CHAP 7 ISOPARAMETRIC ELEMENTS

Introduction

- CST and rectangular elements
  - Shape function in the global coordinates (x-y)
  - Different elements have different shape functions (computationally expensive to build individual shape functions)
- Deriving the shape functions for quadrilaterals in global coordinates is difficult
- It is possible to define shape functions in parametric space
  - Mapping between the physical (x-y) and parametric (s-t) spaces
  - All elements in different geometry share the same shape functions
- Then, we need to interpolate both geometry and displacement
  - Isoparametric element: both geometry and displacement are interpolated using the same shape functions

$$u(s) = \sum_{k=1}^{n} N_k(s) u_k$$

Displacement interpolation

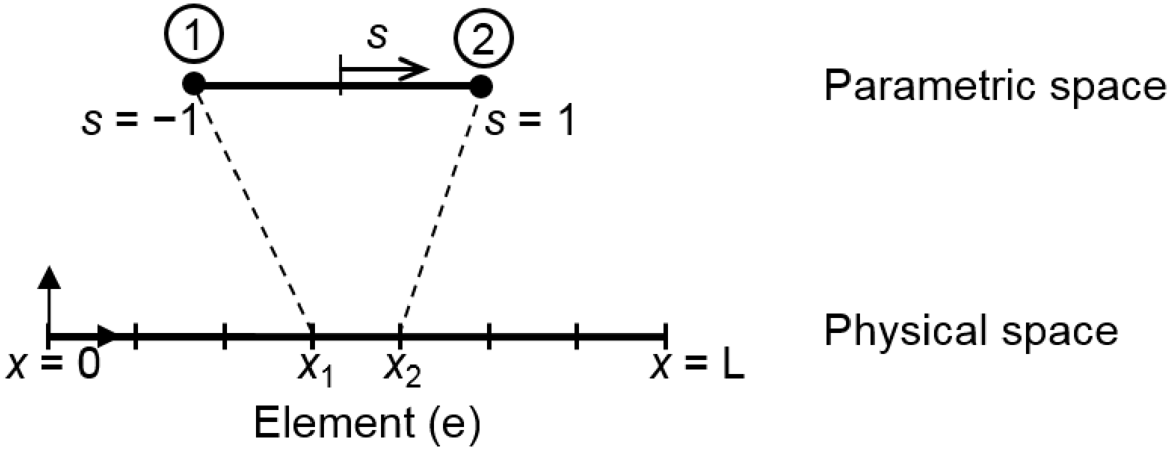
$$x(s) = \sum_{k=1}^{n} N_k(s) x_k$$

Geometry mapping

7.2.1 1D LINEAR ISOPARAMETRIC ELEMENT

2-Node Linear Isoparametric Element

- For 1D bar element or 1D heat transfer element



- Geometry mapping: $x(s=-1)=x_i$

$$x(s=+1)=x_j$$

- Shape functions: $N_1(s)=\dfrac{1-s}{2}$

Same for all elements!!

$$N_2(s)=\frac{1+s}{2}$$

2-Node Linear Isoparametric Element cont.

- Ex) Interpolation of temperature within an element

$$T(s)=\frac{(1-s)}{2}T_1+\frac{(1+s)}{2}T_2=N_1(s)T_1+N_2(s)T_2$$

$$=\{N_1(s)\quad N_2(s)\}\begin{Bmatrix}T_1\\T_2\end{Bmatrix}=\{\mathbf{N}(s)\}^{\mathrm{T}}\{\mathbf{T}^{(e)}\}$$

Function of s

- How to find x corresponding to s?

$$x(s)=x_iN_1(s)+x_jN_2(s)$$

- Derivative w.r.t. s

$$\frac{\mathrm{d}N_1(s)}{\mathrm{d}s}=-\frac{1}{2}$$

$$\frac{\mathrm{d}N_2(s)}{\mathrm{d}s}=\frac{1}{2}$$

2-Node Linear Isoparametric Element cont.

- Jacobian

$$\frac{dx}{ds} = \frac{dN_1}{ds}x_1 + \frac{dN_2}{ds}x_2 = -\frac{1}{2}x_1 + \frac{1}{2}x_2 = \boxed{\frac{L_e}{2}}$$

$$\frac{ds}{dx} = \frac{1}{dx\,/\,ds} = \frac{2}{x_2 - x_1} = \frac{2}{L_e}$$

- How to calculate derivative w.r.t. x?

$$\frac{dT}{dx} = \frac{dT}{ds}\frac{ds}{dx} = \frac{1}{2}(-T_1 + T_2)\frac{2}{L_e} = \frac{1}{L_e}\{-1 \quad 1\}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \boxed{\{\mathbf{B}\}^{\mathrm{T}}\{\mathbf{T}\}}$$

- How to integrate over the element?

$$\int_{x_1}^{x_2} f(s)dx = \int_{-1}^{1} f(s)\frac{dx}{ds}ds = \frac{L_2}{2}\int_{-1}^{1} f(s)ds$$

Ex) Element Conductivity Matrix

$$\int_{x_1}^{x_2} kA \frac{\mathrm{d}\delta T}{\mathrm{d}x}\frac{\mathrm{d}T}{\mathrm{d}x}\mathrm{d}x = \int_{-1}^{1} kA \frac{\mathrm{d}\delta T}{\mathrm{d}x}\frac{\mathrm{d}T}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}s}\mathrm{d}s$$

$$= \{\delta \mathbf{T}\}^{\mathrm{T}} \left[\int_{-1}^{1} kA\{\mathbf{B}\}\{\mathbf{B}\}^{\mathrm{T}} \frac{L_e}{2}\mathrm{d}s \right] \{\mathbf{T}\}$$

$$= \{\delta \mathbf{T}\}^{\mathrm{T}} \left[\int_{-1}^{1} kA \frac{1}{L_e} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \frac{1}{L_e} \{-1 \quad 1\} \frac{L_e}{2}\mathrm{d}s \right] \{\mathbf{T}\}$$

$$= \{\delta \mathbf{T}\}^{\mathrm{T}} \frac{kA}{L_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \{\mathbf{T}\}$$

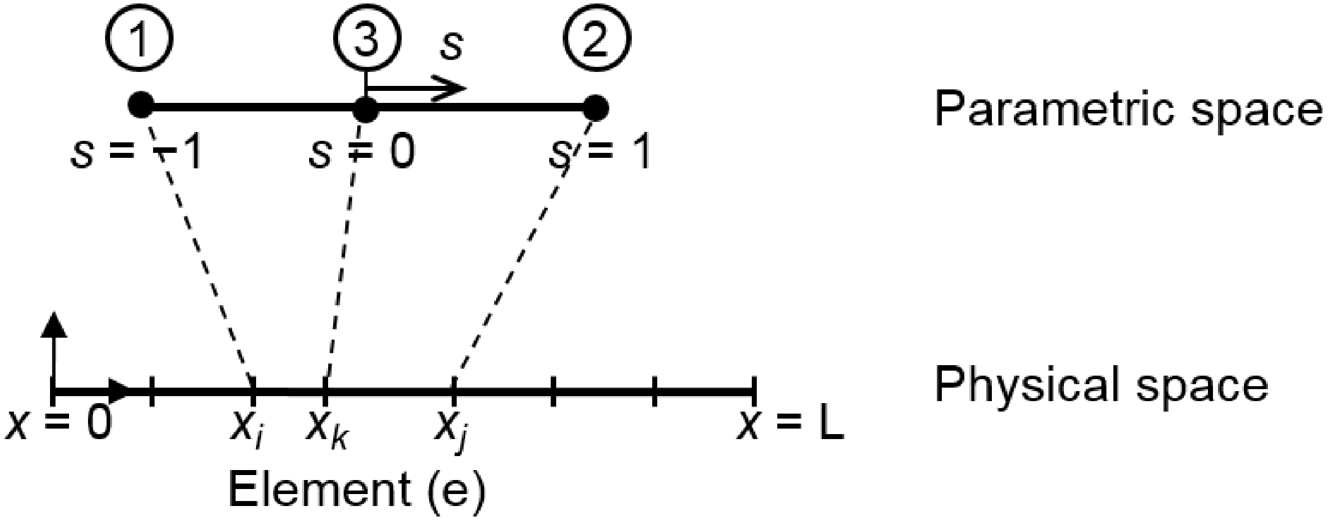
$$= \{\delta \mathbf{T}\}^{\mathrm{T}} [\underset{\uparrow}{\mathbf{k}_T}] \{\mathbf{T}\}$$

Same heat conductivity matrix!!

7.2.2 1D QUADRATIC ISOPARAMETRIC ELEMENT

1D Quadratic Isoparametric Element

- Quadratic element: 1D 3-node (3 DOFs)



$$x(s=-1)=x_1$$
$$x(s=+1)=x_2$$
$$x(s=0)=x_3$$

- Assume quadratic polynomials

$$T(r)=a+bs+cs^2 \quad \Longrightarrow \quad \begin{cases} N_1(s)=-\dfrac{1}{2}s(1-s) \\ N_2(s)=\dfrac{1}{2}s(1+s) \\ N_3(s)=1-s^2 \end{cases}$$

- Mapping

$$x(s)=N_1(s)x_1+N_2(s)x_2+N_3(s)x_3$$

1D Quadratic Isoparametric Element cont.

- Jacobian

$$\frac{dx}{ds} = (s - \tfrac{1}{2})x_1 + (s + \tfrac{1}{2})x_2 - 2sx_3$$

Not a constant!

  - Regular element: $x_3 = \dfrac{x_1 + x_2}{2}$

$$\frac{dx}{ds} = \frac{x_2 - x_1}{2} = \frac{L_e}{2}$$

Constant! Same as 2-node element!

- Derivative w.r.t. x

$$\frac{dT}{ds} = \frac{dT}{ds}\frac{ds}{dx}$$

$$= \left[(s - \tfrac{1}{2})T_1 + (s + \tfrac{1}{2})T_2 - 2sT_3\right]\underset{\uparrow}{\frac{ds}{dx}}$$

$$= \underset{\uparrow}{\{\mathbf{B}\}}^{\mathrm{T}}\{\mathbf{T}\}$$

Not a constant!

Rational function

1D Quadratic Isoparametric Element cont.

- Derivative w.r.t. x (Regular element)

$$\frac{\mathrm{d}T}{\mathrm{d}s} = \{\mathbf{B}\}^{\mathrm{T}}\{\mathbf{T}\} \qquad \{\mathbf{B}\} = \frac{1}{L_e}\begin{Bmatrix} 2s-1 \\ 2s+1 \\ -4s \end{Bmatrix}$$

- Conductivity matrix (Regular element)

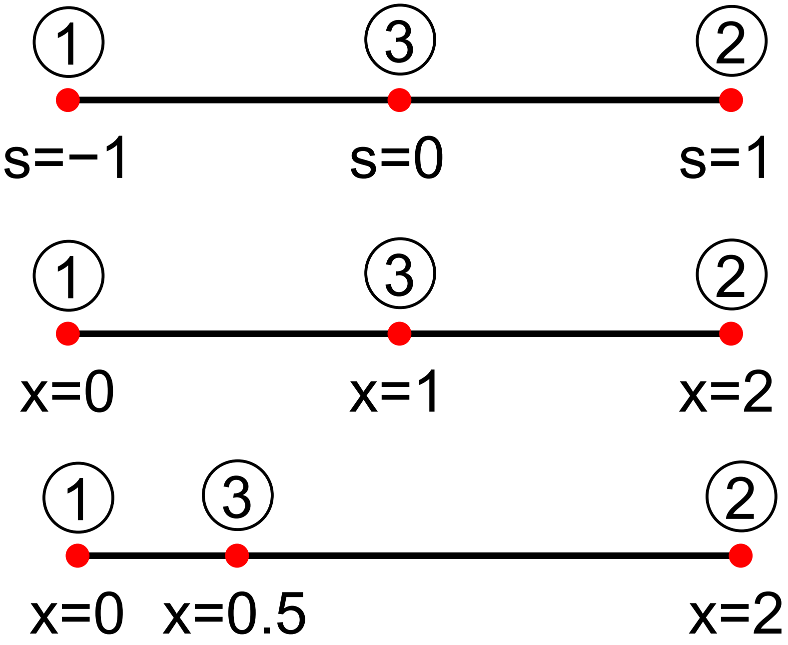
$$\int_{x_1}^{x_2} kA\frac{\mathrm{d}\delta T}{\mathrm{d}x}\frac{\mathrm{d}T}{\mathrm{d}x}\mathrm{d}x = \{\delta\mathbf{T}\}^{\mathrm{T}}\left[\int_{-1}^{1} kA\{\mathbf{B}\}\{\mathbf{B}\}^{\mathrm{T}}\frac{L_e}{2}\mathrm{d}s\right]\{\mathbf{T}\}$$

$$= \{\delta\mathbf{T}\}^{\mathrm{T}}\frac{kA}{3L_e}\begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix}\{\mathbf{T}\}$$

