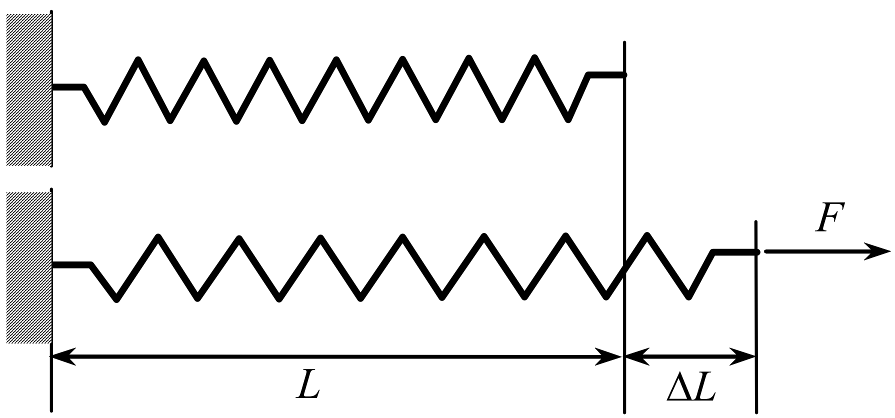
CHAP 5 Review of Solid Mechanics

5.2 STRESS

STRESS

- Stress
  - Fundamental concept related to the safety of a structure
  - Often used as criteria for mechanical design
  - Internal force created by deforming the shape against external loads.



$$F = K\Delta L$$

  - Linear elasticity: the relation between internal force and deformation is linear.

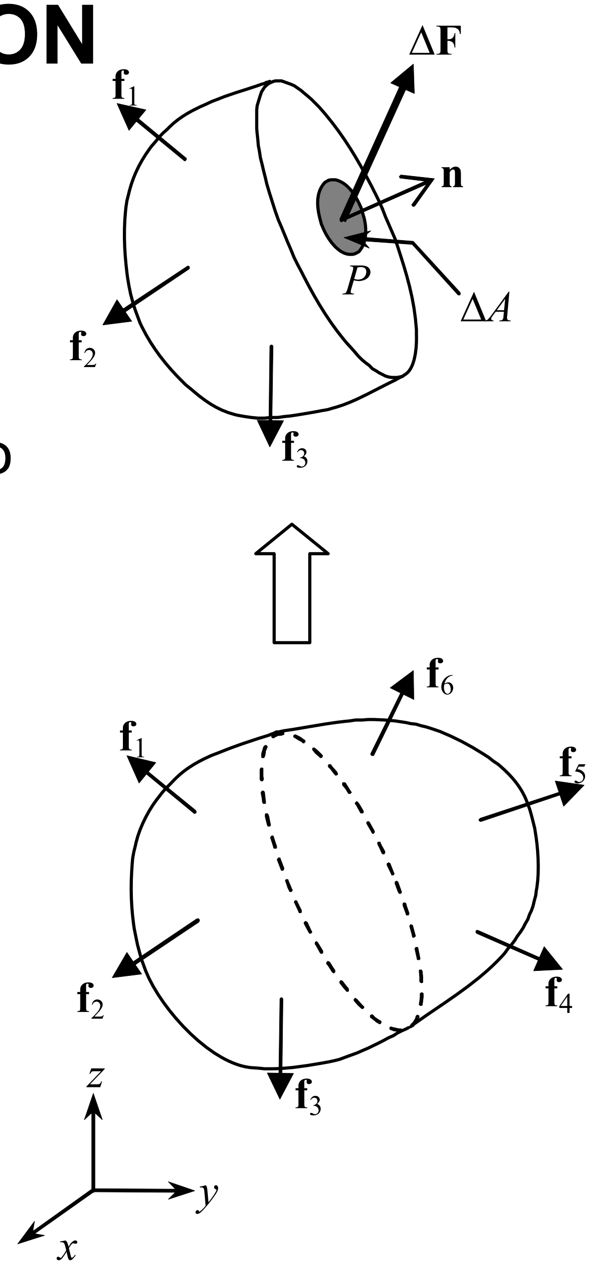
SURFACE TRACTION

- Surface traction (Stress)
  - The entire body is in equilibrium with external forces ($\mathbf{f}_1 \sim \mathbf{f}_6$)
  - The imaginary cut body is in equilibrium due to external forces ($\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3$) and internal forces
  - Internal force acting at a point P on a plane whose unit normal is **n**:

$$\mathbf{T}^{(\mathbf{n})} = \lim_{\Delta A \to 0} \frac{\Delta \mathbf{F}}{\Delta A}$$

  - The surface traction depends on the unit normal direction **n**.
  - Surface traction will change when **n** changes.
  - unit = force per unit area (pressure)

$$\mathbf{T}^{(n)} = T_x\mathbf{i} + T_y\mathbf{j} + T_z\mathbf{k} \qquad \left\|\mathbf{T}^{(n)}\right\| = T = \sqrt{T_x^2 + T_y^2 + T_z^2}$$



NORMAL AND SHEAR STRESSES

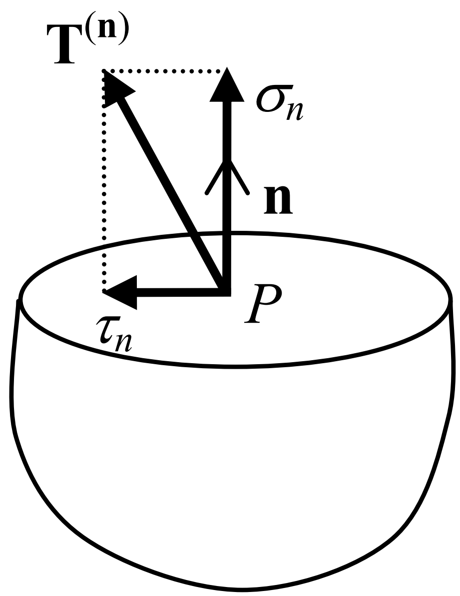
- Normal and shear stresses
  - Decompose $\mathbf{T}^{(\mathbf{n})}$ into normal and tangential components

σ_n	normal stress	stress component parallel to **n**
τ_n	shear stress	stress component perpendicular to **n**

$$\sigma_n = \mathbf{T}^{(\mathbf{n})} \cdot \mathbf{n}$$

$$\left\| \mathbf{T}^{(\mathbf{n})} \right\|^2 = \sigma_n^2 + \tau_n^2$$

$$\tau_n = \sqrt{\left\| \mathbf{T}^{(\mathbf{n})} \right\|^2 - \sigma_n^2}$$



What if $\mathbf{T}^{(\mathbf{n})}$ and **n** are in the same direction?

- Practice Example 1.2 in the textbook

CARTESIAN STRESS COMPONENTS

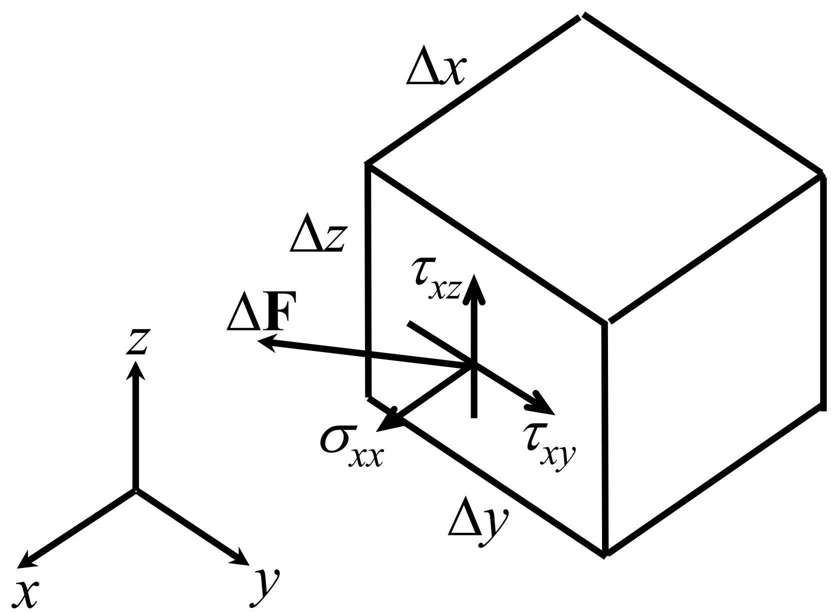
- Surface traction changes according to the direction of the surface.
- Impossible to store stress information for all directions.
- Let's store surface traction parallel to the three coordinate directions.
- Surface traction in other directions can be calculated from them.
- Consider the x-face of an infinitesimal cube

$$\mathbf{T}^{(x)} = T_x^{(x)}\mathbf{i} + T_y^{(x)}\mathbf{j} + T_z^{(x)}\mathbf{k}$$

$$\Delta\mathbf{F} = \Delta F_x\mathbf{i} + \Delta F_y\mathbf{j} + \Delta F_z\mathbf{k}$$

$$\mathbf{T}^{(x)} = \sigma_{xx}\mathbf{i} + \tau_{xy}\mathbf{j} + \tau_{xz}\mathbf{k}$$

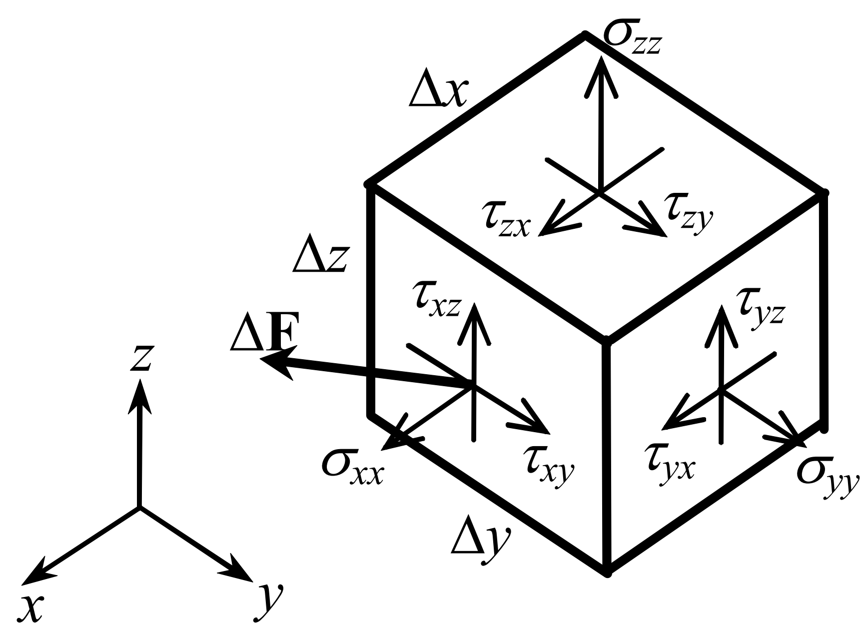
$$\begin{cases} \sigma_{xx} = \lim_{\Delta A_x \to 0} \dfrac{\Delta F_x}{\Delta A_x} \\ \tau_{xy} = \lim_{\Delta A_x \to 0} \dfrac{\Delta F_y}{\Delta A_x} \\ \tau_{xz} = \lim_{\Delta A_x \to 0} \dfrac{\Delta F_z}{\Delta A_x} \end{cases}$$



CARTESIAN COMPONENTS *cont.*

- First index is the face and the second index is its direction
- When two indices are the same, normal stress, otherwise shear stress.
- Continuation for other surfaces.
- Total nine components
- Same stress components are defined for the negative planes.

