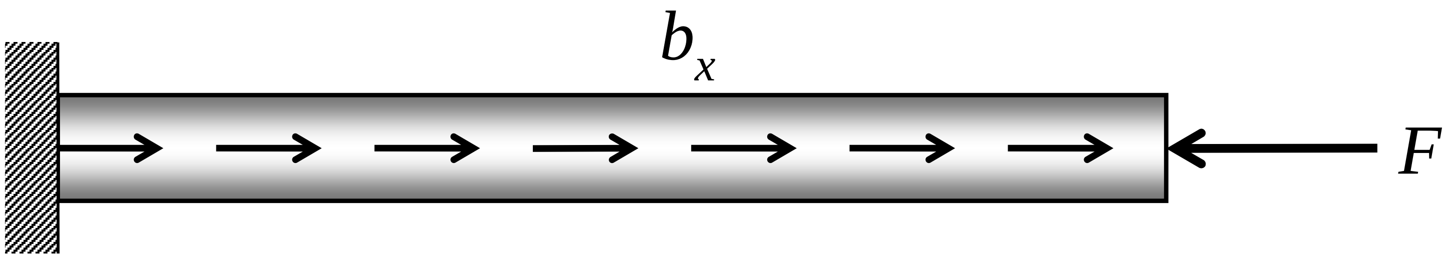
$$\frac{\partial \Pi}{\partial c_i} = 0, \quad i = 1, \ldots n$$

- After finding coefficients, determine displacement and stress

$$u(x) = c_1 f_1(x) + \cdots + c_n f_n(x)$$

$$P(x) = AE \frac{\mathrm{d}u}{\mathrm{d}x} \qquad \text{(for 1D bar)}$$

EXAMPLE



- L = 1m, A = 100mm2, E = 100 GPa, F = 10kN, bx = 10kN/m
- Approximate solution $u(x) = c_1 x + c_2 x^2$
- Strain energy $U = \int_0^L U_L(x)\mathrm{d}x = \int_0^L \frac{1}{2}AE\varepsilon_x^2 \mathrm{d}x = \int_0^L \frac{1}{2}AE\left(\frac{du}{dx}\right)^2 \mathrm{d}x$

$$U(c_1, c_2) = \frac{1}{2}AE\int_0^L (c_1 + 2c_2 x)^2 \mathrm{d}x = \frac{1}{2}AE\left(Lc_1^2 + 2L^2 c_1 c_2 + \frac{4}{3}L^3 c_2^2\right)$$

- Potential energy of forces

$$V(c_1, c_2) = -\int_0^L b_x(x)u(x)\mathrm{d}x - (-F)u(L) = -\int_0^L b_x\left(c_1 x + c_2 x^2\right)\mathrm{d}x + F\left(c_1 L + c_2 L^2\right)$$

$$= c_1\left(FL - b_x\frac{L^2}{2}\right) + c_2\left(FL^2 - b_x\frac{L^3}{3}\right)$$

EXAMPLE *cont.*

- PMPE $\Pi(c_1, c_2) = U + V$

$$\frac{\partial \Pi}{\partial c_1} = AELc_1 + AEL^2 c_2 + FL - b_x \frac{L^2}{2} = 0$$

$$\frac{\partial \Pi}{\partial c_2} = AEL^2 c_1 + \frac{4}{3} AEL^3 c_2 + FL^2 - b_x \frac{L^3}{3} = 0$$

$$\begin{aligned} 10^7 c_1 + 10^7 c_2 &= -5{,}000 \\ 10^7 c_1 + \frac{4 \times 10^7}{3} c_2 &= -6{,}667 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} c_1 &= 0 \\ c_2 &= -0.5 \times 10^{-3} \end{aligned}$$

- Approximate solution $u(x) = -0.5 \times 10^{-3} x^2$
- Axial force $P(x) = AE\mathrm{d}u / \mathrm{d}x = -10{,}000x$
- Reaction force $R = -P(0) = 0$