$$= \{\delta\mathbf{T}\}^{\mathrm{T}}[\mathbf{k}_T]\{\mathbf{T}\}$$

Ex: Isoparametric Quadratic Elements

- Test case: Regular versus irregular element



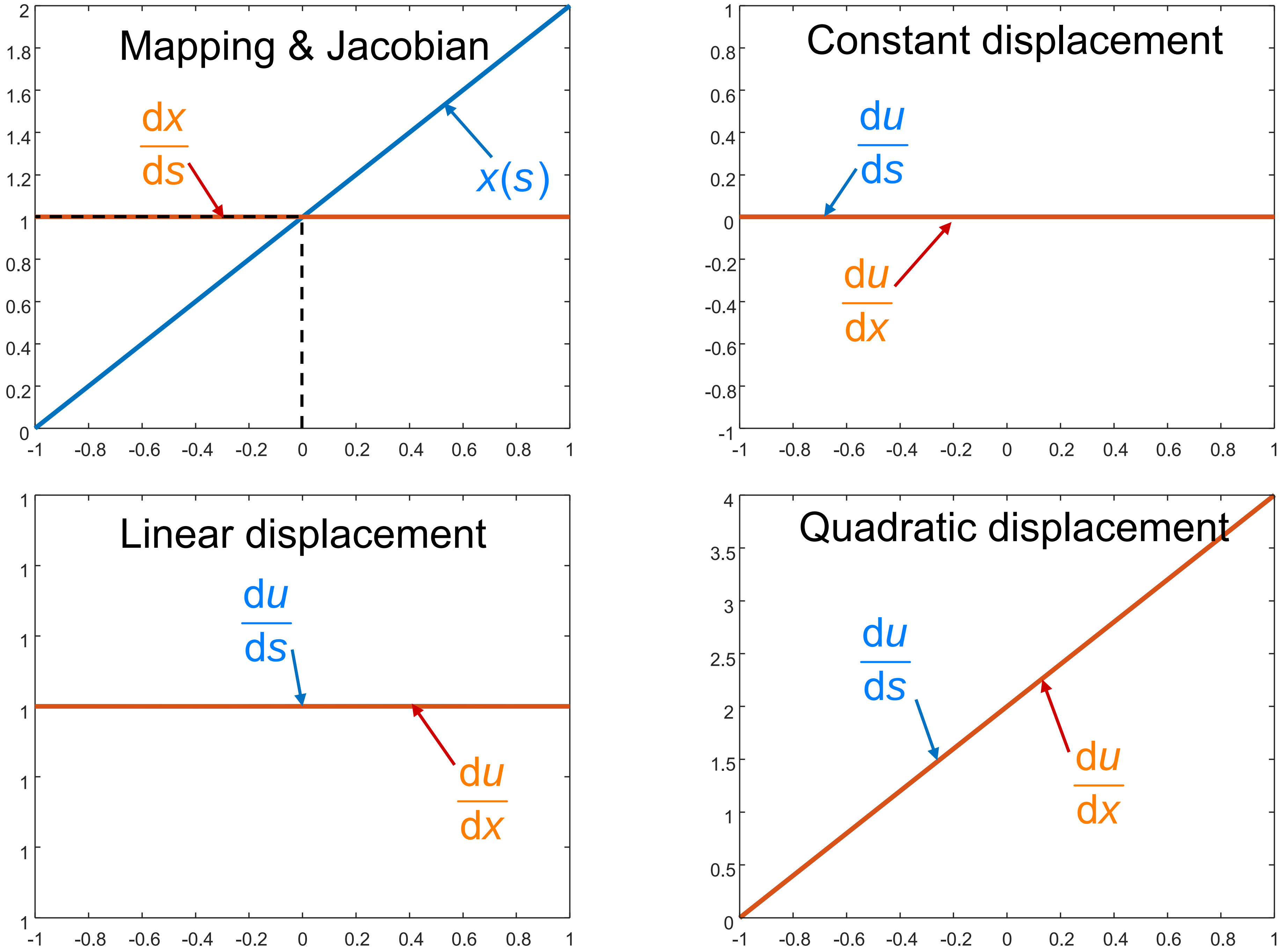
Reference element

Regular physical element

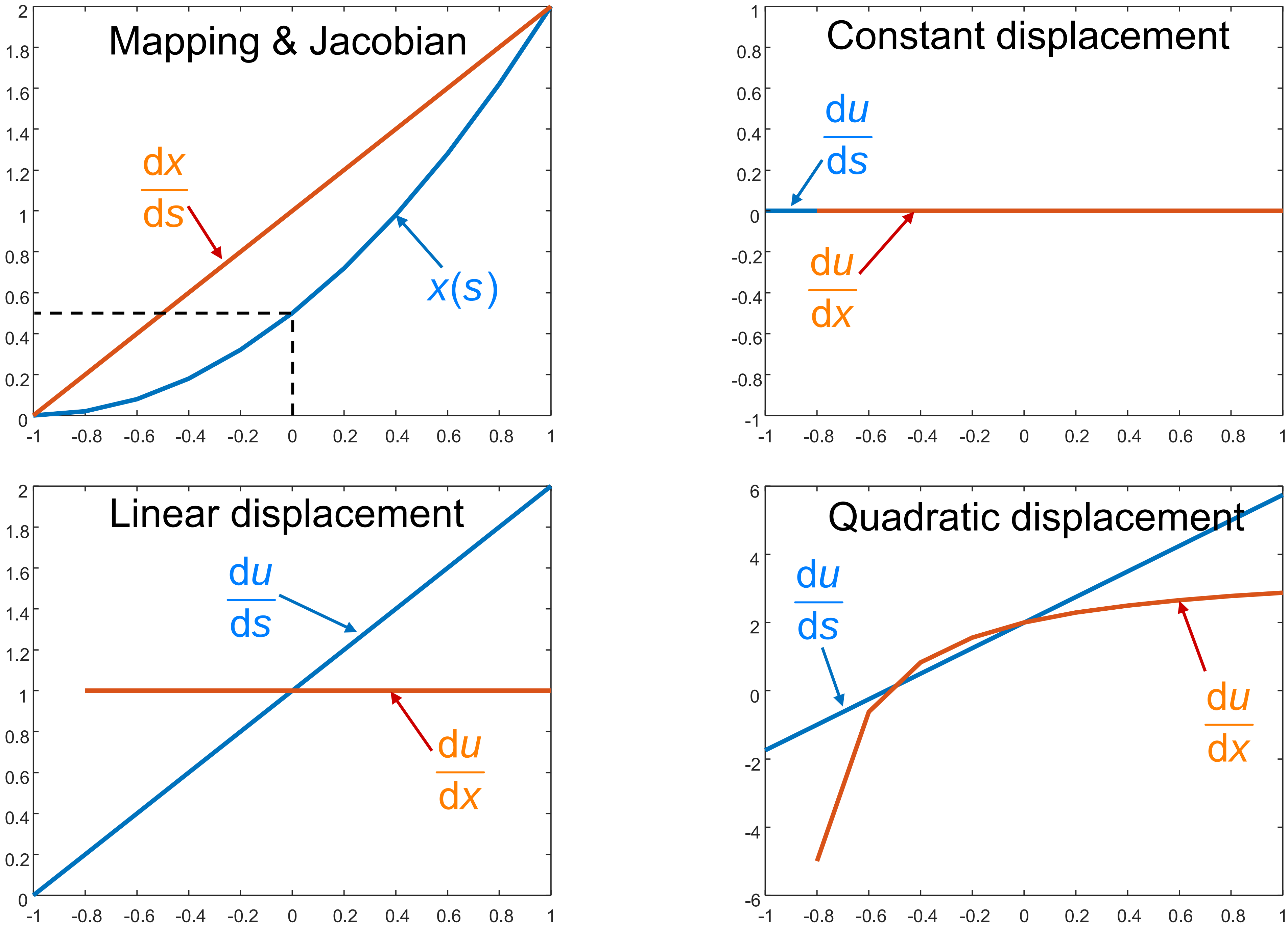
Irregular physical element

- Imposed displacement/strain
  - Zero strain: $u(x) = \text{constant}$ $du\,/\,dx = 0$
  - Constant strain: $u(x) = x$ $du\,/\,dx = 1$
  - Linear strain: $u(x) = x^2$ $du\,/\,dx = 2x$

Ex: Regular Element



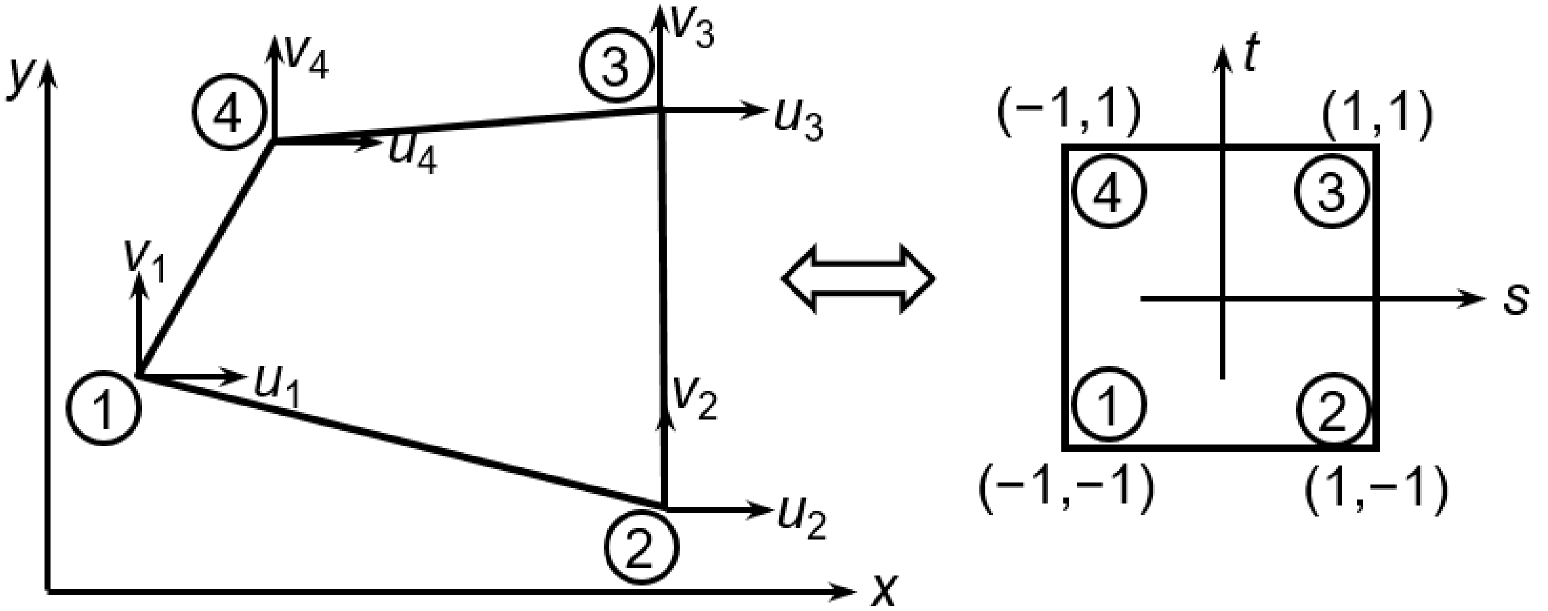
Ex: Irregular Element



7.3 2D ISOPARAMETRIC QUADRILATERAL ELEMENT

ISOPARAMETRIC ELEMENT

- Quadrilateral Shape
  - Most commonly used element (irregular shape)
  - Generalization of rectangular element
  - Use mapping to transform into a square (Reference element).
  - The relationship between (x, y) and (s, t) must be obtained.
  - All formulations are done in the reference element.



Actual Element (x, y)

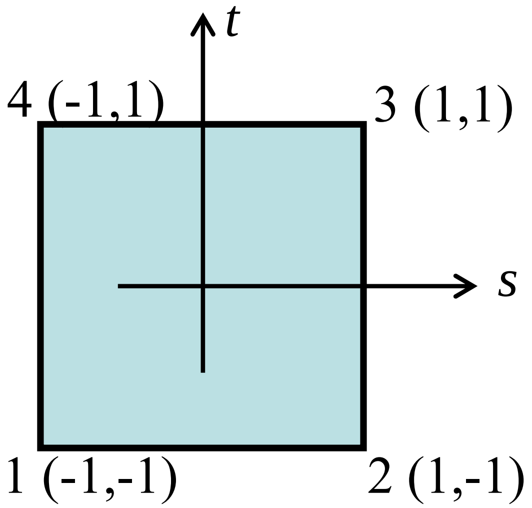
Parent Element (s, t)

ISOPARAMETRIC MAPPING

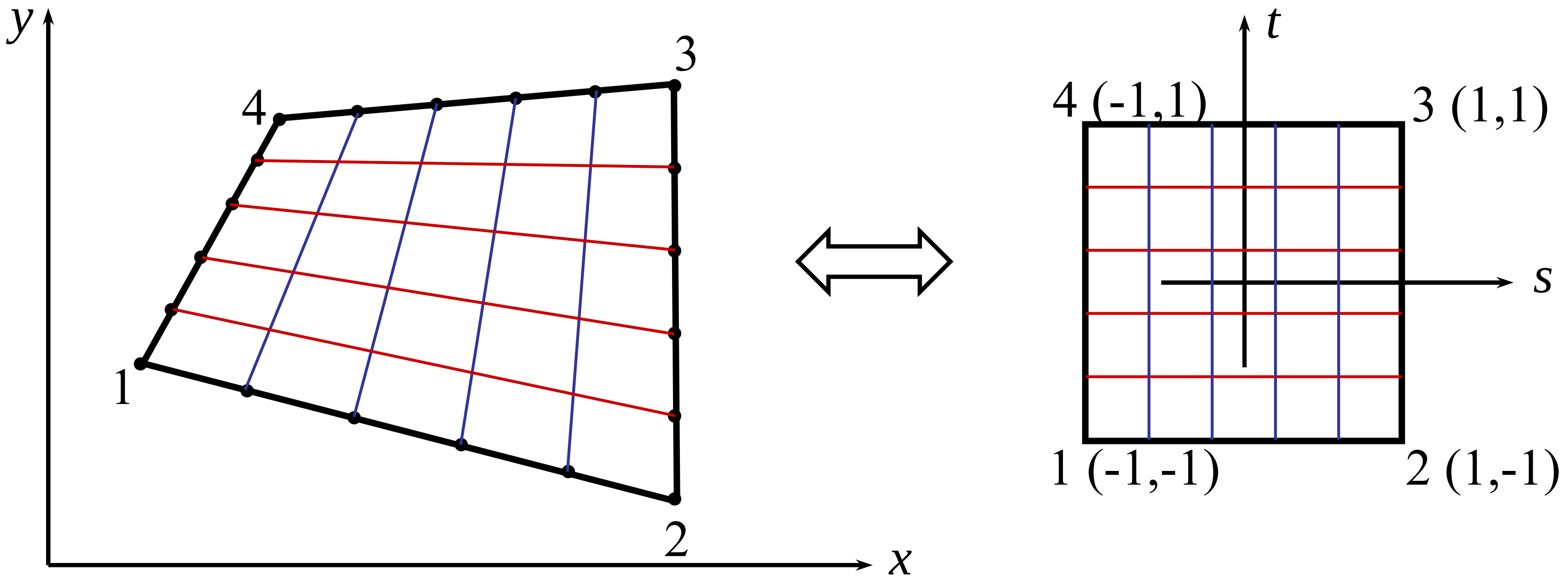
- Definition
  - the same interpolation method is used for displacement and geometry.
- Procedure
  - Construct the shape functions N_1, N_2, N_3, and N_4 at the reference element

$$N_1(s,t) = \frac{s-1}{-1-1} \times \frac{t-1}{-1-1}$$

$$\begin{cases} N_1(s,t) = \frac{1}{4}(1-s)(1-t) \\ N_2(s,t) = \frac{1}{4}(1+s)(1-t) \\ N_3(s,t) = \frac{1}{4}(1+s)(1+t) \\ N_4(s,t) = \frac{1}{4}(1-s)(1+t) \end{cases}$$



ISOPARAMETRIC MAPPING *cont.*



Proportional mapping

For a given (x,y), find corresponding (s,t).