Comp.	Description
σ_{xx}	Normal stress on the *x* face in the *x* dir.
σ_{yy}	Normal stress on the *y* face in the *y* dir.
σ_{zz}	Normal stress on the *z* face in the *z* dir.
τ_{xy}	Shear stress on the *x* face in the *y* dir.
τ_{yx}	Shear stress on the *y* face in the *x* dir.
τ_{yz}	Shear stress on the *y* face in the *z* dir.
τ_{zy}	Shear stress on the *z* face in the *y* dir.
τ_{xz}	Shear stress on the *x* face in the *z* dir.
τ_{zx}	Shear stress on the *z* face in the *x* dir.



CARTESIAN COMPONENTS *cont*.

- Sign convention
  - Positive when tension and negative when compression.
  - Shear stress acting on the positive face is positive when it is acting in the positive coordinate direction.

$$\mathrm{sgn}(\sigma_{xx}) = \mathrm{sgn}(\mathbf{n}) \times \mathrm{sgn}(\Delta F_x)$$
$$\mathrm{sgn}(\tau_{xy}) = \mathrm{sgn}(\mathbf{n}) \times \mathrm{sgn}(\Delta F_y)$$

- Example

STRESS TRANSFORMATION

- **If stress components in xyz-planes are known, it is possible to determine the surface traction acting on any plane.**
- Consider a plane whose normal is **n**.

$$\mathbf{n} = n_x\mathbf{i} + n_y\mathbf{j} + n_z\mathbf{k} = \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}$$

- Surface area ($\Delta ABC = A$)

$$\Delta PAB = An_z;\;\; \Delta PBC = An_x;\;\; \Delta PAC = An_y$$

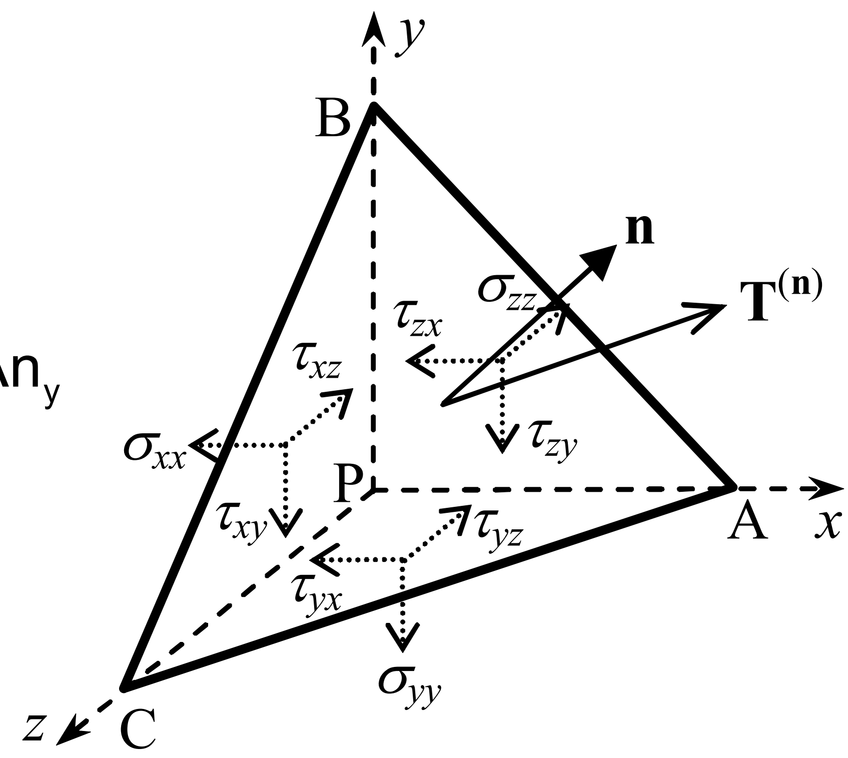
- The surface traction

$$\mathbf{T}^{(\mathbf{n})} = T_x^{(\mathbf{n})}\mathbf{i} + T_y^{(\mathbf{n})}\mathbf{j} + T_z^{(\mathbf{n})}\mathbf{k}$$

- Force balance ($h \rightarrow 0$)

$$\sum F_x = T_x^{(\mathbf{n})}A - \sigma_{xx}An_x - \tau_{yx}An_y - \tau_{zx}An_z = 0$$

$$T_x^{(\mathbf{n})} = \sigma_{xx}n_x + \tau_{yx}n_y + \tau_{zx}n_z$$



STRESS TRANSFORMATION *cont.*

- All three-directions

$$T_x^{(\mathbf{n})} = \sigma_{xx} n_x + \tau_{yx} n_y + \tau_{zx} n_z$$

$$T_y^{(\mathbf{n})} = \tau_{xy} n_x + \sigma_{yy} n_y + \tau_{zy} n_z$$

$$T_z^{(\mathbf{n})} = \tau_{xz} n_x + \tau_{yz} n_y + \sigma_{zz} n_z$$

- Matrix notation

$$\mathbf{T}^{(\mathbf{n})} = [\sigma] \cdot \mathbf{n} \qquad [\sigma] = \begin{bmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix}$$

  - $[\sigma]$: stress matrix; completely characterize the state of stress at a point

- Normal and shear components

$$\sigma_n = \mathbf{T}^{(\mathbf{n})} \cdot \mathbf{n} = \mathbf{n} \cdot [\sigma] \cdot \mathbf{n} \quad \Longleftarrow \quad \{\mathbf{n}\}^T [\sigma] \{\mathbf{n}\}$$

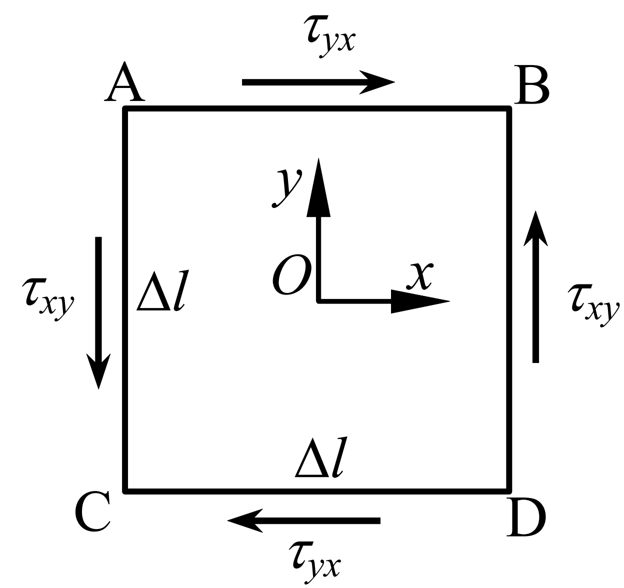
$$\tau_n = \sqrt{\left\| \mathbf{T}^{(\mathbf{n})} \right\|^2 - \sigma_n^2}$$

SYMMETRY OF STRESS TENSOR

- Stress tensor should be symmetric
  9 components ⟹ 6 components
- Equilibrium of the angular moment

$$\sum M = \Delta l(\tau_{xy} - \tau_{yx}) = 0$$

$$\Rightarrow \tau_{xy} = \tau_{yx}$$



- Similarly for all three directions:

$$\tau_{xy} = \tau_{yx}$$

$$\tau_{yz} = \tau_{zy}$$

$$\tau_{xz} = \tau_{zx}$$

- Let's use vector notation: $\{\sigma\} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}$

$$[\sigma] = \begin{bmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix}$$

PRINCIPAL STRESSES

– Can it be possible to find planes that have zero shear stresses?
– Normal stress = principal stress
– Normal direction = principal direction
– Extreme values (max or min) of stress at the point
– Three principal stresses and directions.
– Stress vector ($\mathbf{T}^{(\mathbf{n})}$) // normal vector ($\mathbf{n}$)

$$\mathbf{T}^{(\mathbf{n})} = \sigma_n \mathbf{n}$$

n and σ_n are unknown
σ_n : principal stress
n: principal direction

$$[\sigma] \cdot \mathbf{n} = \sigma_n \mathbf{n}$$

Eigenvalue problem

$$\left([\sigma] - \sigma_n [\mathbf{I}]\right) \cdot \mathbf{n} = \mathbf{0}$$

$$\begin{bmatrix} \sigma_{xx} - \sigma_n & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} - \sigma_n & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} - \sigma_n \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

What would be the solution?

PRINCIPAL STRESSES *cont.*

– **n** = 0 satisfies the equation: trivial solution

– Non-trivial solution when the determinant is zero.

$$\begin{vmatrix} \sigma_{xx} - \sigma_n & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} - \sigma_n & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} - \sigma_n \end{vmatrix} = 0$$

– Expanding the determinant equation:

$$\sigma_n^3 - I_1\sigma_n^2 + I_2\sigma_n - I_3 = 0$$

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

$$I_2 = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$$

$$I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2$$

– I_1, I_2, I_3: invariants of the stress matrix [**σ**], which are independent of coordinate systems.

– Three roots: principal stresses, $\sigma_1 \geq \sigma_2 \geq \sigma_3$

PRINCIPAL DIRECTION

- Calculate principal direction using principal stress.
- Substitute each principal stress at a time.

$$\begin{bmatrix} \sigma_{xx}-\sigma_1 & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy}-\sigma_1 & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz}-\sigma_1 \end{bmatrix} \begin{Bmatrix} n_x^1 \\ n_y^1 \\ n_z^1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

- Since the determinant is zero (i.e., the matrix is singular), three equations are not independent.
- An infinite number of solutions exist.
- Need one more relation to uniquely determine **n**.

$$\left\|\mathbf{n}^i\right\|^2 = (n_x^i)^2 + (n_y^i)^2 + (n_z^i)^2 = 1, \qquad i = 1,2,3$$

- Infinite solutions mean the same direction with different magnitude. We select the one that has unit magnitude

PRINCIPAL DIRECTION *cont.*

– Planes on which the principal stresses act are mutually perpendicular
– Let's consider two principal directions $\mathbf{n}^i$ and $\mathbf{n}^j$, with i ≠ j.

$$[\sigma]\cdot\mathbf{n}^i = \sigma_i\mathbf{n}^i$$
$$[\sigma]\cdot\mathbf{n}^j = \sigma_j\mathbf{n}^j$$

– Scalar products using $\mathbf{n}^j$ and $\mathbf{n}^i$,

$$\mathbf{n}^j\cdot[\sigma]\cdot\mathbf{n}^i = \sigma_i\mathbf{n}^j\cdot\mathbf{n}^i$$
$$\mathbf{n}^i\cdot[\sigma]\cdot\mathbf{n}^j = \sigma_j\mathbf{n}^i\cdot\mathbf{n}^j$$

– Subtract two equations,

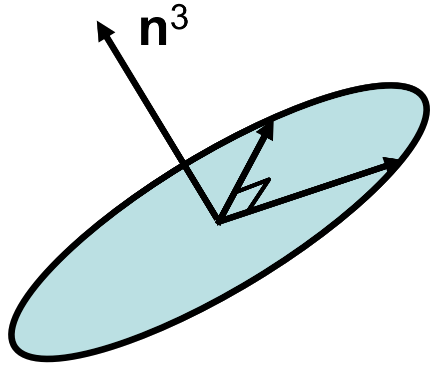
$$(\sigma_i - \sigma_j)\mathbf{n}^i\cdot\mathbf{n}^j = 0$$

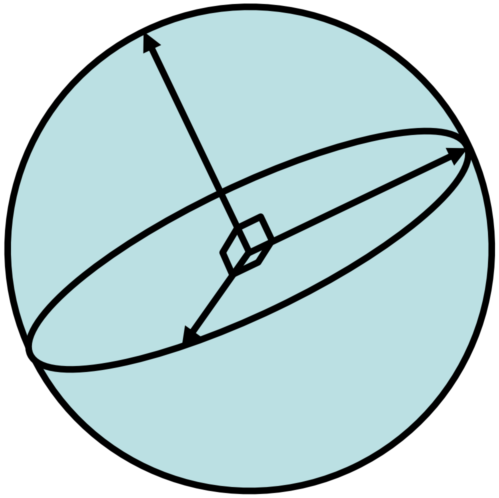
– Since two principal stresses are different,

$$\mathbf{n}^i\cdot\mathbf{n}^j = 0, \quad \text{when } i \neq j$$

PRINCIPAL DIRECTION *cont*.

- There are three cases for principal directions:
  1. σ_1, σ_2, and σ_3 are distinct $\Rightarrow$ principal directions are three unique mutually orthogonal unit vectors.
  2. $\sigma_1 = \sigma_2$ and σ_3 are distinct $\Rightarrow$ $\mathbf{n}^3$ is a unique principal direction, and any two orthogonal directions on the plane that is perpendicular to $\mathbf{n}^3$ are principal directions.
  3. $\sigma_1 = \sigma_2 = \sigma_3$ $\Rightarrow$ any three orthogonal directions are principal directions. This state of stress corresponds to a **hydrostatic pressure**.



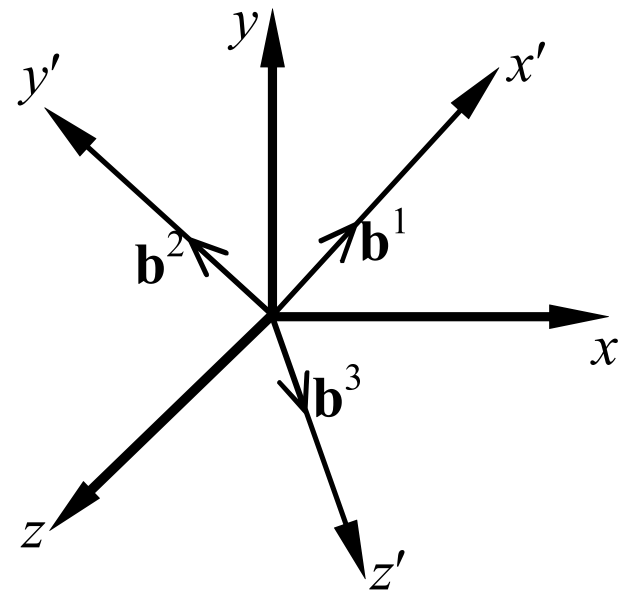


COORDINATE TRANSFORMATION

– When $[\boldsymbol{\sigma}]_{xyz}$ is given, what would be the components in a different coordinate system x′y′z′ (i.e., $[\boldsymbol{\sigma}]_{x'y'z'}$)?

– Unit vectors in x′y′z′-coordinates:

$$\mathbf{b}^1 = \begin{Bmatrix} b_1^1 \\ b_2^1 \\ b_3^1 \end{Bmatrix}, \ \mathbf{b}^2 = \begin{Bmatrix} b_1^2 \\ b_2^2 \\ b_3^2 \end{Bmatrix}, \ \mathbf{b}^3 = \begin{Bmatrix} b_1^3 \\ b_2^3 \\ b_3^3 \end{Bmatrix}$$



– $\mathbf{b}^1 = \{1, 0, 0\}^T$ in x′y′z′ coordinates,
while $\mathbf{b}^1 = \{b_1^1, b_2^1, b_3^1\}$ in xyz coordinates

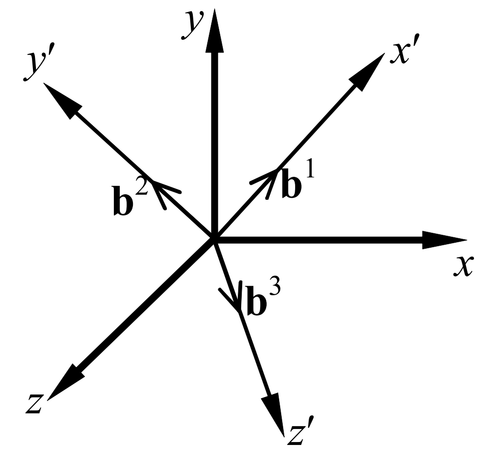
– the rotational transformation matrix

$$[\mathbf{N}] = [\mathbf{b}^1 \quad \mathbf{b}^2 \quad \mathbf{b}^3] = \begin{bmatrix} b_1^1 & b_1^2 & b_1^3 \\ b_2^1 & b_2^2 & b_2^3 \\ b_3^1 & b_3^2 & b_3^3 \end{bmatrix}$$

– Stress does not rotate. The coordinates rotate

COORDINATE TRANSFORMATION *cont*.

– [**N**] transforms a vector in the x′y′z′ coordinates into the xyz coordinates, while [**N**]^T transforms a vector in the xyz coordinates into the x′y′z′ coordinates.



– Consider $\mathbf{b}_{x'y'z'} = \{1, 0, 0\}^T$:

$$\mathbf{b}^1_{xyz} = [\mathbf{N}] \cdot \mathbf{b}^1_{x'y'z'} = \begin{Bmatrix} b^1_1 \\ b^1_2 \\ b^1_3 \end{Bmatrix}$$

– Stress transformation: Using stress vectors,

$$[\mathbf{T}^{(\mathbf{b}^1)} \quad \mathbf{T}^{(\mathbf{b}^2)} \quad \mathbf{T}^{(\mathbf{b}^3)}]_{xyz} = [\sigma]_{xyz}[\mathbf{b}^1 \quad \mathbf{b}^2 \quad \mathbf{b}^3] = [\sigma]_{xyz}[\mathbf{N}]$$

– By multiplying [**N**]^T the stress vectors can be represented in the x′y′z′ coordinates

$$[\sigma]_{x'y'z'} = [\mathbf{N}]^T [\sigma]_{xyz} [\mathbf{N}]$$

– The first [**N**] transforms the plane, while the second transforms the force.

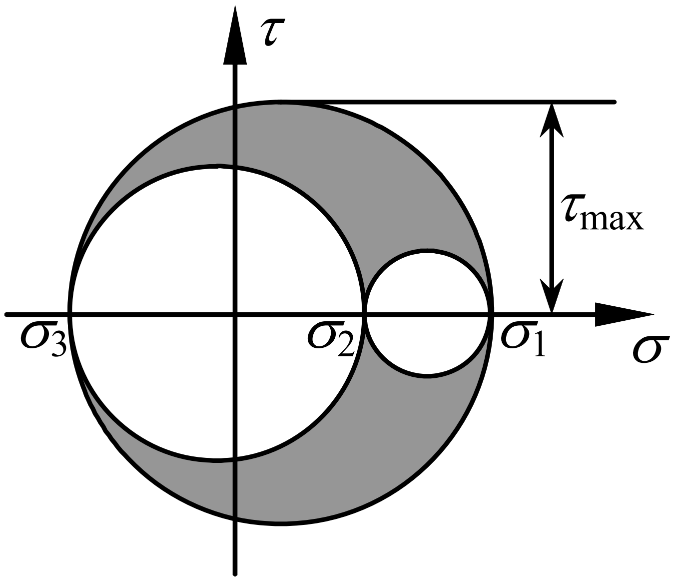
MAXIMUM SHEAR STRESS

- Important in the failure criteria of the material
- Mohr's circle
- maximum shear stress

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2}$$

- Normal stress
  at max shear stress plane

$$\sigma_n = \frac{\sigma_1 + \sigma_3}{2}$$



What Stress Could Be Design Criteria?

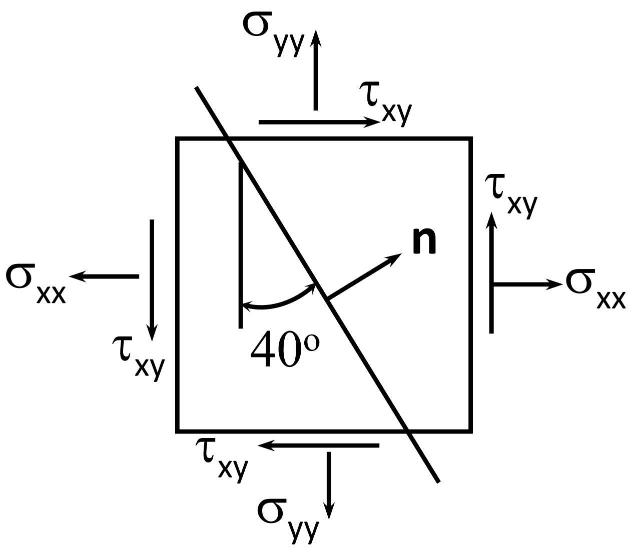
- It must be independent of the coordinate system.
- Stress Invariants
- Principal Stresses
- Maximum Shear Stress

Exercise

2. Direction $n_x:n_y:n_z = 3:4:12$. Determine $T^{(n)}$, magnitude of $T^{(n)}$, normal stress σ_n, shear stress τ_n, angle between $T^{(n)}$ and n.

$$[\sigma] = \begin{bmatrix} 13 & 13 & 0 \\ 13 & 26 & -13 \\ 0 & -13 & -39 \end{bmatrix}$$

4. If $\sigma_{xx} = 90$, $\sigma_{yy} = -45$, $\tau_{xy} = 30$, and $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$, find $T^{(n)}$, σ_n, and τ_n.



Exercise

7. Determine the principal stresses and their associated directions, when the stress matrix at a point is given by

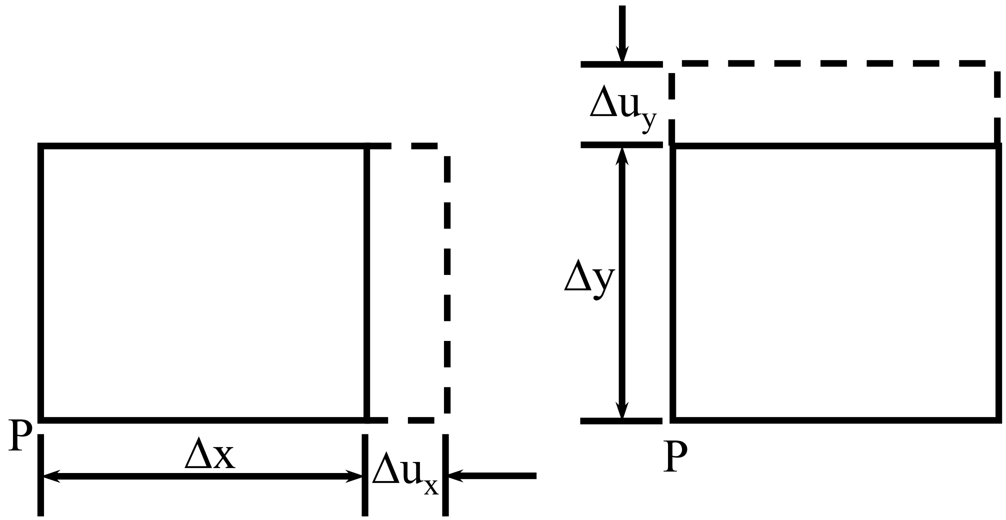
$$[\sigma] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \text{MPa}$$

8. Let $x'y'z'$ coordinate system be defined using the three principal directions obtained from Problem 7. Determine the transformed stress matrix $[\sigma]_{x'y'z'}$ in the new coordinates system

5.3 STRAIN

Elementary Definition of Strain

– Strain is defined as the elongation per unit length



– Tensile (normal) strains in *x*- and *y*-directions

$$\varepsilon_{xx} = \lim_{\Delta x \to 0} \frac{\Delta u_x}{\Delta x} = \frac{\partial u_x}{\partial x}$$

$$\varepsilon_{yy} = \lim_{\Delta y \to 0} \frac{\Delta u_y}{\Delta y} = \frac{\partial u_y}{\partial y}$$