For a given (s,t), find corresponding (x,y).

ISOPARAMETRIC MAPPING *cont.*

- Use the shape functions for interpolating displacement and geometry.
- For a given value of (s,t) in the parent element, the corresponding point (x,y) in the actual element and displacement at that point can be obtained using the mapping relationship.

Displacement interpolation

$$u(s,t) = [N_1 \quad N_2 \quad N_3 \quad N_4]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$v(s,t) = [N_1 \quad N_2 \quad N_3 \quad N_4]\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix}$$

Geometry interpolation

$$x(s,t) = [N_1 \quad N_2 \quad N_3 \quad N_4]\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}$$

$$y(s,t) = [N_1 \quad N_2 \quad N_3 \quad N_4]\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}$$

EXAMPLE

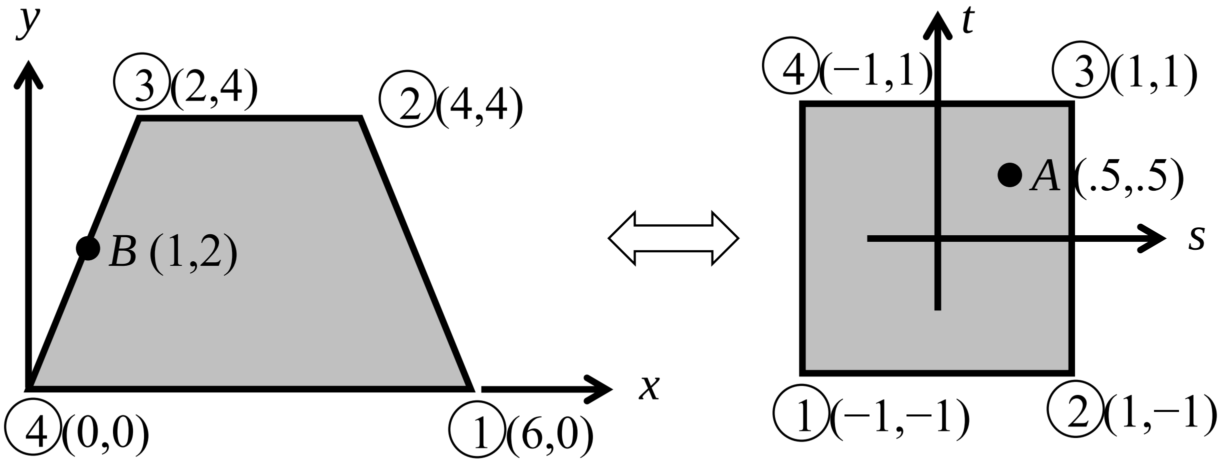
- Find mapping point of A in the physical element
  - At point *A*, (*s*, *t*) = (0.5, 0.5)

$$N_1(\tfrac{1}{2},\tfrac{1}{2}) = \frac{1}{16}, \quad N_2(\tfrac{1}{2},\tfrac{1}{2}) = \frac{3}{16}, \quad N_3(\tfrac{1}{2},\tfrac{1}{2}) = \frac{9}{16}, \quad N_4(\tfrac{1}{2},\tfrac{1}{2}) = \frac{3}{16}$$

  - Physical coord

$$x(\tfrac{1}{2},\tfrac{1}{2}) = \sum_{I=1}^{4} N_I(\tfrac{1}{2},\tfrac{1}{2})x_I = \frac{1}{16}\cdot 6 + \frac{3}{16}\cdot 4 + \frac{9}{16}\cdot 2 + \frac{3}{16}\cdot 0 = 2.25$$

$$y(\tfrac{1}{2},\tfrac{1}{2}) = \sum_{I=1}^{4} N_I(\tfrac{1}{2},\tfrac{1}{2})y_I = \frac{1}{16}\cdot 0 + \frac{3}{16}\cdot 4 + \frac{9}{16}\cdot 4 + \frac{3}{16}\cdot 0 = 3$$



EXAMPLE *cont.*

- Find mapping point of B in the reference element
  - At point *B*, (*x*, *y*) = (1, 2)

$$x = 1 = \sum_{I=1}^{4} N_I(s,t)x_I = \tfrac{1}{4}(1-s)(1-t)\cdot 6 + \tfrac{1}{4}(1+s)(1-t)\cdot 4$$

$$+\tfrac{1}{4}(1+s)(1+t)\cdot 2 + \tfrac{1}{4}(1-s)(1+t)\cdot 0$$

$$= st - 2t + 3$$

$$y = 2 = \sum_{I=1}^{4} N_I(s,t)y_I = \tfrac{1}{4}(1-s)(1-t)\cdot 0 + \tfrac{1}{4}(1+s)(1-t)\cdot 4$$

$$+\tfrac{1}{4}(1+s)(1+t)\cdot 4 + \tfrac{1}{4}(1-s)(1+t)\cdot 0$$

$$= 2 + 2s$$

  - Thus, (*s*, *t*) = (0, 1)

JACOBIAN OF MAPPING

Shape functions are given in (s,t). But, we want to differentiate w.r.t. (x,y) in order to calculate strain and stress. Use chain rule of differentiation.

$$\frac{\partial N_I}{\partial s} = \frac{\partial N_I}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial N_I}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial N_I}{\partial t} = \frac{\partial N_I}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial N_I}{\partial y}\frac{\partial y}{\partial t}$$

In Matrix Form

$$\begin{Bmatrix} \dfrac{\partial N_I}{\partial s} \\ \dfrac{\partial N_I}{\partial t} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N_I}{\partial x} \\ \dfrac{\partial N_I}{\partial y} \end{Bmatrix} = [\mathbf{J}] \begin{Bmatrix} \dfrac{\partial N_I}{\partial x} \\ \dfrac{\partial N_I}{\partial y} \end{Bmatrix}$$

Matrix [**J**] is called the Jacobian matrix of mapping.

How to calculate matrix [**J**]?

$$\frac{\partial x}{\partial s} = \sum_{I=1}^{4} \frac{\partial N_I}{\partial s} x_I = \frac{1}{4}(-x_1 + x_2 + x_3 - x_4) + \frac{t}{4}(x_1 - x_2 + x_3 - x_4)$$

$$\frac{\partial x}{\partial t} = \sum_{I=1}^{4} \frac{\partial N_I}{\partial t} x_I = \frac{1}{4}(-x_1 - x_2 + x_3 + x_4) + \frac{s}{4}(x_1 - x_2 + x_3 - x_4)$$

JACOBIAN OF MAPPING *cont.*

- Derivatives of shape functions w.r.t. (x,y) coordinates:

$$\begin{Bmatrix} \dfrac{\partial N_I}{\partial x} \\ \dfrac{\partial N_I}{\partial y} \end{Bmatrix} = [\mathbf{J}]^{-1} \begin{Bmatrix} \dfrac{\partial N_I}{\partial s} \\ \dfrac{\partial N_I}{\partial t} \end{Bmatrix} = \frac{1}{|\mathbf{J}|} \begin{bmatrix} \dfrac{\partial y}{\partial t} & -\dfrac{\partial y}{\partial s} \\ -\dfrac{\partial x}{\partial t} & \dfrac{\partial x}{\partial s} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N_I}{\partial s} \\ \dfrac{\partial N_I}{\partial t} \end{Bmatrix}$$

- Determinant |**J**|: Jacobian

$$|\mathbf{J}| = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}$$

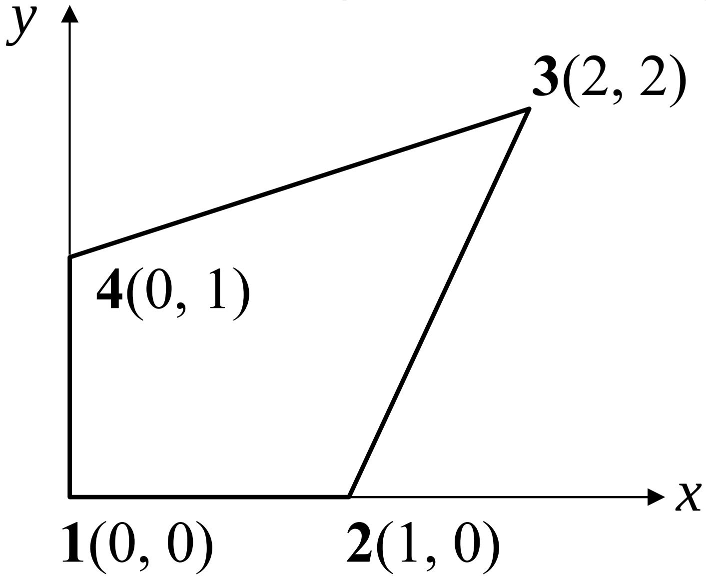
- What happen if |**J**| = 0 or |**J**| < 0?
  - shape function derivative cannot be obtained if the |**J**| = 0 anywhere in the element
  - Mapping relation between (*x*, *y*) and (*s*, *t*) is not valid if |**J**| = 0 or |**J**| < 0 anywhere in the element (–1 ≤ *s*, *t* ≤ 1).

JACOBIAN OF MAPPING *cont*.

- Jacobian is an important criterion for evaluating the validity of mapping, as well as the quality of element
- Every point in the reference element should be mapped into the interior of the physical element
- When an interior point in (s, t) coord. is mapped into an exterior point in the (x, y) coord., the Jacobian becomes negative
- If multiple points in (s, t) coordinates are mapped into a single point in (x, y) coordinates, the Jacobian becomes zero at that point
- It is important to maintain the element shape so that the Jacobian is positive everywhere in the element

EXAMPLE (JACOBIAN)

Jacobian must not be zero anywhere in the domain ($-1 \le s, t \le 1$)



- Nodal Coordinates

$$x_1 = 0, \quad x_2 = 1, \quad x_3 = 2, \quad x_4 = 0$$
$$y_1 = 0, \quad y_2 = 0, \quad y_3 = 2, \quad y_4 = 1$$

- Iso-Parametric Mapping

$$x = \sum_{I=1}^{4} N_I x_I = N_2 + 2N_3 = \frac{1}{4}(3 + 3s + t + st)$$
$$y = \sum_{I=1}^{4} N_I y_I = 2N_3 + N_4 = \frac{1}{4}(3 + s + 3t + st)$$

- Jacobian Matrix

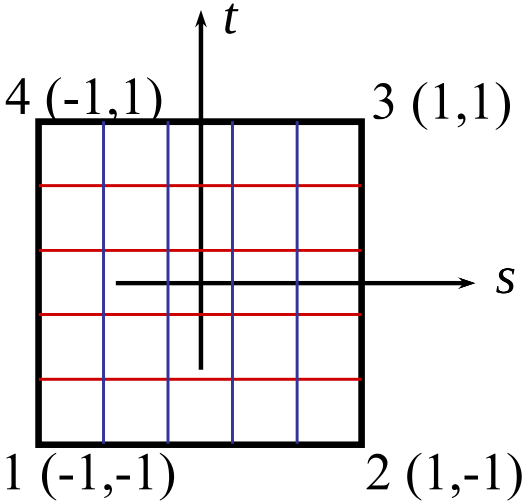
$$[\mathbf{J}] = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix} = \boxed{\frac{1}{4}\begin{bmatrix} 3+t & 1+t \\ 1+s & 3+s \end{bmatrix}}$$

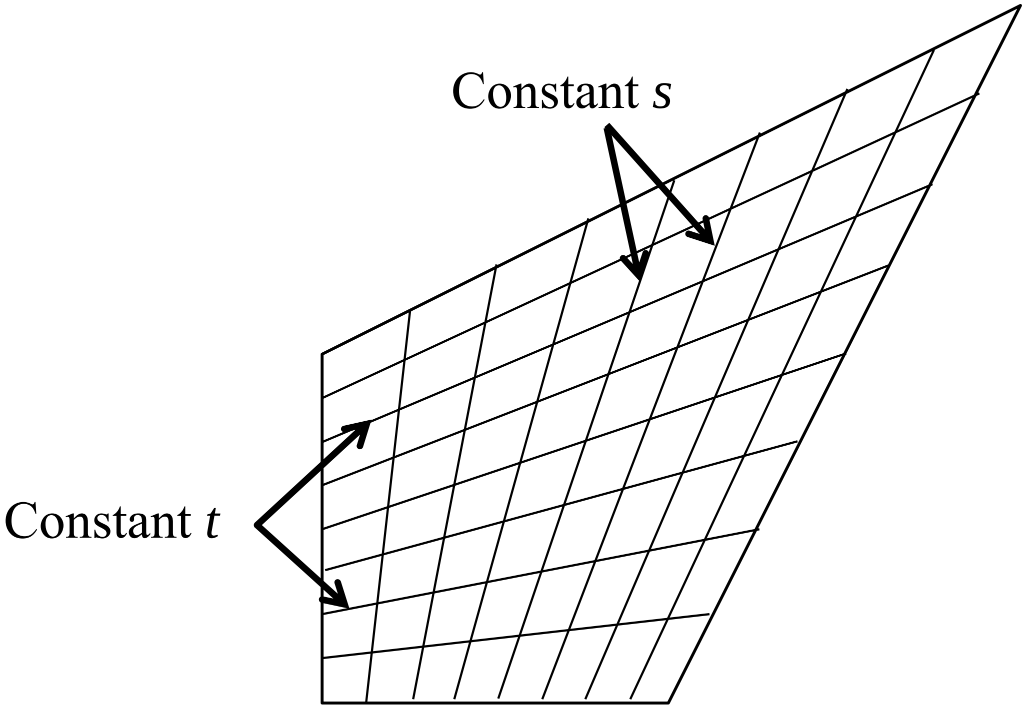
EXAMPLE (JACOBIAN) *cont.*

- Jacobian

$$|\mathbf{J}| = \frac{1}{4}[(3+t)(3+s)-(1+t)(1+s)] = \frac{1}{2}+\frac{1}{8}s+\frac{1}{8}t$$

  - It is clear that |**J**| > 0 for –1 ≤ *s* ≤ 1 and –1 ≤ *t* ≤ 1.