Textbook has different, but more rigorous derivations

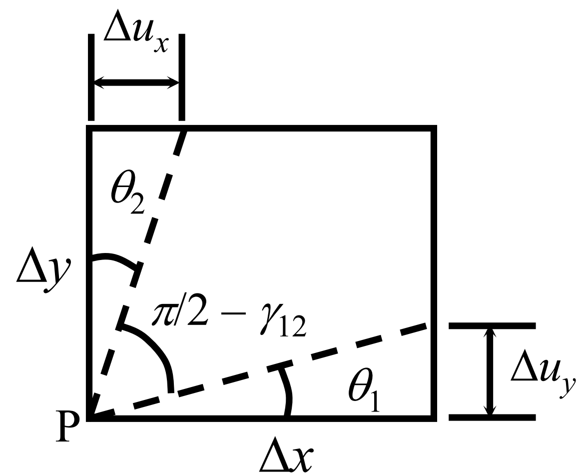
– Strain is a dimensionless quantity. Positive for elongation and negative for compression

Elementary Definition of Strain

– Shear strain is the tangent of the change in angle between two originally perpendicular axes

$$\theta_1 \sim \tan\theta_1 = \frac{\Delta u_y}{\Delta x}$$

$$\theta_2 \sim \tan\theta_2 = \frac{\Delta u_x}{\Delta y}$$



– Shear strain (change of angle)

$$\gamma_{xy} = \theta_1 + \theta_2 = \lim_{\Delta x \to 0} \frac{\Delta u_y}{\Delta x} + \lim_{\Delta y \to 0} \frac{\Delta u_x}{\Delta y} = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}$$

$$\varepsilon_{xy} = \frac{1}{2}\gamma_{xy} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)$$

– Positive when the angle between two positive (or two negative) faces is reduced and negative when the angle is increased.

– Valid for small deformation

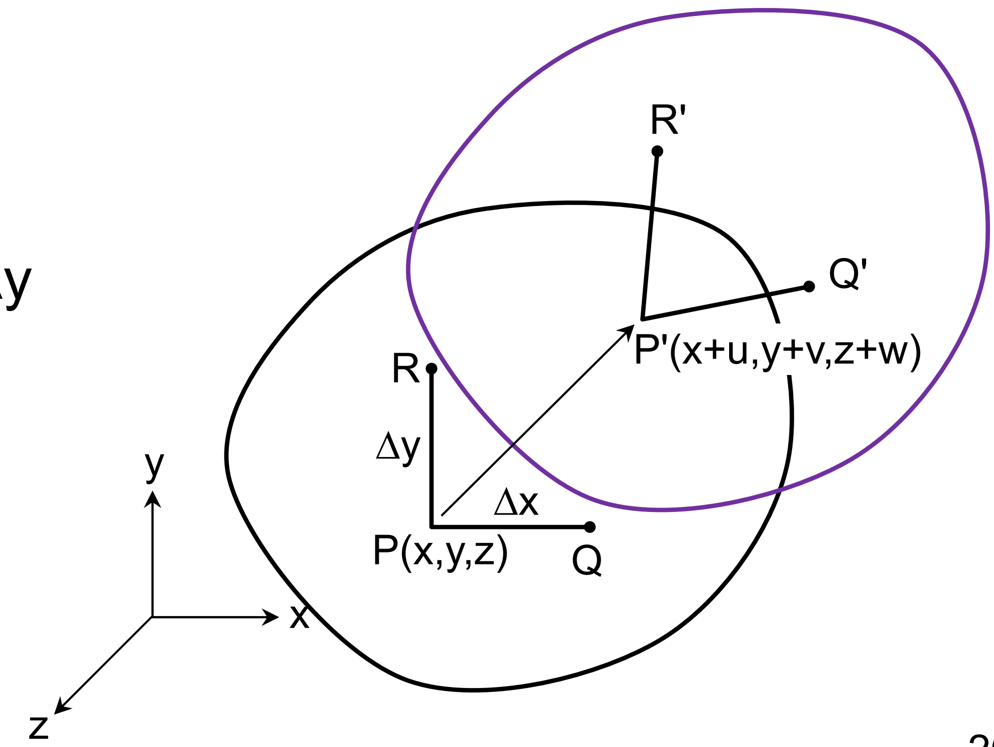
Rigorous Definition of Strain

- Strain: a quantitative measure of deformation
  - Normal strain: change in length of a line segment
  - Shear strain: change in angle between two perpendicular line segments
- Displacement of P = (u, v, w)
- Displacement of Q & R

$$u_Q = u + \frac{\partial u}{\partial x}\Delta x \qquad u_R = u + \frac{\partial u}{\partial y}\Delta y$$

$$v_Q = v + \frac{\partial v}{\partial x}\Delta x \qquad v_R = v + \frac{\partial v}{\partial y}\Delta y$$

$$w_Q = w + \frac{\partial w}{\partial x}\Delta x \qquad w_R = w + \frac{\partial w}{\partial y}\Delta y$$



Displacement Field

- The coordinates of P, Q, and R before and after deformation

$$P:(x,y,z)$$

$$Q:(x+\Delta x,y,z)$$

$$R:(x,y+\Delta y,z)$$

$$P':(x+u_P,y+v_P,z+w_P)=(x+u,y+v,z+w)$$

$$Q':(x+\Delta x+u_Q,y+v_Q,z+w_Q)$$

$$=(x+\Delta x+u+\frac{\partial u}{\partial x}\Delta x,y+v+\frac{\partial v}{\partial x}\Delta x,z+w+\frac{\partial w}{\partial x}\Delta x)$$

$$R':(x+u_R,y+\Delta y+v_R,z+w_R)$$

$$=(x+u+\frac{\partial u}{\partial y}\Delta y,y+\Delta y+v+\frac{\partial v}{\partial y}\Delta y,z+w+\frac{\partial w}{\partial y}\Delta y)$$

- Length of the line segment P'Q'

$$P'Q'=\sqrt{(x_{P'}-x_{Q'})^2+(y_{P'}-y_{Q'})^2+(z_{P'}-z_{Q'})^2}$$

Deformation Field

- Length of the line segment P'Q'

$$P'Q' = \Delta x \sqrt{\left(1+\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2}$$

$$= \Delta x \left(1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right)^{1/2}$$

$$\approx \Delta x \left(\underbrace{1 + \frac{\partial u}{\partial x}}_{\text{Linear}} + \underbrace{\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2}_{\text{Nonlinear}}\right) \qquad \approx \Delta x \left(1 + \frac{\partial u}{\partial x}\right)$$

Linear

Nonlinear ⟹ Ignore H.O.T. when displacement gradients are small

- Linear normal strain

$$\varepsilon_{xx} = \frac{P'Q' - PQ}{PQ} = \frac{\partial u}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y}, \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

Deformation Field

- Shear strain γ_{xy}
  - change in angle between two lines originally parallel to x– and y–axes

$$\theta_1 = \frac{y_{Q'} - y_Q}{\Delta x} = \frac{\partial v}{\partial x} \qquad \theta_2 = \frac{x_{R'} - x_R}{\Delta y} = \frac{\partial u}{\partial y}$$

$$\gamma_{xy} = \theta_1 + \theta_2 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

Engineering shear strain

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

STRAIN MATRIX

- Strain matrix and strain vector

$$[\boldsymbol{\varepsilon}] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \qquad \{\boldsymbol{\varepsilon}\} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix}$$

- Normal component: $\varepsilon_{nn} = \mathbf{n} \cdot [\varepsilon] \cdot \mathbf{n}$
- Coordinate transformation: $[\varepsilon]_{x'y'z'} = [\mathbf{N}]^T [\varepsilon]_{xyz} [\mathbf{N}]$
- Principal strain: $[\varepsilon] \cdot \mathbf{n} = \lambda \mathbf{n}$
- $\varepsilon_1 \geq \varepsilon_2 \geq \varepsilon_3$
- Maximum shear strain: $\dfrac{\gamma_{max}}{2} = \dfrac{\varepsilon_1 - \varepsilon_3}{2}$

- Will the principal direction of strain be the same as that of stress?

STRESS VS STRAIN

$[\sigma]$ is a symmetric 3×3 matrix	$[\varepsilon]$ is a symmetric 3×3 matrix
Normal stress in the direction **n** is $\sigma_{nn} = \mathbf{n} \cdot [\sigma] \cdot \mathbf{n}$	Normal strain in the direction **n** is $\varepsilon_{nn} = \mathbf{n} \cdot [\varepsilon] \cdot \mathbf{n}$
Transformation of stress $[\sigma]_{x'y'z'} = [\mathbf{N}]^T [\sigma]_{xyz} [\mathbf{N}]$	Transformation of strain $[\varepsilon]_{x'y'z'} = [\mathbf{N}]^T [\varepsilon]_{xyz} [\mathbf{N}]$
Three mutually perpendicular principal directions and principal stresses can be computed as eigenvalues and eigenvectors of the stress matrix as $[\sigma] \cdot \mathbf{n} = \lambda \mathbf{n}$	Three mutually perpendicular principal directions and principal strains can be computed as eigenvalues and eigenvectors of the strain matrix as $[\varepsilon] \cdot \mathbf{n} = \lambda \mathbf{n}$

Compatibility Conditions

- 3 displacements (u, v, w) versus six strain components
  - Physically suitable 3 displacement fields can be used to determine six strain components
  - However, six arbitrary strain fields may not physically possible to produce continuous 3 displacement fields
  - Six strain components must satisfy compatibility conditions to yield non-discontinuous and non-overlapping displacement fields
- 2D compatibility condition

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2}$$

  - Take home message: all strain components are not independent
- Ex) $u(x,y) = x^4 + y^4, \quad v(x,y) = x^2 y^2$

$$\varepsilon_{xx} = 4x^3, \quad \varepsilon_{yy} = 2x^2 y, \quad \gamma_{xy} = 4y^3 + 2xy^2$$

Exercise

- The displacement field in a solid is given by $u=kx^2$, $v=2kxy^2$, and $w=k(x + y)z$, where k is a constant. (a) Write the strain matrix and (b) what is normal strain in the direction of $\mathbf{n} = \{1, 1, 1\}$?

5.4 STRESS-STRAIN RELATIONS

STRESS-STRAIN RELATIONSHIP

- Applied Load ⇨ shape change (strain) ⇨ stress
- There must be a relation between stress and strain
- Linear Elasticity: Simplest and most commonly used
- Uni-axial Stress:
  - Axial force F will generate stress $\sigma_{zz} = F / A$
  - In the elastic range, the relation between stress and strain is

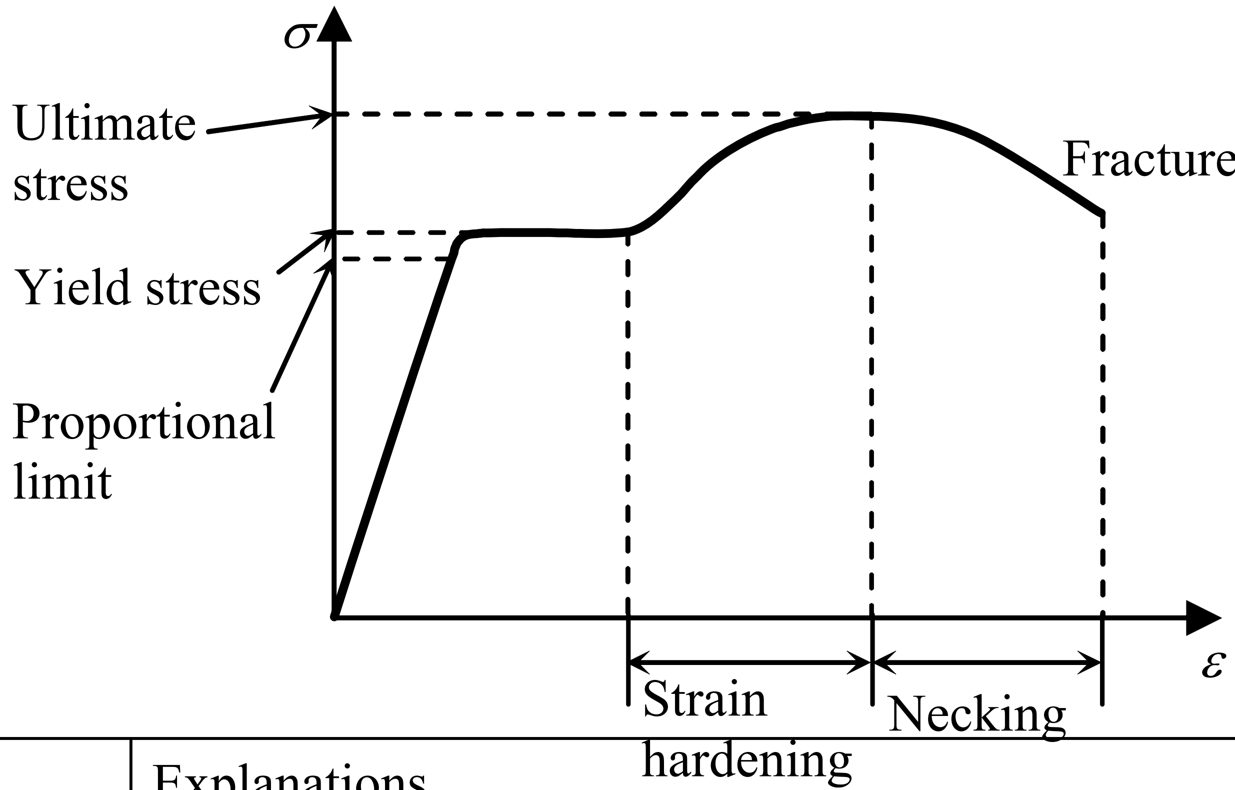
$$\sigma_{zz} = E\varepsilon_{zz}$$

  - Reduction of cross-section

$$\varepsilon_{xx} = \varepsilon_{yy} = -\nu\varepsilon_{zz}$$

  - E: Young's modulus, ν: Poisson's ratio

UNI-AXIAL TENSION TEST



Terms	Explanations
Proportional limit	The greatest stress for which stress is still proportional to strain
Elastic limit	The greatest stress without resulting in any permanent strain
Yield stress	The stress required to produce 0.2% plastic strain
Strain hardening	A region where more stress is required to deform the material
Ultimate stress	The maximum stress the material can resist
Necking	Cross section of the specimen reduces during deformation
Fracture	Material failure

LINEAR ELASTICITY (HOOKE'S LAW)

- When the material is in the Proportional Limit (or Elastic Limit)
- In General 3-D Relationship

$$\{\sigma\} = [\mathbf{C}] \cdot \{\varepsilon\}$$

$$\{\sigma\} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}, \quad [\mathbf{C}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix}, \quad \{\varepsilon\} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix}$$

Stress-Strain Matrix

  - For homogeneous, isotropic material 36 constants can be reduced to 2 independent constants.

LINEAR ELASTICITY (HOOKE'S LAW) *cont.*

- Isotropic Material:
  - Stress in terms of strain: $\{\sigma\} = [\mathbf{C}] \cdot \{\varepsilon\}$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix}$$

$$\tau_{xy} = G\gamma_{xy}, \quad \tau_{yz} = G\gamma_{yz}, \quad \tau_{zx} = G\gamma_{zx}$$

  - Strain in terms of stress

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}, \quad \gamma_{yz} = \frac{\tau_{yz}}{G}, \quad \gamma_{zx} = \frac{\tau_{zx}}{G}$$

Shear Modulus

$$G = \frac{E}{2(1+\nu)}$$

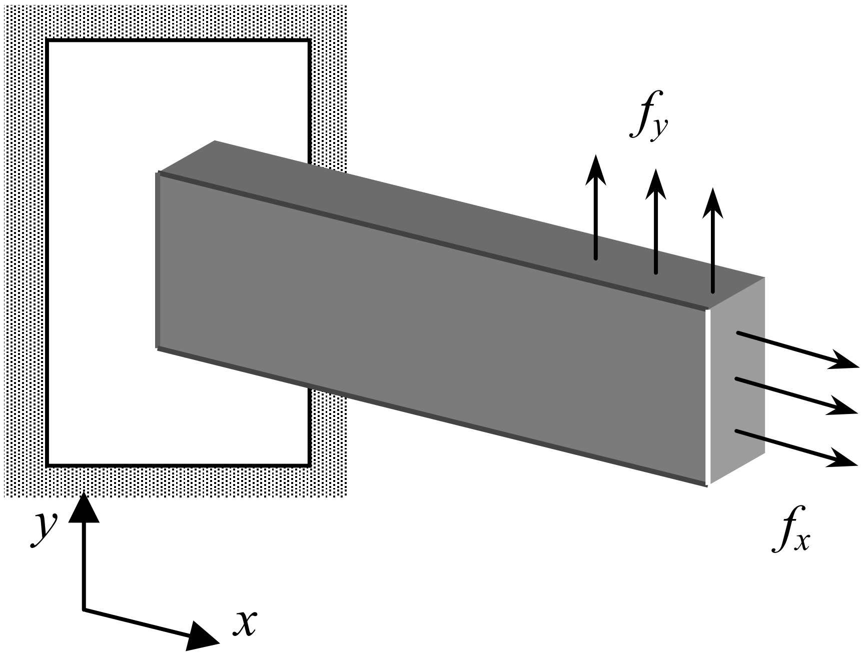
Simplified Relationships for Plane Solids

- Plane Solids
  - All engineering problems are 3-D. It is the engineer who approximates the problem using 1-D (beam or truss) or 2-D (plane stress or strain).
  - Stress and strain are either zero or constant in the direction of the thickness.
  - System of coupled second-order partial differential equation
  - Plane stress and plane strain: different constraints imposed in the thickness direction
  - **Plane stress**: zero stresses in the thickness direction (thin plate with in-plane forces)
  - **Plane strain**: zero strains in the thickness direction (thick solid with constant thickness, gun barrel)
  - Main variables: *u* (*x*-displacement) and *v* (*y*-displacement)

PLANE STRESS PROBLEM

- Plane Stress Problem:
  - Thickness is much smaller than the length and width dimensions
  - Thin plate or disk with applied in-plane forces
  - z-directional stresses are zero at the top and bottom surfaces
  - Thus, it is safe to assume that they are also zero along the thickness

$$\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$$



  - Non-zero stress components:
    σ_{xx}, σ_{yy}, τ_{xy}
  - Non-zero strain components:
    ε_{xx}, ε_{yy}, ε_{xy}, ε_{zz}

PLANE STRESS PROBLEM *cont.*

- Stress-strain relation

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \Leftrightarrow \quad \{\sigma\} = [\mathbf{C}_\sigma]\{\varepsilon\}$$

  – Even if ε_{zz} is not zero, it is not included in the stress-strain relation because it can be calculated from the following relation:

$$\varepsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy})$$

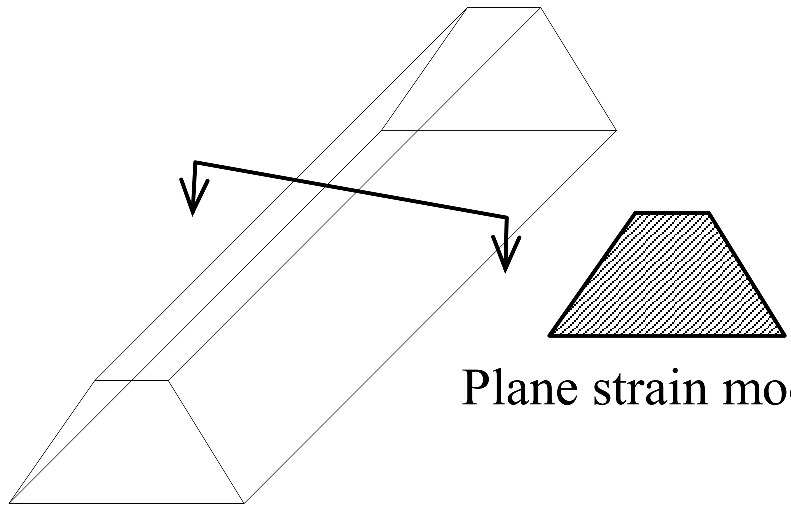
- How to derive plane stress relation?
  – Solve for ε_{zz} in terms of ε_{xx} and ε_{yy} from the relation of σ_{zz} = 0 and Eq. (1.57)
  – Write σ_{xx} and σ_{yy} in terms of ε_{xx} and ε_{yy}

PLANE STRAIN PROBLEM

- Plane Strain Problem
  - Thickness dimension is much larger than other two dimensions.
  - Deformation in the thickness direction is constrained.
  - Strain in z-dir is zero

$$\varepsilon_{zz} = 0,\ \varepsilon_{xz} = 0,\ \varepsilon_{yz} = 0$$

  - Non-zero stress components: σ_{xx}, σ_{yy}, τ_{xy}, σ_{zz}.
  - Non-zero strain components: ε_{xx}, ε_{yy}, ε_{xy}.



Plane strain model

PLANE STRAIN PROBLEM *cont.*

- Plan Strain Problem
  - Stress-strain relation

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \Leftrightarrow \{\sigma\} = [\mathbf{C}_{\varepsilon}]\{\varepsilon\}$$

  - Even if σ_{zz} is not zero, it is not included in the stress-strain relation because it can be calculated from the following relation:

$$\sigma_{zz} = \frac{E\nu}{(1+\nu)(1-2\nu)}(\varepsilon_{xx} + \varepsilon_{yy})$$

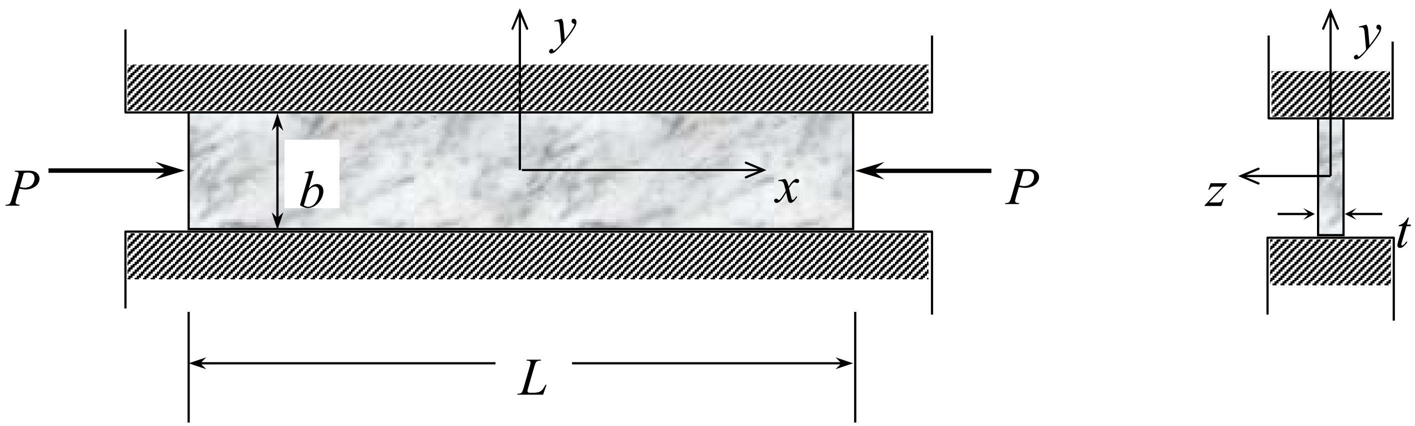
EQUIVALENCE

- A single program can be used to solve both the plane stress and plane strain problems by converting material properties.