EXAMPLE (JACOBIAN) *cont.*

- Nodal Coordinates

$$x_1 = 0, \quad x_2 = 1, \quad x_3 = 5, \quad x_4 = 0$$
$$y_1 = 0, \quad y_2 = 4, \quad y_3 = 5, \quad y_4 = 5$$

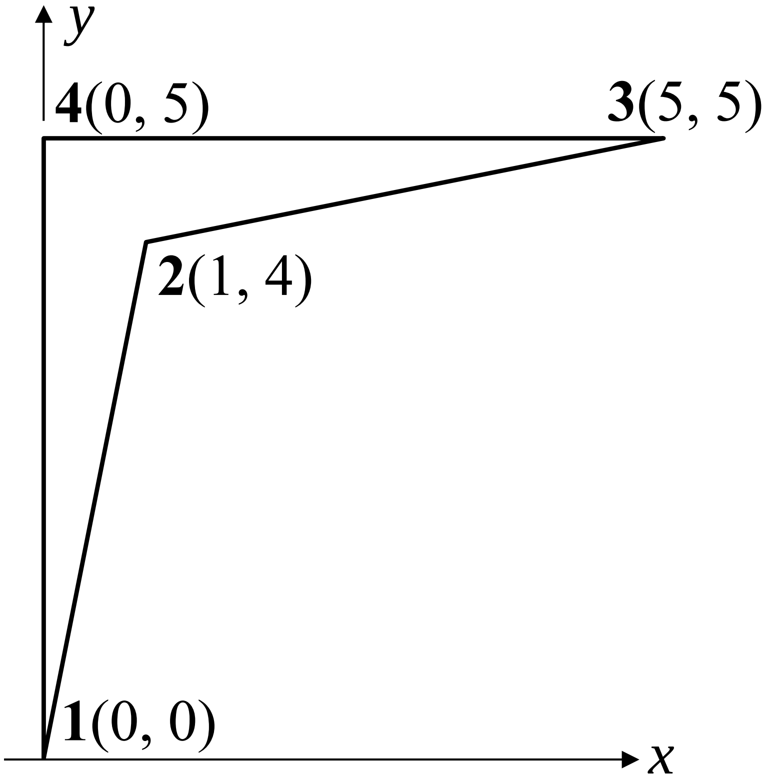
- Mapping

$$x = \sum_{I=1}^{4} N_I x_I = \frac{1}{2}(1+s)(3+2t)$$
$$y = \sum_{I=1}^{4} N_I y_I = \frac{1}{2}(7+2s+3t-2st)$$

- Jacobian

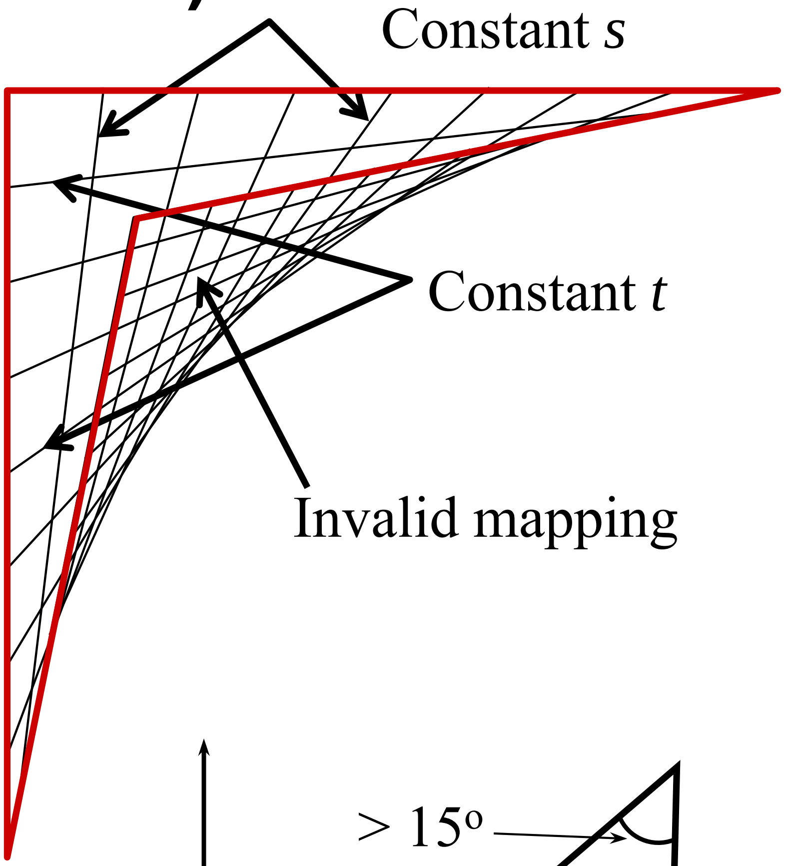
$$|\mathbf{J}| = \frac{1}{4}(5-10s+10t)$$

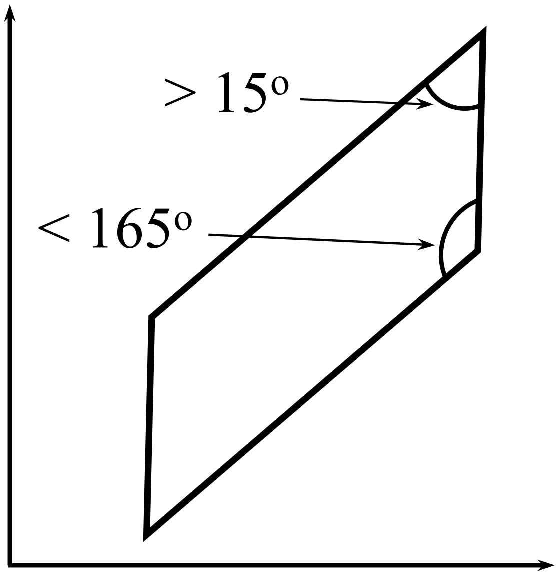
|**J**| = 0 at 5 – 10s + 10t = 0; i.e., s – t = 1/2



EXAMPLE 8.3 (JACOBIAN) *cont.*

- In general the element geometry is invalid if the Jacobian is either zero or negative anywhere in the element.
- Problems also arise when the Jacobian matrix is nearly singular either due to round-off errors or due to badly shaped elements.
- To avoid problems due to badly shaped elements, it is suggested that the inside angles in quadrilateral elements be > 15° and < 165°





INTERPOLATION

- Displacement Interpolation (8-DOF)

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = [\mathbf{N}]\{\mathbf{q}\}$$

  - the interpolation is done in the reference coordinates (*s*, *t*)
  - The behavior of the element is similar to that of the rectangular element because both of them are based on the bilinear Lagrange interpolation

INTERPOLATION *cont.*

- Strain

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \partial u/\partial x \\ \partial u/\partial y \\ \partial v/\partial x \\ \partial v/\partial y \end{Bmatrix}$$

$$\begin{Bmatrix} \partial u/\partial x \\ \partial u/\partial y \\ \partial v/\partial x \\ \partial v/\partial y \end{Bmatrix} = \frac{1}{|\mathbf{J}|} \begin{bmatrix} \partial y/\partial t & -\partial y/\partial s & 0 & 0 \\ -\partial x/\partial t & \partial x/\partial s & 0 & 0 \\ 0 & 0 & \partial y/\partial t & -\partial y/\partial s \\ 0 & 0 & -\partial x/\partial t & \partial x/\partial s \end{bmatrix} \begin{Bmatrix} \partial u/\partial s \\ \partial u/\partial t \\ \partial v/\partial s \\ \partial v/\partial t \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{|\mathbf{J}|} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \partial y/\partial t & -\partial y/\partial s & 0 & 0 \\ -\partial x/\partial t & \partial x/\partial s & 0 & 0 \\ 0 & 0 & \partial y/\partial t & -\partial y/\partial s \\ 0 & 0 & -\partial x/\partial t & \partial x/\partial s \end{bmatrix} \begin{Bmatrix} \partial u/\partial s \\ \partial u/\partial t \\ \partial v/\partial s \\ \partial v/\partial t \end{Bmatrix} \equiv [\mathbf{A}] \begin{Bmatrix} \partial u/\partial s \\ \partial u/\partial t \\ \partial v/\partial s \\ \partial v/\partial t \end{Bmatrix}$$

INTERPOLATION *cont.*

- Strain cont.

$$\begin{Bmatrix} \partial u/\partial s \\ \partial u/\partial t \\ \partial v/\partial s \\ \partial v/\partial t \end{Bmatrix} = \frac{1}{4}\begin{bmatrix} -1+t & 0 & 1-t & 0 & 1+t & 0 & -1-t & 0 \\ -1+s & 0 & -1-s & 0 & 1+s & 0 & 1-s & 0 \\ 0 & -1+t & 0 & 1-t & 0 & 1+t & 0 & -1-t \\ 0 & -1+s & 0 & -1-s & 0 & 1+s & 0 & 1-s \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \equiv [\mathbf{G}]\{\mathbf{q}\}$$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = [\mathbf{A}]\begin{Bmatrix} \partial u/\partial s \\ \partial u/\partial t \\ \partial v/\partial s \\ \partial v/\partial t \end{Bmatrix} = [\mathbf{A}][\mathbf{G}]\{\mathbf{q}\} \equiv [\mathbf{B}]\{\mathbf{q}\}$$

Strain-displacement matrix

  - The expression of [**B**] is not readily available because the matrix [**A**] involves the inverse of Jacobian matrix
  - The strain-displacement matrix [**B**] is not constant

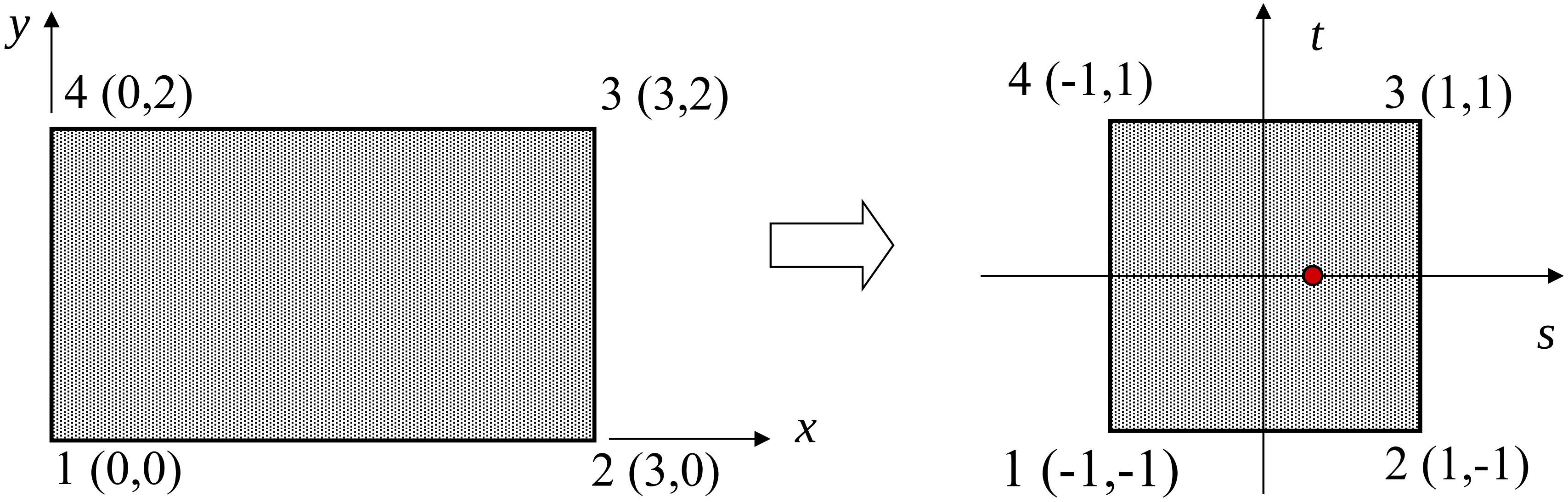
EXAMPLE

- $\{u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4\} = \{0, 0, 1, 0, 2, 1, 0, 2\}$
- Displacement and strain at (s,t)=(1/3, 0)?
- Shape Functions

$$N_1(s,t) = \frac{1}{4}(1-s)(1-t) \qquad N_2(s,t) = \frac{1}{4}(1+s)(1-t)$$

$$N_3(s,t) = \frac{1}{4}(1+s)(1+t) \qquad N_4(s,t) = \frac{1}{4}(1-s)(1+t)$$

- At (s,t)=(1/3, 0) $\quad N_1 = \frac{1}{6}, \quad N_2 = \frac{1}{3}, \quad N_3 = \frac{1}{3}, \quad N_4 = \frac{1}{6}$



EXAMPLE *cont.*

- Location at the Actual Element

$$\begin{cases} x = \sum_{I=1}^{4} N_I x_I = \frac{1}{6}\cdot 0 + \frac{1}{3}\cdot 3 + \frac{1}{3}\cdot 3 + \frac{1}{6}\cdot 0 = 2 \\ y = \sum_{I=1}^{4} N_I y_I = \frac{1}{6}\cdot 0 + \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 2 + \frac{1}{6}\cdot 2 = 1 \end{cases}$$

- Displacement at (s,t) = (1/3,0)

$$\begin{cases} u = \sum_{I=1}^{4} N_I u_I = \frac{1}{6}\cdot 0 + \frac{1}{3}\cdot 1 + \frac{1}{3}\cdot 2 + \frac{1}{6}\cdot 0 = 1 \\ v = \sum_{I=1}^{4} N_I v_I = \frac{1}{6}\cdot 0 + \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 1 + \frac{1}{6}\cdot 2 = \frac{2}{3} \end{cases}$$

EXAMPLE *cont.*

- Derivatives of the shape functions w.r.t. *s* and *t*.

$$\begin{cases} \dfrac{\partial N_1}{\partial s} = -\dfrac{1}{4}(1-t) = -\dfrac{1}{4} \\ \dfrac{\partial N_2}{\partial s} = \dfrac{1}{4}(1-t) = \dfrac{1}{4} \\ \dfrac{\partial N_3}{\partial s} = \dfrac{1}{4}(1+t) = \dfrac{1}{4} \\ \dfrac{\partial N_4}{\partial s} = -\dfrac{1}{4}(1+t) = -\dfrac{1}{4} \end{cases} \qquad \begin{cases} \dfrac{\partial N_1}{\partial t} = -\dfrac{1}{4}(1-s) = -\dfrac{1}{6} \\ \dfrac{\partial N_2}{\partial t} = -\dfrac{1}{4}(1+s) = -\dfrac{1}{3} \\ \dfrac{\partial N_3}{\partial t} = \dfrac{1}{4}(1+s) = \dfrac{1}{3} \\ \dfrac{\partial N_4}{\partial t} = \dfrac{1}{4}(1-s) = \dfrac{1}{6} \end{cases}$$

- But, we need the derivatives w.r.t. x and y. How to convert?