From $\rightarrow$ To	E	ν
Plane strain $\rightarrow$ Plane stress	$E\left[1-\left(\frac{\nu}{1+\nu}\right)^2\right]$	$\frac{\nu}{1+\nu}$
Plane stress $\rightarrow$ Plane strain	$\frac{E}{1-\left(\frac{\nu}{1-\nu}\right)^2}$	$\frac{\nu}{1-\nu}$

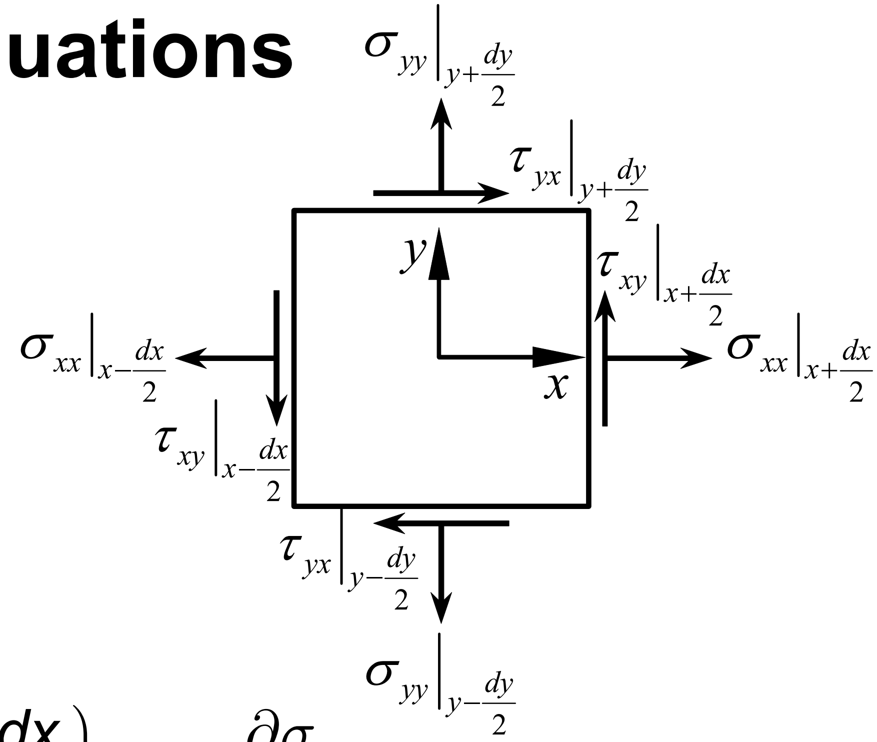
Exercise

- A thin plate of width b, thickness t, and length L is placed between two frictionless rigid walls a distance b apart and is acted on by an axial force P. The material properties are Young’s modulus E and Poisson’s ratio v. (a) Find the stress and strain components in the xyz coordinate system, and (b) find the displacement field.



5.5 BOUNDARY VALUE PROBLEMS

Equilibrium Equations



- Stress field in differential element
- Equilibrium in x-direction:

$$\left(\sigma_{xx}|_{x+\frac{dx}{2}}\right)dy-\left(\sigma_{xx}|_{x-\frac{dx}{2}}\right)dy$$

$$+\left(\tau_{yx}|_{y+\frac{dy}{2}}\right)dx-\left(\tau_{yx}|_{y-\frac{dy}{2}}\right)dx=0$$

$$(1)=\left(\sigma_{xx}|_{x}+\frac{\partial\sigma_{xx}}{\partial x}\frac{dx}{2}\right)dy-\left(\sigma_{xx}|_{x}-\frac{\partial\sigma_{xx}}{\partial x}\frac{dx}{2}\right)dy=\frac{\partial\sigma_{xx}}{\partial x}dxdy$$

$$(2)=\left(\tau_{yx}|_{y}+\frac{\partial\tau_{yx}}{\partial y}\frac{dy}{2}\right)dx-\left(\tau_{yx}|_{y}-\frac{\partial\tau_{yx}}{\partial y}\frac{dy}{2}\right)dx=\frac{\partial\tau_{yx}}{\partial y}dxdy$$

- After deleting dxdy, we get equilibrium equation:

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}=0$$

- In y-direction:

$$\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}=0$$

Equilibrium Equations

- Extension to 3D differential element

$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y} + \dfrac{\partial \tau_{zx}}{\partial z} = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + \dfrac{\partial \tau_{zy}}{\partial z} = 0 \\ \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma_{zz}}{\partial z} = 0 \end{cases}$$

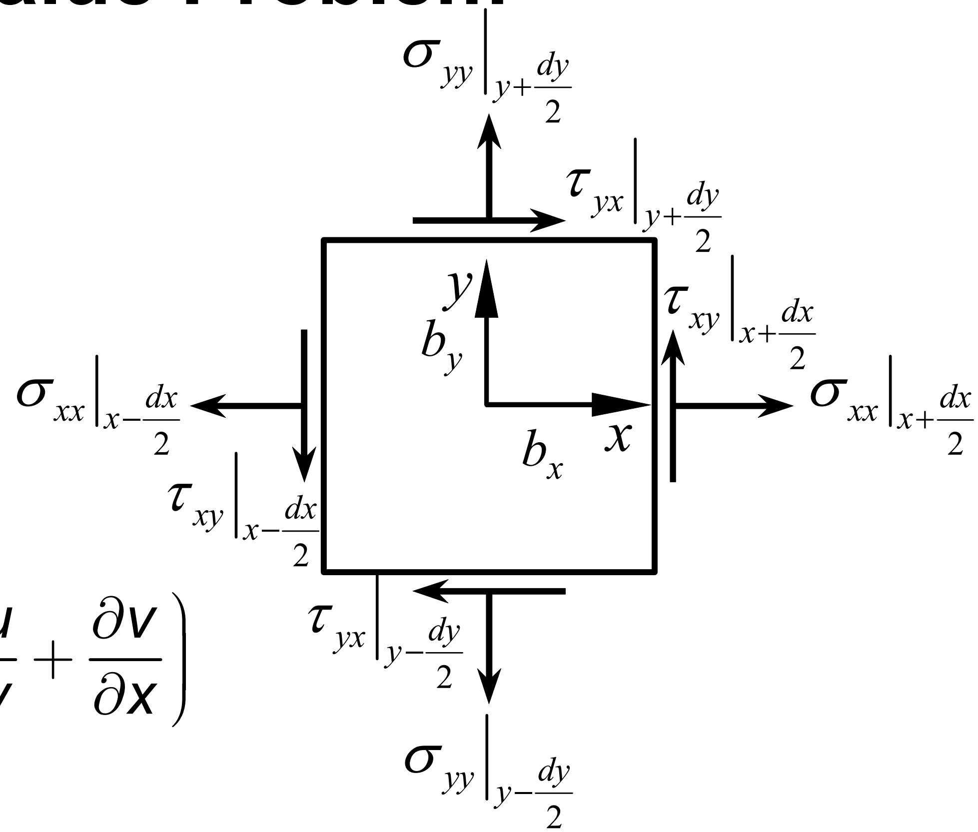
- Traction (stress) boundary conditions
  - The condition that the stress field must satisfy on the boundary

$$\begin{aligned} \sigma_{xx} n_x + \tau_{yx} n_y + \tau_{zx} n_z &= t_x \\ \tau_{xy} n_x + \sigma_{yy} n_y + \tau_{zy} n_z &= t_y \\ \tau_{xz} n_x + \tau_{yz} n_y + \sigma_{zz} n_z &= t_z \end{aligned} \quad \Longrightarrow \quad [\boldsymbol{\sigma}]\{\mathbf{n}\} = \{\mathbf{T}^{(\mathbf{n})}\} = \mathbf{t}$$

2D Boundary Value Problem

- Governing D.E.

$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + b_x = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + b_y = 0 \end{cases}$$



- Definition of strain

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \qquad \gamma_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

- Stress-Strain Relation

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \Leftrightarrow \{\sigma\} = [\mathbf{C}]\{\varepsilon\}$$

  – Since stress involves first-order derivative of displacements, the governing differential equation is the second-order

2D Boundary Value Problem

- Compatibility condition

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2}$$

- Boundary Conditions
  - All differential equations must be accompanied by boundary conditions

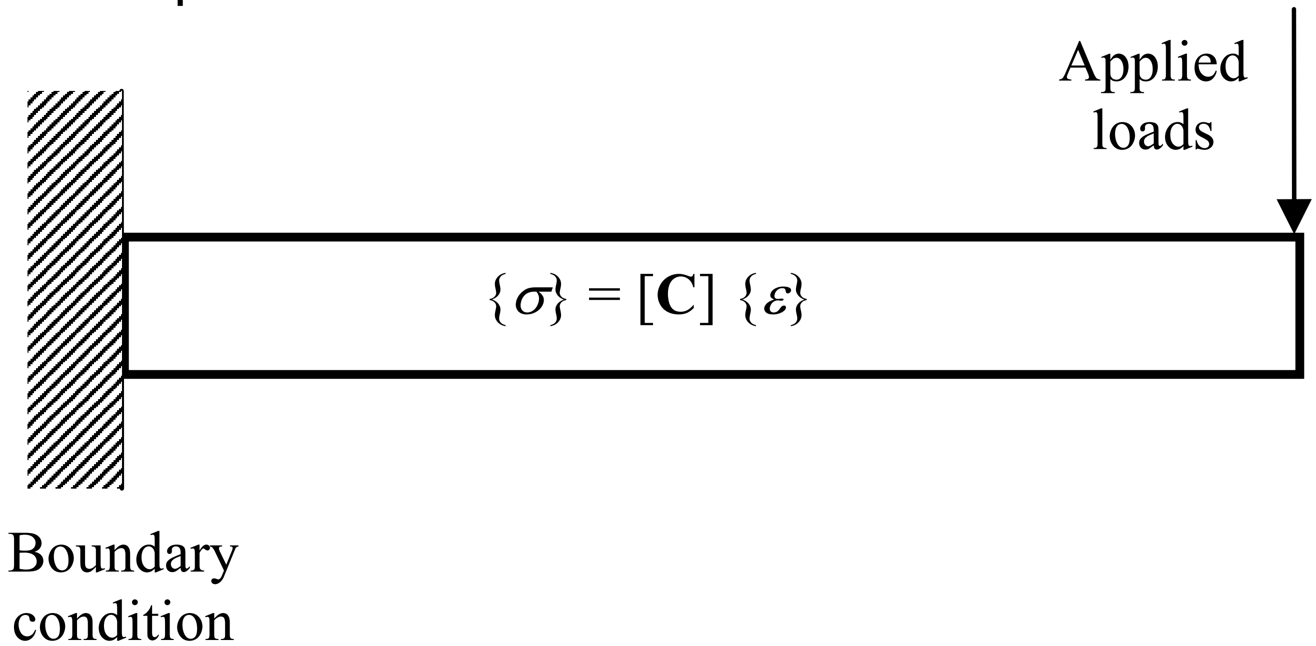
$$\mathbf{u} = \mathbf{g}, \qquad \text{on } S_g$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{T}, \qquad \text{on } S_T$$

  - S_g is the essential boundary and S_T is the natural boundary
  - **g**: prescribed (specified) displacement (usually zero for linear problem)
  - **T**: prescribed (specified) surface traction force
- **Objective**: to determine the displacement fields u(x, y) and v(x, y) that satisfy the D.E. and the B.C.

BOUNDARY-VALUE PROBLEM

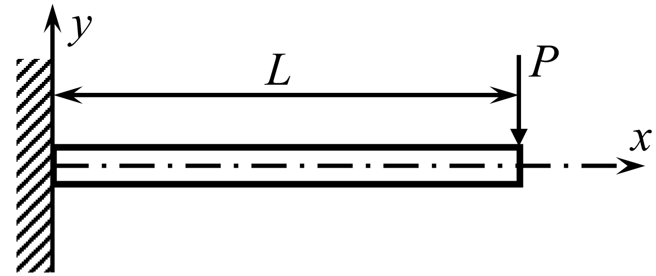
- When boundary conditions are given, how can we calculate the displacement, stress, and strain of the structure?
  - Solve for displacement



  - Equilibrium equation
  - Constitutive equation (Stress-strain relation)
  - Strain definition
  - Load and boundary conditions
  - Compatibility conditions

Example: Cantilevered Beam Bending

- Displacement field



$$u(x,y) = \frac{P}{EI}(Lx - \frac{x^2}{2})y - \frac{\nu P}{6EI}y^3$$

$$v(x,y) = \frac{-\nu P}{2EI}(L-x)y^2 - \frac{P}{EI}(\frac{Lx^2}{2} - \frac{x^3}{6})$$

- Strain field

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{P}{EI}(L-x)y \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} = \frac{-\nu P}{EI}(L-x)y$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \left[\frac{\nu P y^2}{2EI} - \frac{P}{EI}(Lx - \frac{x^2}{2})\right] + \left[\frac{P}{EI}(Lx - \frac{x^2}{2})y - \frac{\nu P y^2}{2EI}\right] = 0$$

- Stress field

$$\sigma_{xx} = \frac{E}{1-\nu^2}\left[\frac{P}{EI}(L-x)y - \frac{\nu^2 P}{EI}(L-x)y\right] = \frac{P}{I}(L-x)y$$

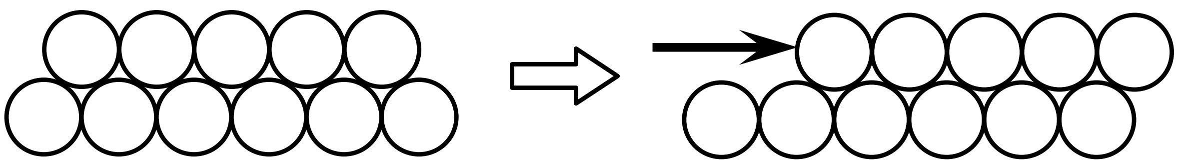
$$\sigma_{yy} = \frac{E}{1-\nu^2}\left[-\frac{\nu P}{EI}(L-x)y + \frac{\nu P}{EI}(L-x)y\right] = 0$$

$$\tau_{xy} = 0$$

5.6 FAILURE THEORIES

FAILURE THEORIES

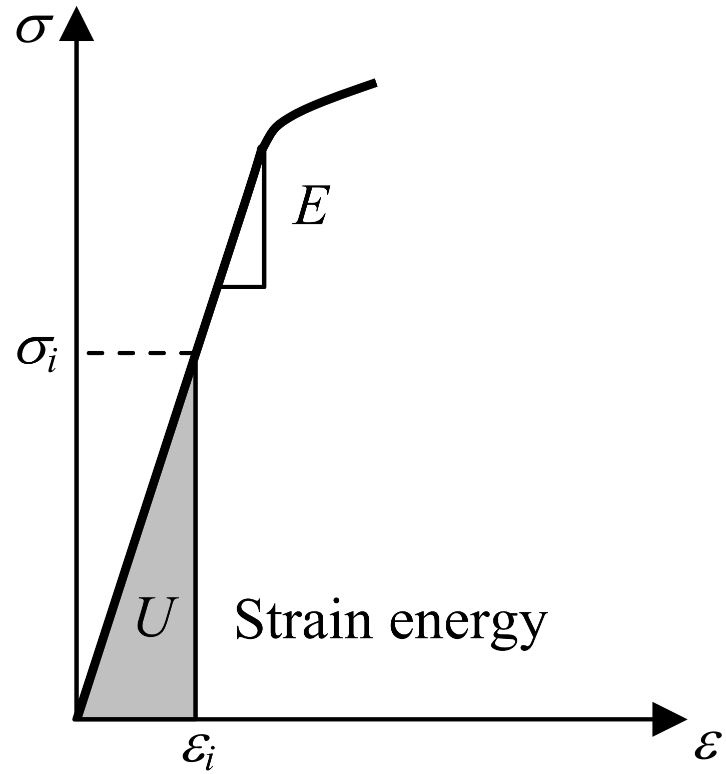
- Materials fail because the stress exceed the strength
  - → Need to specify the exact stress type to determine failure
  - → Design Criteria
- Material failure
  - → Ductile materials (metals): yield stress
  - → Brittle materials (ceramics): ultimate stress, fracture
- Materials don't fail by changing volume (inter-atomic distance)
- Shear stress (distortion of shape) is related to material failure.



- Two Categories: stress-based and energy-based

STRAIN ENERGY

- Force ➡ Deformation ➡ Stress ➡ Stored Energy
- Strain Energy Density: $U_0 = \frac{1}{2}\sigma\varepsilon$
- 3-D situation: $U_0 = \frac{1}{2}(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3)$
- Use principal stress-strain relation



$$\begin{cases} \varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2 - \nu\sigma_3) \\ \varepsilon_2 = \frac{1}{E}(\sigma_2 - \nu\sigma_1 - \nu\sigma_3) \\ \varepsilon_3 = \frac{1}{E}(\sigma_3 - \nu\sigma_1 - \nu\sigma_2) \end{cases}$$

- Strain Energy Density: $U_0 = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)\right]$

in terms of principal stresses

DECOMPOSITION OF ENERGY

- Hydrostatic Stress (Volumetric stress)
  - Hydrostatic pressure does not contribute to failure
  - Thus, subtract the volumetric strain energy from total strain energy.
  - Hydrostatic pressure: same for all directions

$$\sigma_h = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3}$$

  - Strain energy density caused by σ_h:

$$U_h = \frac{1}{2E}\left[\sigma_h^2 + \sigma_h^2 + \sigma_h^2 - 2\nu(\sigma_h\sigma_h + \sigma_h\sigma_h + \sigma_h\sigma_h)\right] = \frac{3}{2}\frac{(1-2\nu)}{E}\sigma_h^2$$

$$U_h = \frac{3}{2}\frac{(1-2\nu)}{E}\left(\frac{\sigma_1 + \sigma_2 + \sigma_3}{3}\right)^2$$

$$= \frac{1-2\nu}{6E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)\right]$$

DECOMPOSITION OF ENERGY *cont.*

- Distortion Energy Density

$$U_d = U_0 - U_h$$

$$= \frac{1+\nu}{3E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_1\sigma_3\right]$$

$$= \frac{1+\nu}{3E}\sigma_{VM}^2$$

Von Mises Stress

$$\sigma_{VM} = \sqrt{\frac{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2}{2}}$$

This energy contributes to material failure

DISTORTION ENERGY THEORY

- Von Mises (1913)
  - Material fails when the distortion energy reaches a certain level.
  - Material yields in the tensile test when $\sigma_{xx} = \sigma_Y$, and all others are zero
  - Distortion energy when the material yields in tensile test

$$U_d = \frac{1+\nu}{3E}\sigma_Y^2$$

  - In general stress status, the material yields when the distortion energy is greater than that of the tensile test at yielding:

$$\frac{1+\nu}{3E}\sigma_{VM}^2 \geq \frac{1+\nu}{3E}\sigma_Y^2$$

  - Without calculating distortion energy, just compare the von Mises stress with yield stress of the tensile test:

$$\therefore \quad \sigma_{VM} \geq \sigma_Y$$

DISTORTION ENERGY THEORY *cont.*

– 3D stress status

$$\sigma_{VM} = \sqrt{\frac{(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}{2}}$$

$$\sigma_{VM} = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_1\sigma_3}$$

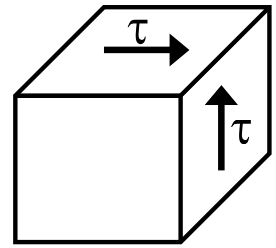
– 2D (when $\sigma_3 = 0$)

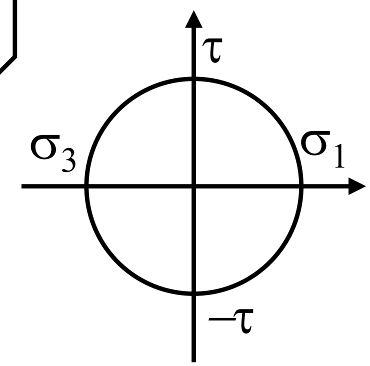
$$\sigma_{VM} = \sqrt{\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2}$$

$$\sigma_{VM} = \sqrt{\sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} + 3\tau_{xy}^2}$$

DISTORTION ENERGY THEORY *cont.*

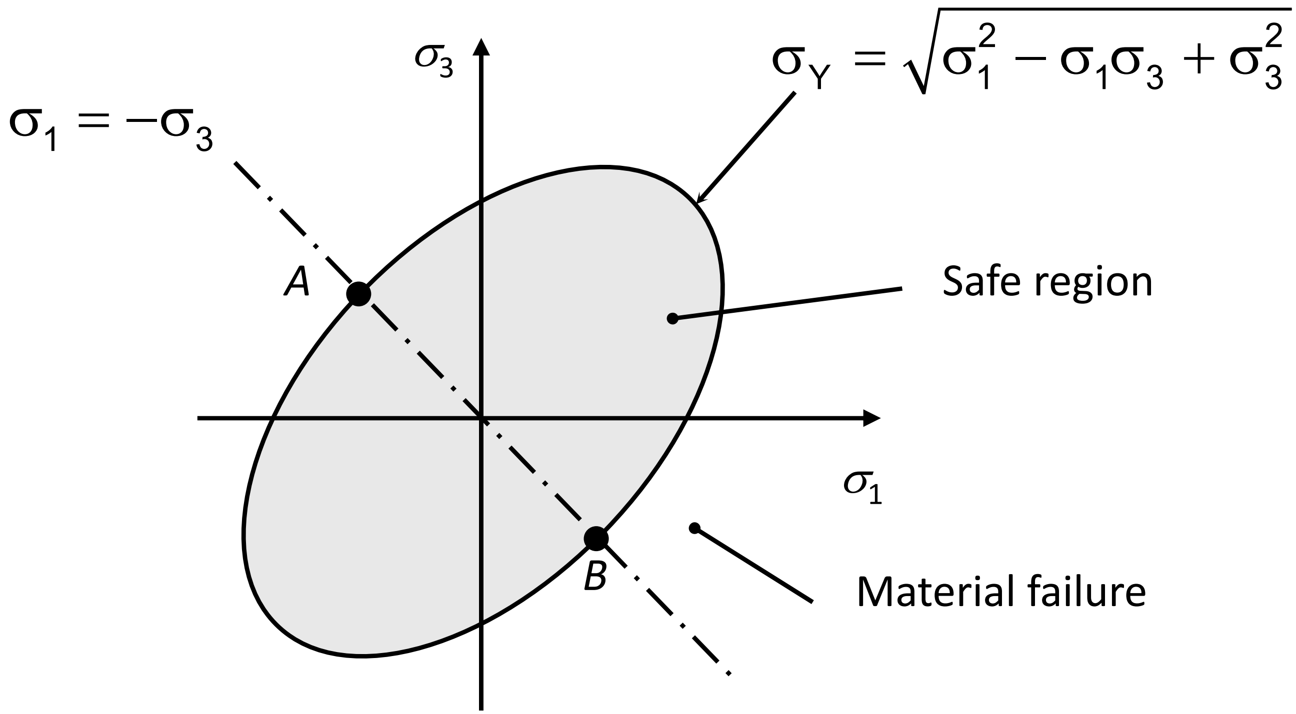
- Example: Pure Shear Problem
  - $\sigma_1 = \tau = -\sigma_3$ and $\sigma_2 = 0$
  - straight line through the origin at –45º