EXAMPLE *cont.*

- Jacobian Matrix

$$\begin{cases} \dfrac{\partial x}{\partial s} = -\dfrac{1}{4}\cdot 0 + \dfrac{1}{4}\cdot 3 + \dfrac{1}{4}\cdot 3 - \dfrac{1}{4}\cdot 0 = \dfrac{3}{2} \\ \dfrac{\partial y}{\partial s} = -\dfrac{1}{4}\cdot 0 + \dfrac{1}{4}\cdot 0 + \dfrac{1}{4}\cdot 2 - \dfrac{1}{4}\cdot 2 = 0 \\ \dfrac{\partial x}{\partial t} = -\dfrac{1}{6}\cdot 0 - \dfrac{1}{3}\cdot 3 + \dfrac{1}{3}\cdot 3 + \dfrac{1}{6}\cdot 0 = 0 \\ \dfrac{\partial y}{\partial t} = -\dfrac{1}{6}\cdot 0 - \dfrac{1}{3}\cdot 0 + \dfrac{1}{3}\cdot 2 + \dfrac{1}{6}\cdot 2 = 1 \end{cases}$$

$$[\mathbf{J}] = \begin{bmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} \end{bmatrix} = \begin{bmatrix} \dfrac{3}{2} & 0 \\ 0 & 1 \end{bmatrix}, \qquad [\mathbf{J}]^{-1} = \begin{bmatrix} \dfrac{2}{3} & 0 \\ 0 & 1 \end{bmatrix}$$

  - Jacobian is positive, and the mapping is valid at this point
  - Jacobian matrix is constant throughout the element
  - Jacobian matrix only has diagonal components, which means that the physical element is a rectangle

EXAMPLE *cont.*

- Derivative of the shape functions w.r.t. *x* and *y*.

$$\begin{cases} \dfrac{\partial N_I}{\partial s} = \dfrac{\partial N_I}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial N_I}{\partial y}\dfrac{\partial y}{\partial s} \\ \dfrac{\partial N_I}{\partial t} = \dfrac{\partial N_I}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial N_I}{\partial y}\dfrac{\partial y}{\partial t} \end{cases} \Rightarrow \begin{Bmatrix} \dfrac{\partial N_I}{\partial s} \\ \dfrac{\partial N_I}{\partial t} \end{Bmatrix} = [\mathbf{J}] \begin{Bmatrix} \dfrac{\partial N_I}{\partial x} \\ \dfrac{\partial N_I}{\partial y} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial N_I}{\partial x} \\ \dfrac{\partial N_I}{\partial y} \end{Bmatrix} = [\mathbf{J}]^{-1} \begin{Bmatrix} \dfrac{\partial N_I}{\partial s} \\ \dfrac{\partial N_I}{\partial t} \end{Bmatrix} = \begin{bmatrix} \dfrac{2}{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N_I}{\partial s} \\ \dfrac{\partial N_I}{\partial t} \end{Bmatrix} = \begin{Bmatrix} \dfrac{2}{3}\dfrac{\partial N_I}{\partial s} \\ \dfrac{\partial N_I}{\partial t} \end{Bmatrix}$$

- Strain

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \sum_{I=1}^{4} \frac{\partial N_I}{\partial x} u_I = \sum_{I=1}^{4} \frac{2}{3}\frac{\partial N_I}{\partial s} u_I = \frac{2}{3}\left(-\frac{1}{4}\cdot 0 + \frac{1}{4}\cdot 1 + \frac{1}{4}\cdot 2 - \frac{1}{4}\cdot 0\right) = \frac{1}{2}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \sum_{I=1}^{4} \frac{\partial N_I}{\partial y} v_I = \sum_{I=1}^{4} \frac{\partial N_I}{\partial t} v_I = -\frac{1}{6}\cdot 0 - \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot 1 + \frac{1}{6}\cdot 2 = \frac{2}{3}$$

FINITE ELEMENT EQUATION

- Element stiffness matrix from strain energy expression

$$U^{(e)} = \frac{h}{2}\iint_A \{\varepsilon\}^T[\mathbf{C}]\{\varepsilon\}\, dA^{(e)}$$

$$= \frac{h}{2}\{\mathbf{q}^{(e)}\}^T \iint_A [\mathbf{B}]^T_{8\times 3}[\mathbf{C}]_{3\times 3}[\mathbf{B}]_{3\times 8}\, dA\{\mathbf{q}^{(e)}\}$$

$$\equiv \frac{1}{2}\{\mathbf{q}^{(e)}\}^T[\mathbf{k}^{(e)}]_{8\times 8}\{\mathbf{q}^{(e)}\}$$

  - $[\mathbf{k}^{(e)}]$ is the element stiffness matrix
  - Integration domain is a general quadrilateral shape
  - Displacement–strain matrix **[B]** is written in (*s*, *t*) coordinates
  - we can perform the integration in the reference element

$$\boxed{[\mathbf{k}^{(e)}] = h\iint_A [\mathbf{B}]^T[\mathbf{C}][\mathbf{B}]\, dA \equiv h\int_{-1}^{1}\int_{-1}^{1}[\mathbf{B}]^T[\mathbf{C}][\mathbf{B}]\left|\mathbf{J}\right| dsdt}$$

$$\uparrow$$

$$dA = \left|\mathbf{J}\right| dsdt$$

7.4 NUMERICAL INTEGRATION

NUMERICAL INTEGRATION

- Stiffness matrix and distributed load calculations involve integration over the domain
- In many cases, analytical integration is very difficult
- Numerical integration based on Gauss Quadrature is commonly used in finite element programs
- Gauss Quadrature:

$$\boxed{I = \int_{-1}^{1} f(s)ds \approx \sum_{i=1}^{n} w_i f(s_i)}$$

  - Integral is evaluated using function values and weights.
  - s_i: Gauss integration points, w_i: integration weights
  - $f(s_i)$: function value at the Gauss point
  - n: number of integration points.

NUMERICAL INTEGRATION *cont.*

- Constant Function: $f(s) = 4$
  - Use one integration point $s_1 = 0$ and weight $w_1 = 2$

$$I = \int_{-1}^{1} 4ds = w_1 f(s_1) = 2 \times 4 = 8$$

  - The numerical integration is exact.
- Linear Function: $f(s) = 2s + 1$
  - Use one integration point $s_1 = 0$ and weight $w_1 = 2$

$$I = \int_{-1}^{1} (2s + 1)ds = w_1 f(s_1) = 2 \times 1 = 2$$

  - The numerical integration is exact.
- One-point Gauss Quadrature can integrate constant and linear functions exactly.

NUMERICAL INTEGRATION *cont*.

- Quadratic Function: $f(s) = 3s^2 + 2s + 1$
  - Let's use one-point Gauss Quadrature

$$I = \int_{-1}^{1} (3s^2 + 2s + 1)ds = 4$$

$$w_1 f(s_1) = 2 \times 1 = 2$$

  - One-point integration is not accurate for quadratic function
  - Let's use two-point integration with $w_1 = w_2 = 1$ and $-s_1 = s_2 = 1/\sqrt{3}$

$$w_1 f(s_1) + w_2 f(s_2) = 1 \times f(-\tfrac{1}{\sqrt{3}}) + 1 \times f(\tfrac{1}{\sqrt{3}})$$

$$= 3 \times \tfrac{1}{3} - \tfrac{2}{\sqrt{3}} + 1 + 3 \times \tfrac{1}{3} + \tfrac{2}{\sqrt{3}} + 1 = 4$$

- Gauss Quadrature points and weights are selected such that n integration points can integrate ($2n$ – 1)-order polynomial exactly.

NUMERICAL INTEGRATION *cont*.

- Gauss Quadrature Points and Weights

n	Integration Points (s_i)	Weights (w_i)	Exact for polynomial of degree
1	0.0	2.0	1
2	±.5773502692	1.0	3
3	±.7745966692 0.0	.5555555556 .8888888889	5
4	±.8611363116 ±.3399810436	.3478546451 .6521451549	7
5	±.9061798459 ±.5384693101 0.0	.2369268851 .4786286705 .5688888889	9

$$\sqrt{\frac{1}{3}} = 0.5773502692$$

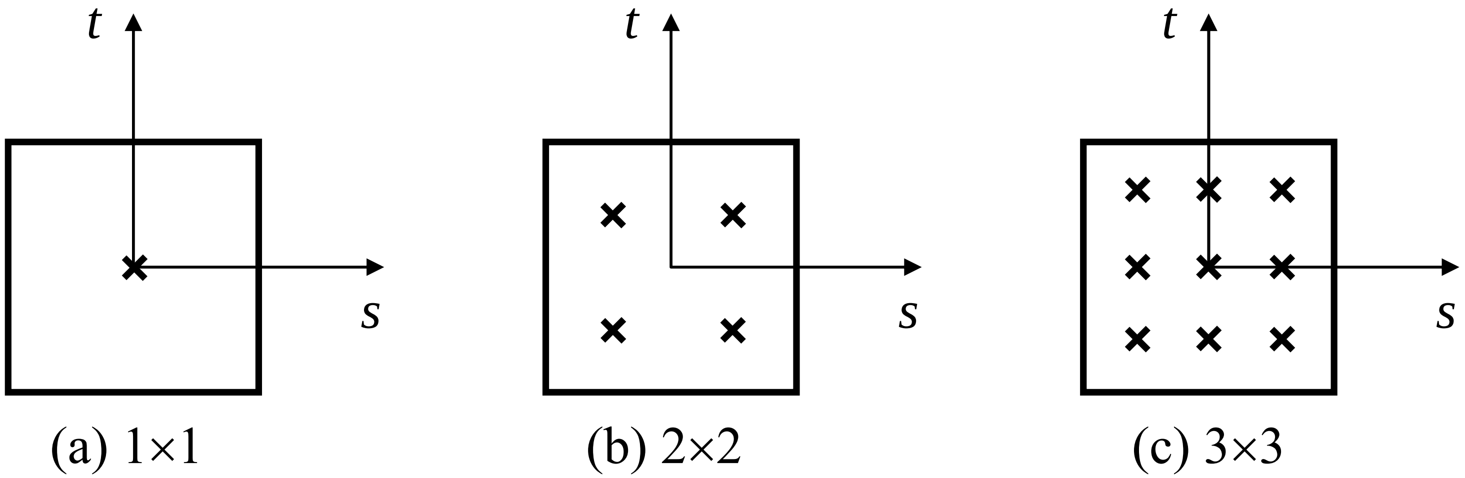
$$\sqrt{\frac{3}{5}} = 0.7745966692$$

NUMERICAL INTEGRATION *cont.*

- 2-Dimensional Integration
  - multiplying two one-dimensional Gauss integration formulas

$$I=\int_{-1}^{1}\int_{-1}^{1} f(s,t)\,ds\,dt \approx \int_{-1}^{1}\sum_{i=1}^{m} w_i f\left(s_i,t\right)dt=\sum_{j=1}^{n}\sum_{i=1}^{m} w_i w_j f(s_i,t_j)$$

  - Total number of integration points = *m*×*n*.



(a) 1×1 (b) 2×2 (c) 3×3

NUMERICAL INTEGRATION EXAMPLE

- Integrate the following polynomial:

$$I = \int_{-1}^{1} (8x^7 + 7x^6)dx = 2$$

  - One-point formula

$$s_1 = 0, \qquad f(s_1) = 0, \qquad w_1 = 2$$
$$I = w_1 f(s_1) = 2 \times 0 = 0$$

  - Two-point formula

$$s_1 = -.577, \quad f(s_1) = 8(-.577)^7 + 7(-.577)^6 = .0882, w_1 = 1$$
$$s_2 = \ \ .577, \quad f(s_2) = 8(.577)^7 + 7(.577)^6 = .4303, \qquad w_2 = 1$$
$$I = w_1 f(x_1) + w_2 f(x_2) = .0882 + .4303 = .5185$$

NUMERICAL INTEGRATION EXAMPLE *cont.*

- 3-point formula

$$s_1 = -.577, \quad f(s_1) = 8(-.577)^7 + 7(-.577)^6 = .0882, \qquad w_1 = 1$$

$$s_2 = \ \ .577, \quad f(s_2) = 8(.577)^7 + 7(.577)^6 = .4303, \qquad w_2 = 1$$

$$I = w_1 f(x_1) + w_2 f(x_2) = .0882 + .4303 = .5185$$

- 4-point formula

$$s_1 = -.8611, \ f(s_1) = .0452, \qquad w_1 = .3479$$

$$s_2 = -.3400, f(s_2) = .0066, \qquad w_2 = .6521$$

$$s_3 = \ \ .3400, f(s_3) = .0150, \qquad w_3 = .6521$$

$$s_3 = \ \ .8611, \ f(s_3) = 5.6638, \qquad w_3 = .3479$$

$$I = w_1 f(s_1) + w_2 f(s_2) + w_3 f(s_3) + w_4 f(s_4) = 2.0$$

- 4-point formula yields the exact solution. Why?