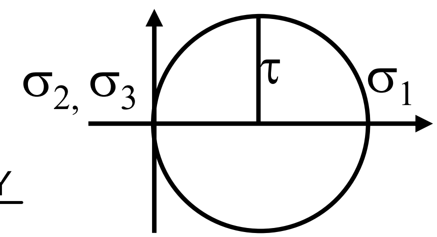
$$\sigma_Y^2 = \sigma_1^2 + \sigma_1\sigma_1 + \sigma_1^2 = 3\sigma_1^2 = 3\tau^2$$

$$\tau = \sigma_1 = \frac{\sigma_Y}{\sqrt{3}} = 0.577\sigma_Y$$

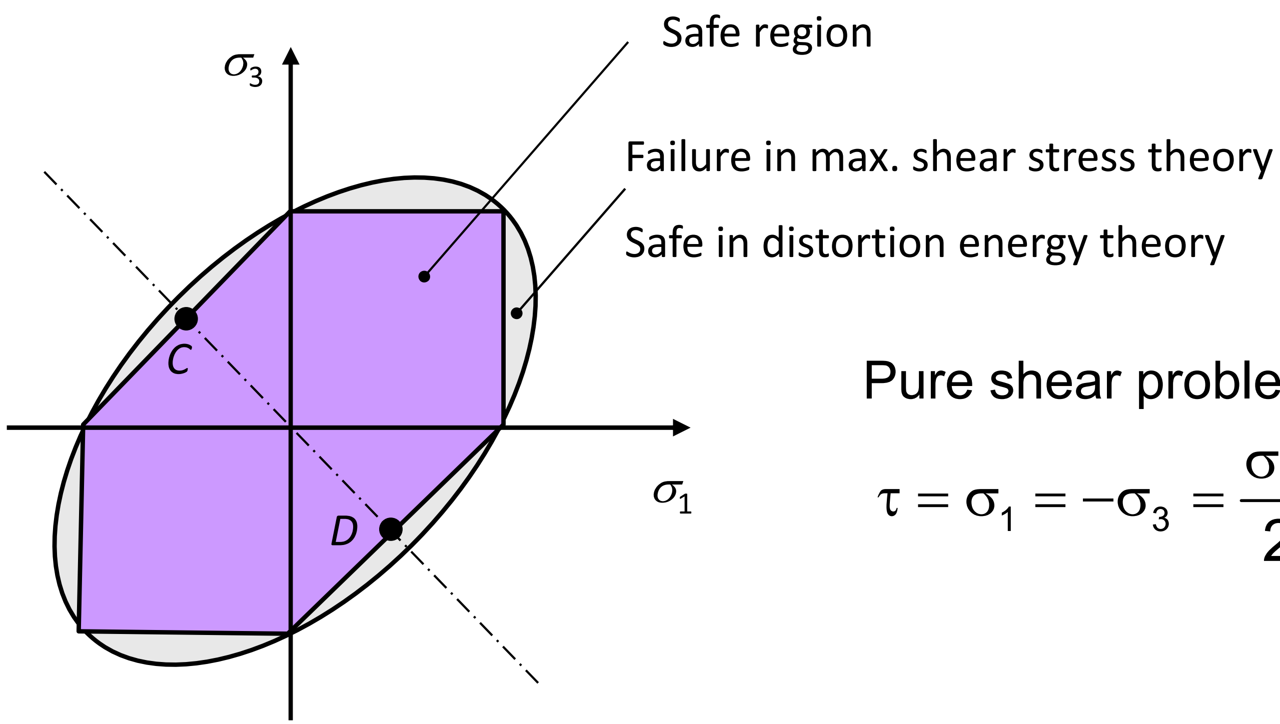


MAX SHEAR STRESS THEORY

- Tresca (1864)
  - Material fails when the max. shear stress exceeds the shear stress in a tensile specimen at yield.
  - In tensile test, $\sigma_1 = \sigma_Y$, $\sigma_2 = \sigma_3 = 0$:
  - $\tau_Y = \dfrac{\sigma_Y}{2}$ $\qquad \tau_{max} = \dfrac{\sigma_1 - \sigma_3}{2} \geq \tau_Y = \dfrac{\sigma_Y}{2}$



  - Tresca theory is more conservative than the distortion energy theory



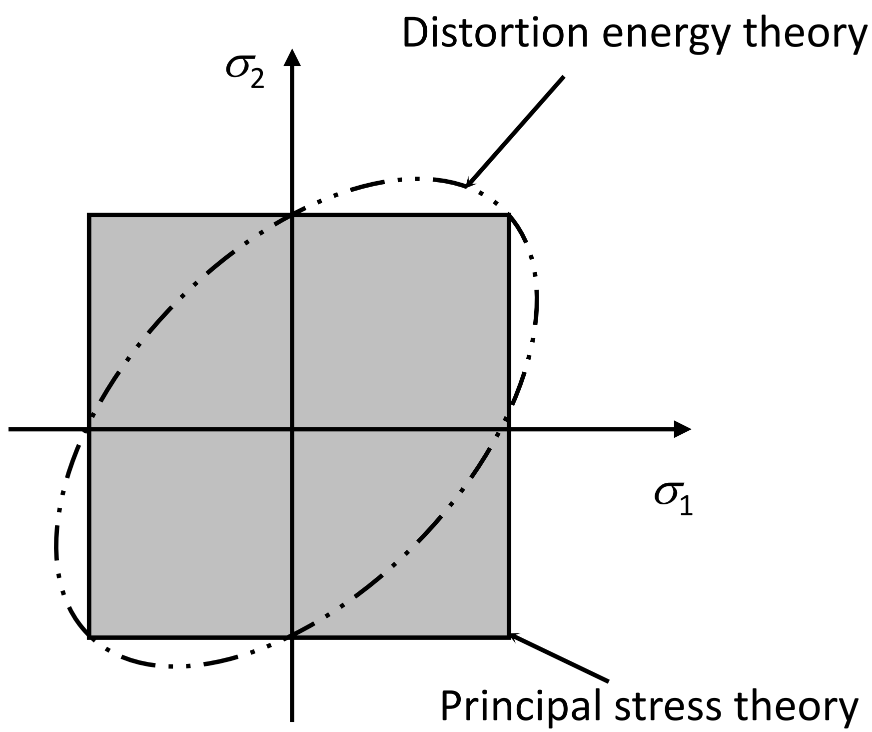
Pure shear problem

$$\tau = \sigma_1 = -\sigma_3 = \frac{\sigma_Y}{2}$$

MAX PRINCIPAL STRESS THEORY

- Rankine
  - Material fails when the principal stress reaches some limit on normal stress such as tensile yield stress or ultimate tensile stress.
  - This theory frequently used for brittle materials.

$$\sigma_1 \geq \sigma_U$$



SAFETY FACTOR

- For design purposes it is convenient to include a chosen safety factor *N* so that the stress will be safely inside the failure-stress envelope.
- In many engineering applications, N = 1.1 – 1.5.
- Safety factor in the distortion energy theory:

$$N_{VM} = \frac{\sigma_Y}{\sigma_{VM}}$$

- safety factor in the maximum shear stress theory:

$$N_{\tau} = \frac{\tau_Y}{\tau_{max}} = \frac{\sigma_Y / 2}{\tau_{max}}$$

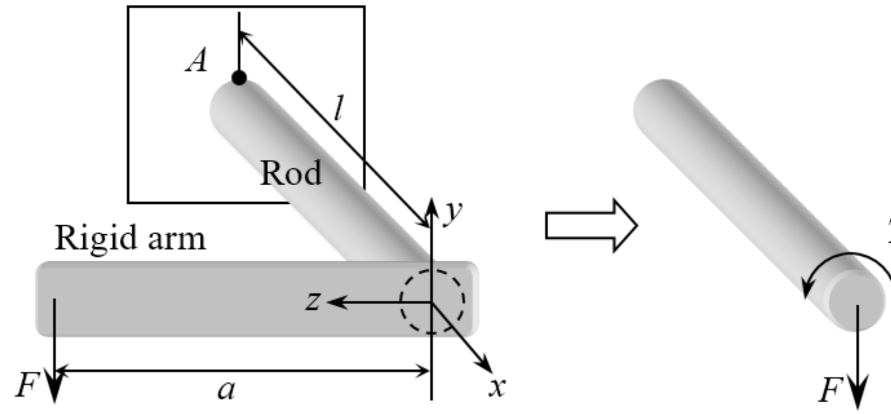
- Note:

$$N_{VM} \geq N_{\tau}$$

$$\frac{\sigma_Y}{\sigma_{VM}} \geq \frac{\tau_Y}{\tau_{max}}$$

Example: Safety Factor of a Bracket

- Stress at point A



$$\sigma_{xx} = \frac{M \cdot r}{I} = \frac{F \cdot l \cdot r}{I} = 1$$

$$\tau_{xz} = \frac{T \cdot r}{J} = \frac{5\sqrt{2} \cdot 0.1}{0.5} = \sqrt{2}$$

$$\sigma_{yy} = \sigma_{zz} = \tau_{xy} = \tau_{yz} = 0.0$$

$$[\sigma] = \begin{bmatrix} 1 & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \end{bmatrix}$$

- Principal stresses

$$([\sigma] - \lambda[\mathbf{I}]) \cdot \mathbf{n} = \mathbf{0}$$

$$\begin{vmatrix} 1-\lambda & 0 & \sqrt{2} \\ 0 & -\lambda & 0 \\ \sqrt{2} & 0 & -\lambda \end{vmatrix} = 0$$

$$-\lambda[-\lambda(1-\lambda) - 2] = -\lambda(\lambda^2 - \lambda - 2) = 0$$

$$\Rightarrow -\lambda(\lambda - 2)(\lambda + 1) = 0,$$

$$\therefore \lambda = 2, 0, -1$$

$$\sigma_1 = 2, \qquad \sigma_2 = 0, \qquad \sigma_3 = -1$$

Example: Safety Factor of a Bracket

- Max shear stress $\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = 1.5$
  - Safety factor $N = \frac{\tau_Y}{\tau_{\max}} = \frac{1.4}{1.5} = 0.9333$
- Von Mises stress $\sigma_{VM} = \sqrt{4 + 2 + 1} = \sqrt{7}$
  - Safety factor $N = \frac{\sigma_Y}{\sigma_{VM}} = \frac{2.8}{\sqrt{7}} = 1.0583$
- The bracket is safety under von Mises criterion, while unsafe under max shear stress criterion
  - Max shear stress criterion is more conservative

Exercise

- The circular stepped shaft is fixed at both ends and is made of an alloy that behaves in a linear elastic manner with Young's modulus E and shear modulus G. A torque T_0 is applied at the junction. (a) Determine the maximum shear strain at location A in terms of the given parameters T_0, d, E, G, K_t. (b) When the yield stress of the material is S_Y and the safety factor is N, using the distortion energy theory determine the allowable torque T_0 in terms of d, E, G, K_t, S_Y, N.