NUMERICAL INTEGRATION

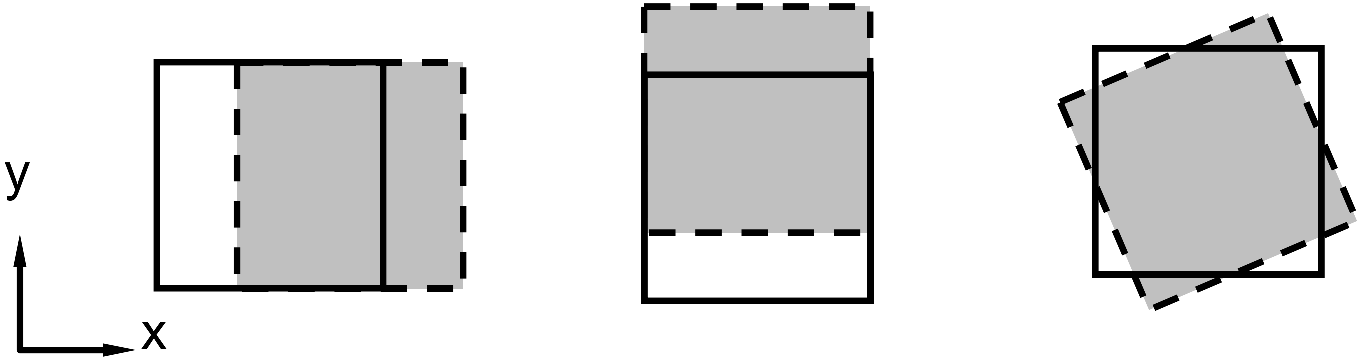
- Application to Stiffness Matrix Integral

$$[\mathbf{k}^{(e)}] = h\int_{-1}^{1}\int_{-1}^{1}[\mathbf{B}]^T[\mathbf{C}][\mathbf{B}]\left|\mathbf{J}\right|dsdt$$

$$\approx h\sum_{i=1}^{2}\sum_{j=1}^{2}w_i w_j[\mathbf{B}(s_i,t_j)]^T[\mathbf{C}][\mathbf{B}(s_i,t_j)]\left|\mathbf{J}(s_i,t_j)\right|$$

Extra Zero-Energy Modes

- Element deforms without changing strain energy (Something must be wrong here!)
- Only rigid-body motions without changing strain energy

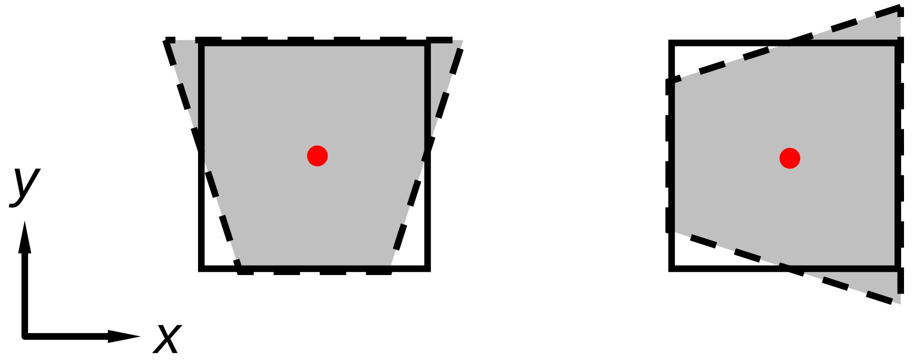


x-translation y-translation z-rotation

- Why do extra-zero energy modes occur?
  - The order of numerical integration is not appropriate
  - Numerical integration can't catch the difference in deformation

Extra Zero-Energy Modes

- Ex) 1x1 Gauss quadrature for rectangular element



- The center point does not move!!

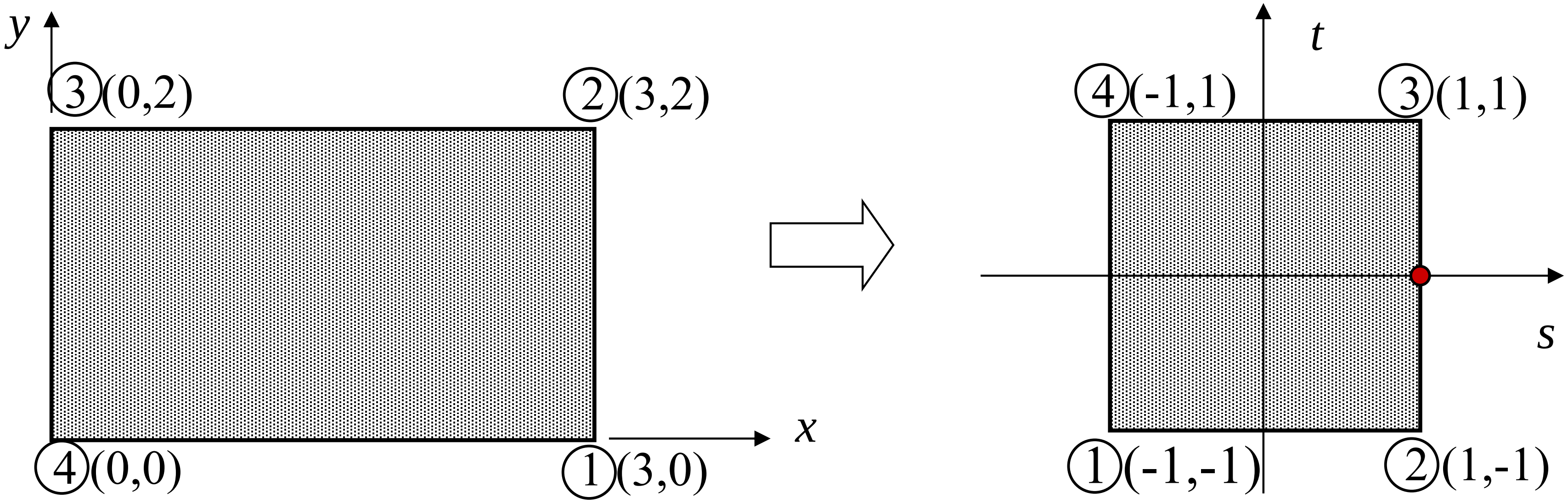
- How to find zero-energy modes?
  - Element stiffness matrix should have 3 rigid-body modes(2D) or 6(3D)
  - The same number of zero eigenvalues
  - 1x1 Gauss quadrature will yield 5 zero eigenvalues (2 extra zero-energy modes!!)

Exercise

- For the isoparametric element shown in the figure, (1) write the Jacobian matrix (you can calculate it based on it's shape) and (2) calculate strains ε_{xx}, ε_{yy}, γ_{xy} at (s,t) = (1,0) when nodal displacements are {**q**}={0.1, -0.1, 0, -0.1, 0, 0, 0.1, 0}^{T}.

$$N_1(s,t)=\frac{1}{4}(1-s)(1-t),\quad N_2(s,t)=\frac{1}{4}(1+s)(1-t)$$

$$N_3(s,t)=\frac{1}{4}(1+s)(1+t),\quad N_4(s,t)=\frac{1}{4}(1-s)(1+t)$$



7.5 HIGHER-ORDER QUADRILATERAL ELEMENT

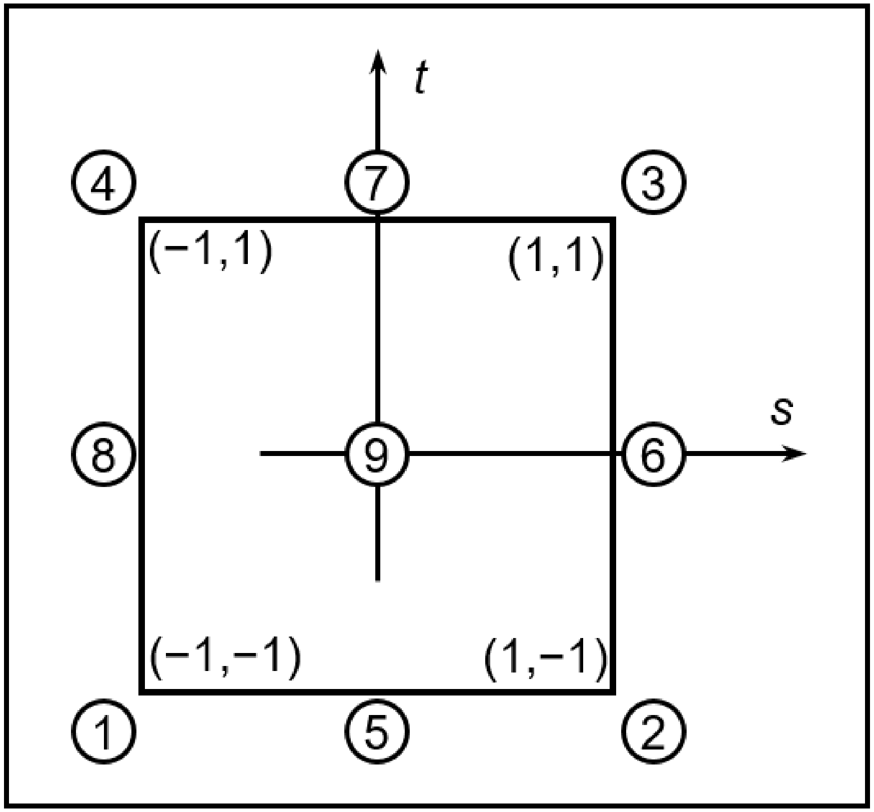
9-Node Lagrange Element

- Interpolation scheme

$$u(s,t) = a_1 + a_2 s + a_3 t + a_4 st + a_5 s^2$$
$$+ a_6 t^2 + a_7 s^2 t + a_8 t^2 s + a_9 s^2 t^2$$

$$\Downarrow$$

$$u(s,t) = \sum_{i=1}^{9} N_i(s,t) u_i$$



$$N_1(s,t) = \tfrac{1}{4} st(1-s)(1-t)$$

$$N_2(s,t) = -\tfrac{1}{4} st(1+s)(1-t)$$

$$N_3(s,t) = \tfrac{1}{4} st(1+s)(1+t)$$

$$N_4(s,t) = -\tfrac{1}{4} st(1-s)(1+t)$$

9-Node Lagrange Element

$$N_1(s,t) = \tfrac{1}{4}st(1-s)(1-t)$$

$$N_2(s,t) = -\tfrac{1}{4}st(1+s)(1-t)$$

$$N_3(s,t) = \tfrac{1}{4}st(1+s)(1+t)$$

$$N_4(s,t) = -\tfrac{1}{4}st(1-s)(1+t)$$

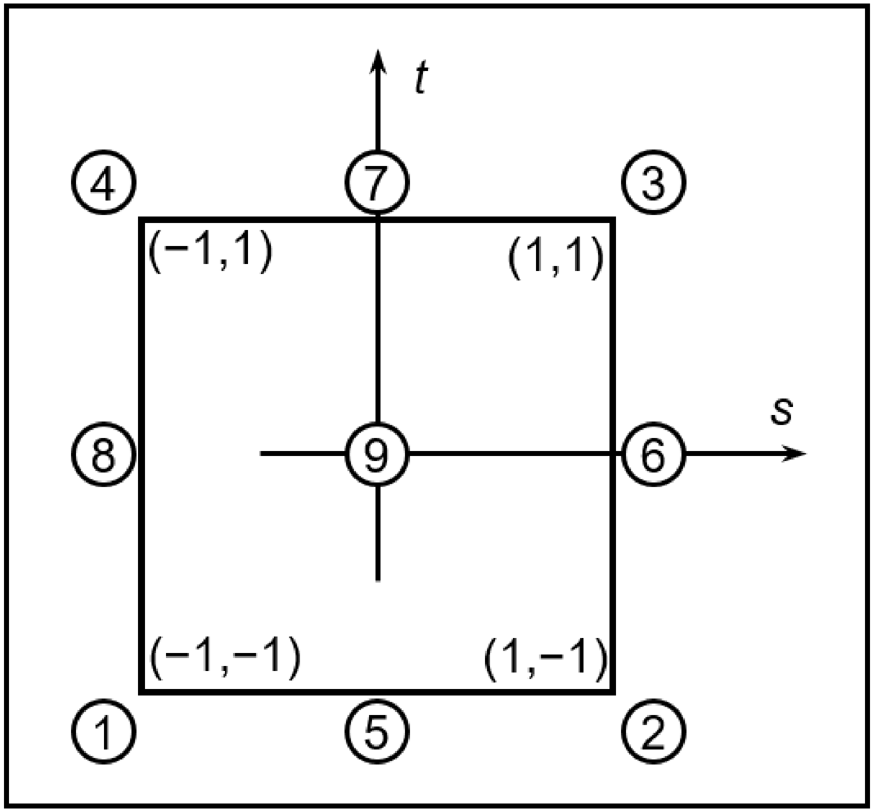
$$N_5(s,t) = \tfrac{1}{2}t(1-s^2)(1-t)$$

$$N_6(s,t) = \tfrac{1}{2}s(1-t^2)(1+s)$$

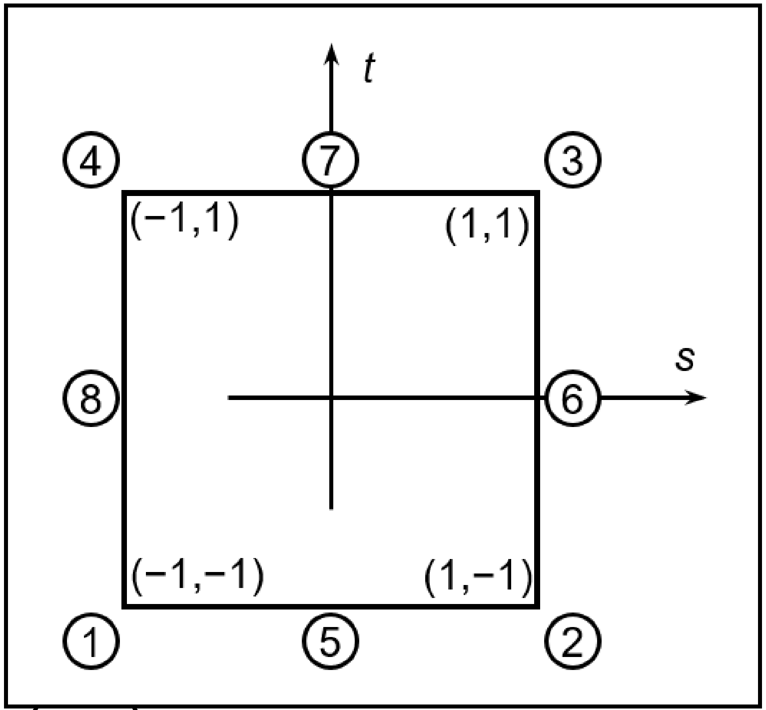
$$N_7(s,t) = \tfrac{1}{2}t(1-s^2)(1+t)$$

$$N_8(s,t) = \tfrac{1}{2}s(1-t^2)(1-s)$$

$$N_9(s,t) = (1-s^2)(1-t^2)$$



8-Node Serendipity Element



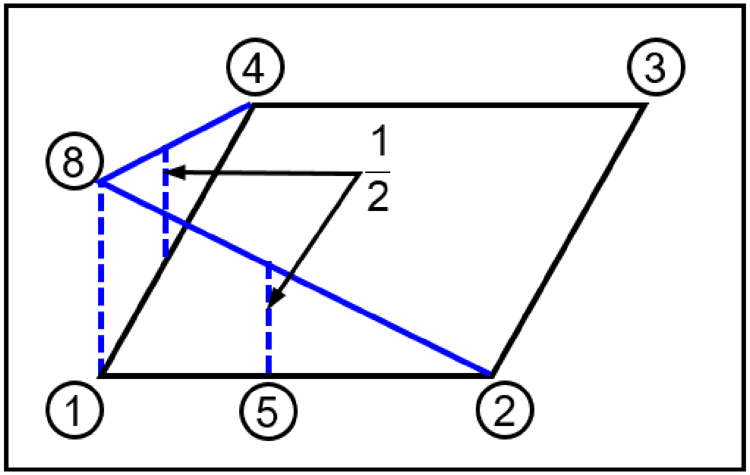
$$N_5(s,t) = \tfrac{1}{2}(1-s^2)(1-t)$$

$$N_6(s,t) = \tfrac{1}{2}(1-t^2)(1+s)$$

$$N_7(s,t) = \tfrac{1}{2}(1-s^2)(1+t)$$

$$N_8(s,t) = \tfrac{1}{2}(1-t^2)(1-s)$$

$$N_1(s,t) = \tfrac{1}{4}(1-s)(1-t) - \tfrac{1}{2}N_5(s,t) - \tfrac{1}{2}N_8(s,t)$$



$$N_1(s,t) = -\tfrac{1}{4}(1-s)(1-t)(s+t+1)$$

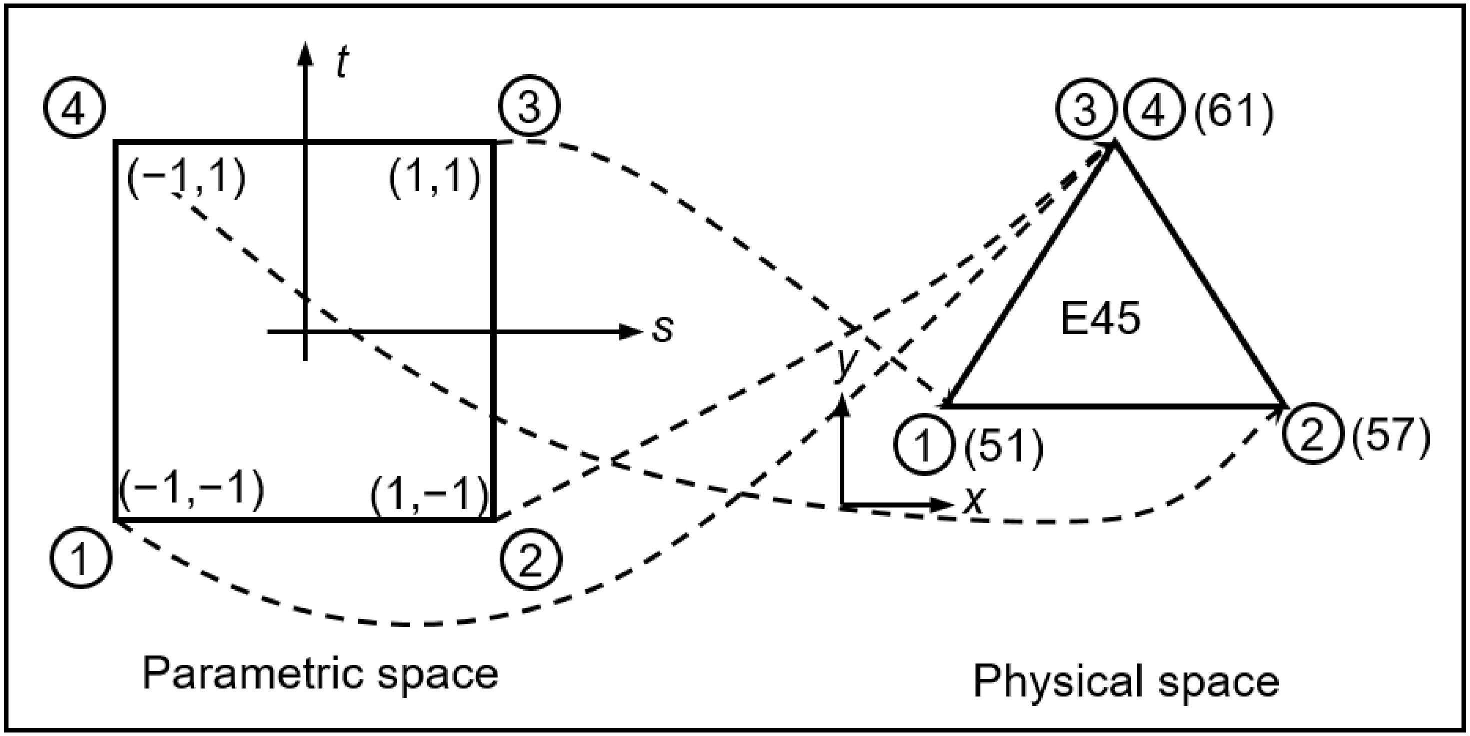
$$N_2(s,t) = \tfrac{1}{4}(1+s)(1-t)(s-t-1)$$

$$N_3(s,t) = \tfrac{1}{4}(1+s)(1+t)(s+t-1)$$

$$N_4(s,t) = \tfrac{1}{4}(1-s)(1+t)(t-s-1)$$

7.6 TRIANGULAR ELEMENT

Collapsed 4-Node Quadrilateral Element

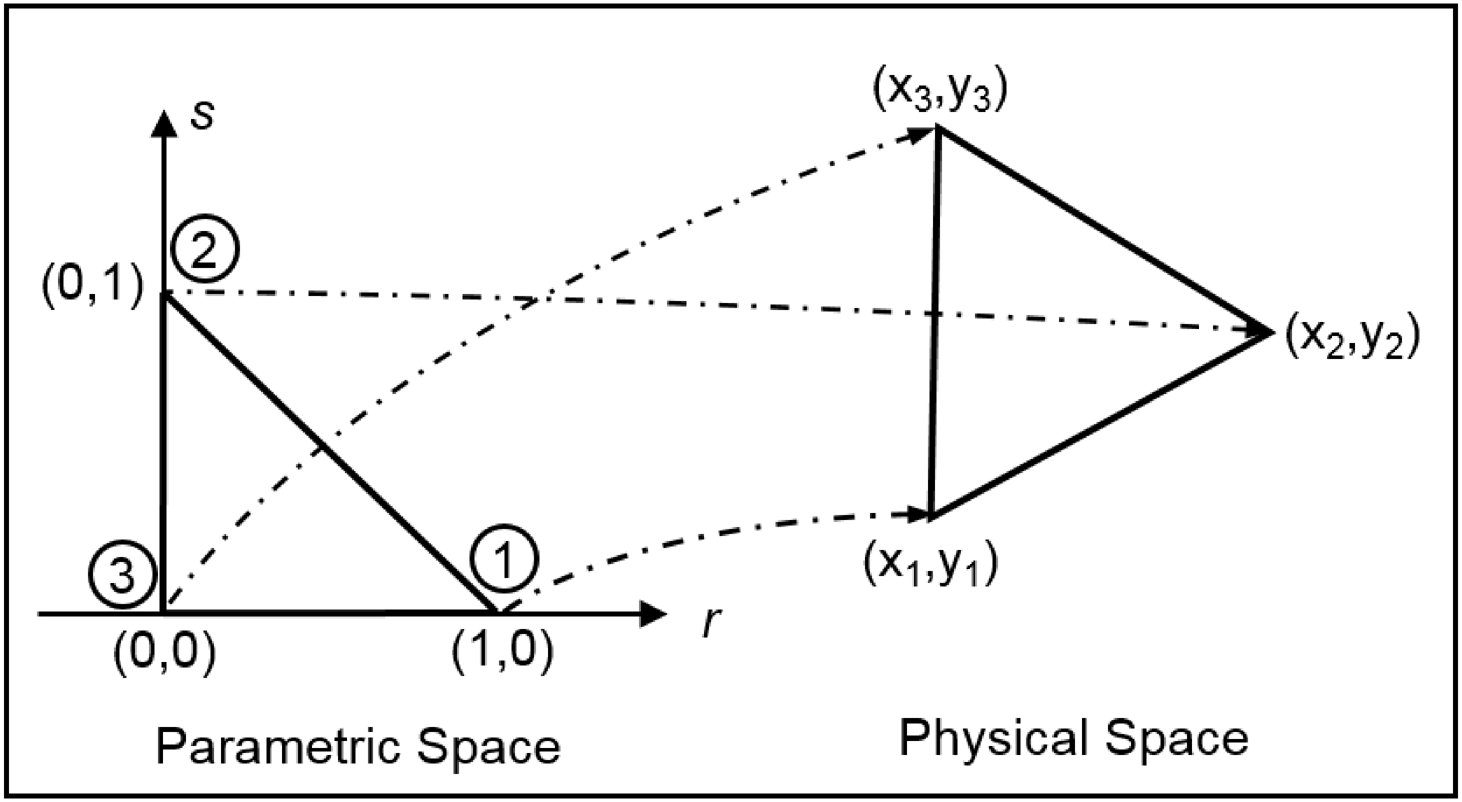


$$T(s,t) = N_1(s,t)T_1 + N_2(s,t)T_2 + (N_3(s,t) + N_4(s,t))T_3$$
$$= N_1(s,t)T_1 + N_2(s,t)T_2 + N_3'(s,t)T_3$$

$$N_1 = \frac{1}{4}(1-s)(1-t) \qquad N_2 = \frac{1}{4}(1+s)(1-t)$$

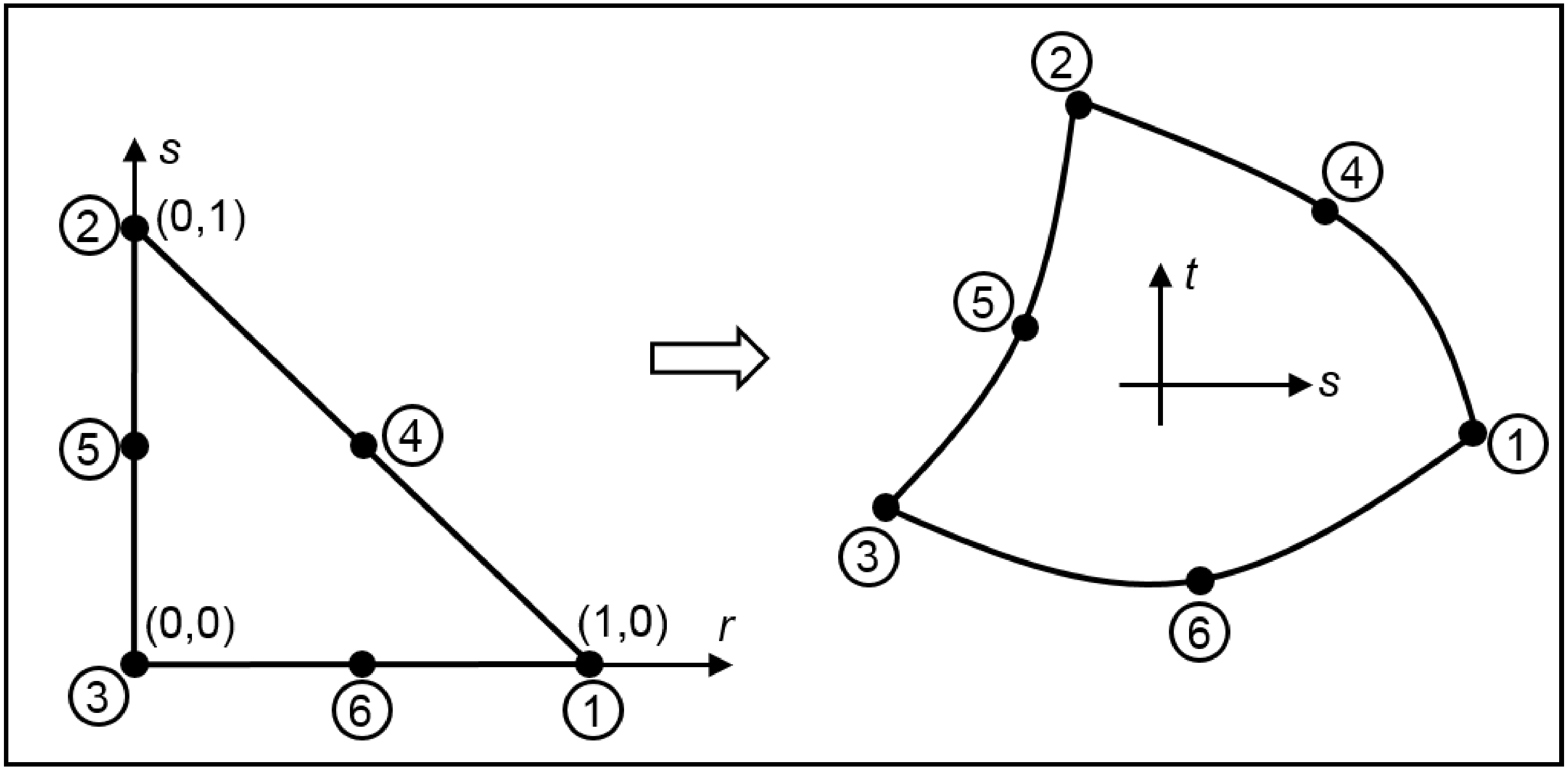
$$N_3' = N_3 + N_4 = \frac{1}{2}(1+t)$$

3-Node Linear Triangular Element



$$N_1 = r \qquad N_2 = s \qquad N_3 = 1 - r - s = t$$

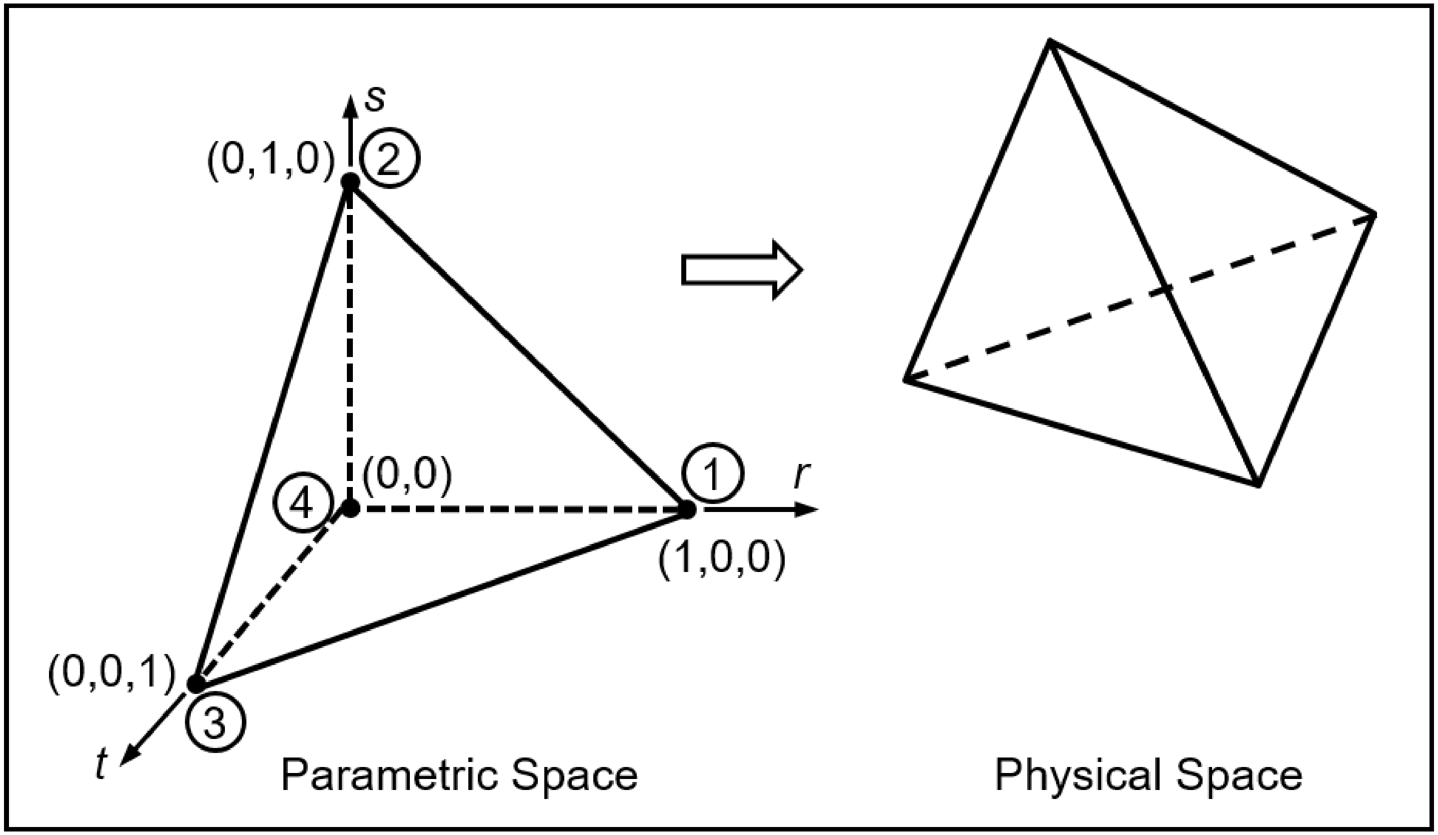
6-Node Quadratic Triangular Element



$$\begin{cases} N_1(r,s,t) = r(2r-1) \\ N_2(r,s,t) = s(2s-1) \\ N_3(r,s,t) = t(2t-1) \\ N_4(r,s,t) = 4rs \\ N_5(r,s,t) = 4st \\ N_6(r,s,t) = 4rt \end{cases}$$

7.8 3D SOLID ELEMENT

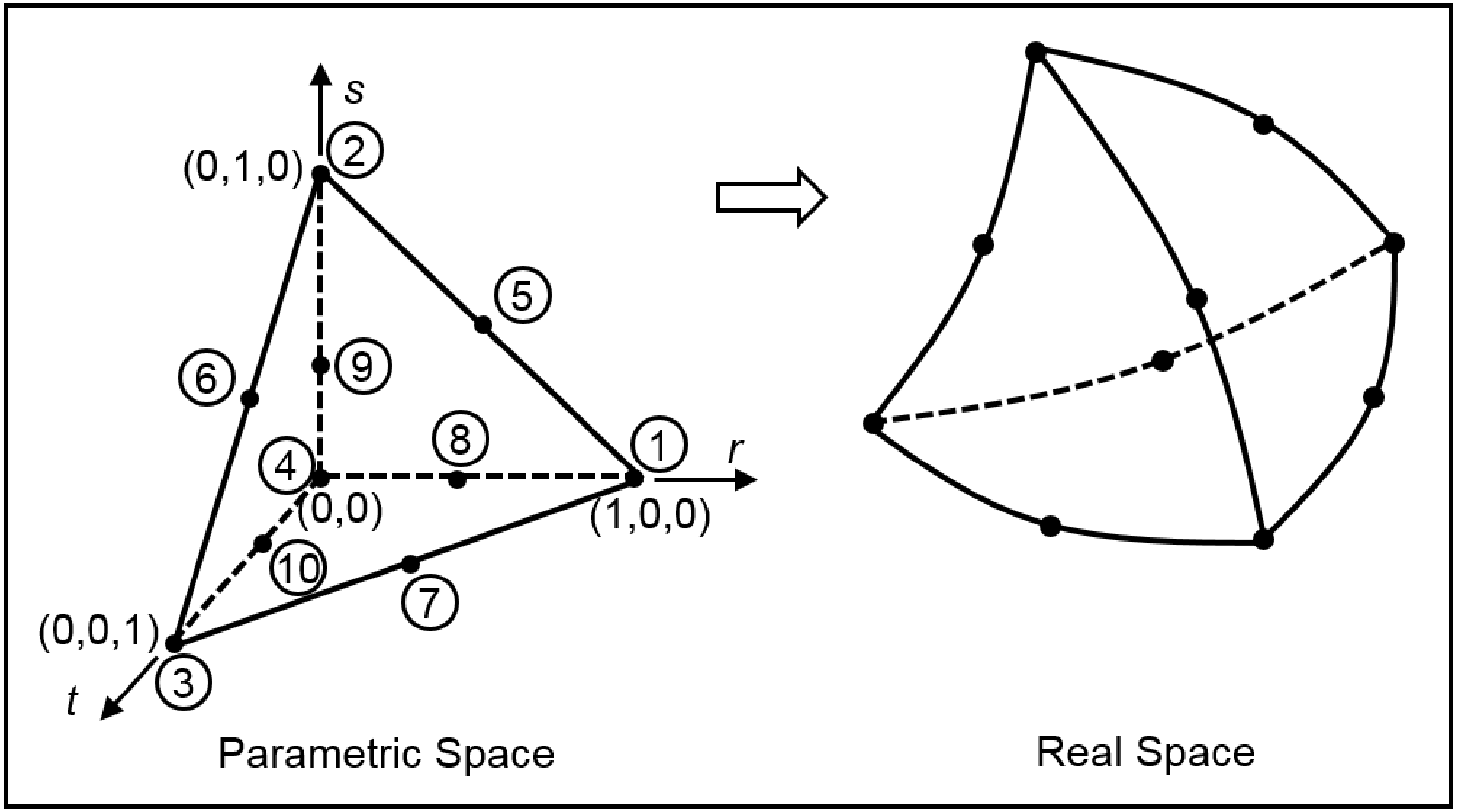
4-Node Tetrahedral Element



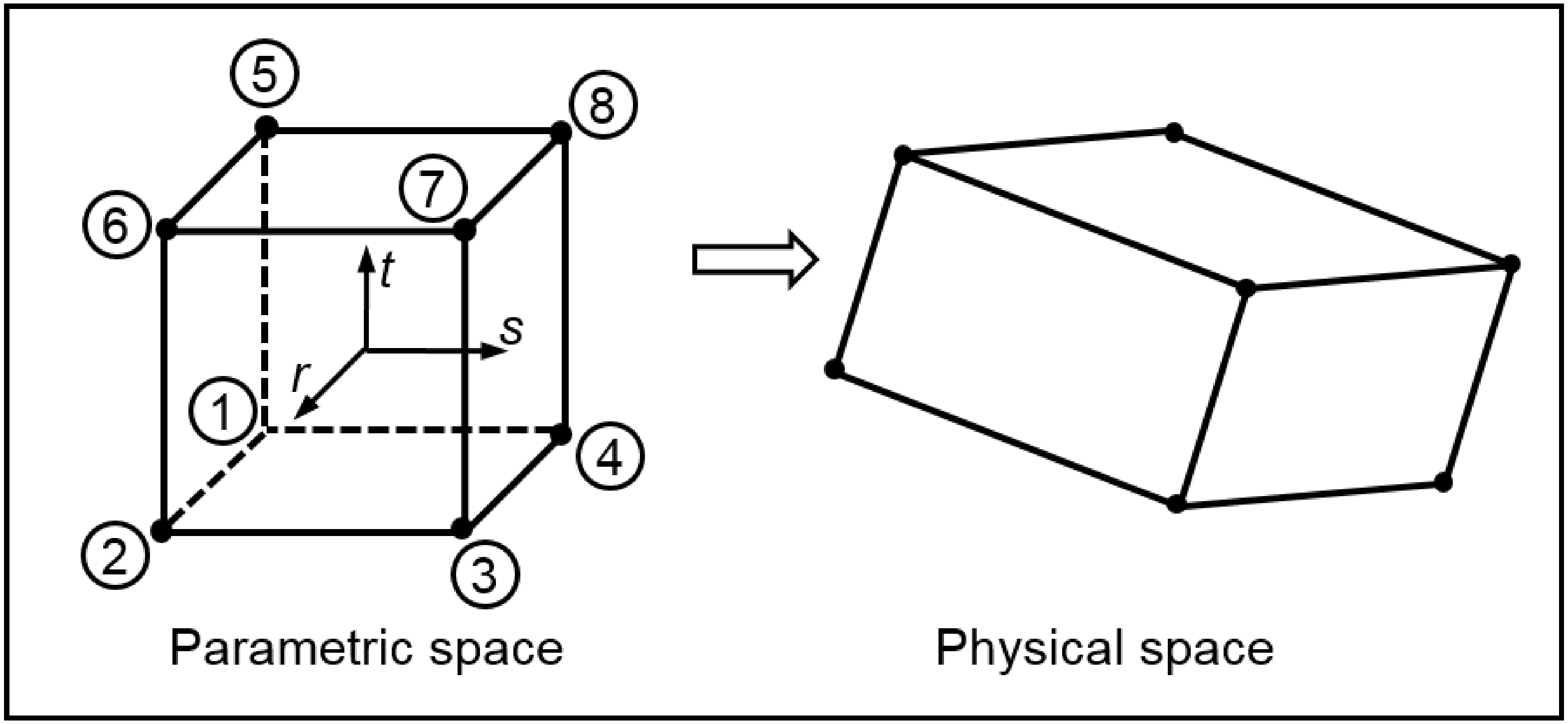
$$\begin{cases} N_1 = r \\ N_2 = s \\ N_3 = t \\ N_4 = 1 - r - s - t \end{cases}$$

10-Node (Quadratic) Tetrahedral Element

$$
\begin{cases}
N_1 = r(2r-1) \\
N_2 = s(2s-1) \\
N_3 = t(2t-1) \\
N_4 = u(2u-1) \\
N_5 = 4rs \\
N_6 = 4st \\
N_7 = 4rt \\
N_8 = 4ru \\
N_9 = 4su \\
N_{10} = 4tu
\end{cases}
$$



8-Node Hexahedral (Brick Element)



$$N_i(r,s,t) = \frac{1}{8}(1 + r_i r)(1 + s_i s)(1 + t_i t)$$