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Multiple tail median approach for high reliability estimation

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ABSTRACT

Sampling-based reliability estimation with expensive computer models may be computationally prohibitive when the probability of failure is low (or high reliability). One way to alleviate the computational expense is to extrapolate reliability estimates from observed levels to unobserved levels. Classical tail modeling techniques, two of which are discussed in this paper provide extrapolation models using asymptotic theory by approximating the tail of the cumulative distribution function (CDF). This paper proposes three additional tail extrapolation techniques in performance space. The proposed tail extrapolations are based on the application of nonlinear transformation to the CDF of the performance measure. The proposed approach called the multiple tail median employs all the five techniques simultaneously and uses the median as the best estimate. The range of the five estimates is used as an estimate of the order of magnitude of error in the median. The method is demonstrated on four standard statistical distributions and two engineering examples. It is found that the best tail model changes for different distributions. Also, for the same distribution no single model performed best at different extrapolation levels. Thus, no single tail model is preferable. We also find that the median is usually much closer to the best of the five estimates than to the worst, and that the range mostly varies between 2 and 10 times the magnitude of the error in the median. Therefore the median estimate serves as insurance against bad predictions if one was to use a single estimate. For the examples studied, the use of tail modeling reduced the number of samples required for given accuracy by one to three orders of magnitude.

1. Introduction

Approaches available for reliability assessment and analysis can be widely classified as analytical and simulation approaches. Analytical approaches use available knowledge of the system but are often limited to single failure modes, whereas simulation methods like Monte Carlo simulation (MCS) are computationally intensive but can handle multiple failure modes. Moreover, they can handle any type of performance functions that dictate the behavior of the system, unlike analytical approaches which are mostly appropriate for mildly nonlinear performance functions. Many real life applications exhibit multiple failure modes and the performance function is not available explicitly in a closed form, rather is available through an algorithm such as finite element analysis. Since there is no information on nonlinearity of the performance function, MCS is the obvious choice in such situations. However, reliability analysis using MCS is computationally prohibitive. Researchers develop variants of MCS or other approximation methods like response surface or surrogate metamodels that replace the expensive simulations.

High reliability, typical of aerospace applications translates to small probability of failure, determined by the tails of the statistical distributions. In some cases, the safety levels can vary by an order of magnitude with slight modifications in the tails of the response variables [1]. Therefore, the tails need to be modeled accurately. Limitations in computational power prevent us in employing MCS to model the tails. One way to alleviate the computational expense is to extrapolate into high reliability levels with limited data at lower reliability levels. Statistical techniques from extreme value theory (referred to as classical tail modeling techniques here) are available to perform this extrapolation. The basic idea in tail modeling techniques is to approximate the conditional cumulative distribution function (CDF) above a certain threshold by the Generalized Pareto Distribution (GPD) [2]. In order to do this, one needs to estimate the parameters of GPD. There are several competing methods available for parameter estimation. This paper uses the maximum likelihood and least-square regression techniques.

In addition to the GPD-based techniques, we propose three additional extrapolation techniques in the performance space. The first technique applies a nonlinear transformation to the CDF of the performance measure and approximates the tail of the

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Nomenclature

a, b	distribution parameters for S_r data
a_0, b_0	distribution parameters for native g_r data
D	tip displacement, Eq. (17)
D_0	allowable deflection, Eq. (17)
E	Young's modulus, Eq. (17)
e_i	error in individual methods, Eq. (13)
e_{MTM}	error in the MTM estimate, Eq. (14)
$F_G(g)$	CDF of G, Eq. (6)
$F_G(u)$	CDF of G at u, Eq.(6)
$F_u(z)$	conditional CDF, Eq. (4)
$\hat{F}_{\xi,\psi}$	approximated conditional CDF, Eq. (3)
F_X	load in X direction, Eq. (16)
F_Y	load in Y direction, Eq. (16)
G	performance measure, Eq. (2)
G_d	displacement performance measure, Eq. (17)
G_p	$(p \times N)$th quantile of G, Eq. (7)
G_s	stress performance measure, Eq. (16)
g_c	capacity, Eq. (8)
g_r	response, Eq. (8)
L	length, Eq. (17)
^5m	median of five estimates, Eq. (14)
$^{1000}m_e$	median of MTM error over 1000 repetitions
$^{1000}m_\eta$	median of η over 1000 repetitions
N	total number of samples
N_{ex}	number of exceedances (samples in tail region)
P_i	empirical CDF, Eq. (10)
p	probability
P_f	failure probability, Eq. (1)
R	yield strength, Eq. (16)
^5r	range of five estimates, Eq. (12)
S_r	reciprocal of conventional safety factor, Eq. (8)
t	thickness, Eq. (16)
u	threshold for samples assumed to lie in tail region, Eq.(2)
w	width, Eq. (16)
$\bar{x}$	mean of data
$\hat{\bar{x}}$	normalized mean
z	exceedance, Eq. (2)
β	reliability index, Eq. (1)
ε_2	transverse strain in first ply, Eq. (18)
ε_2^U	upper bound of allowable strain, Eq. (18)
η	mean(error in MTM)/range of the five estimates, Eq. (15)
ξ	shape parameter, Eq. (3)
$^{1000}\mu_e$	mean of MTM error over 1000 repetitions
$^{1000}\mu_\eta$	mean of η over 1000 repetitions
σ_{comp}	computed stress, Eq. (16)
Φ	standard normal cumulative distribution function (CDF)
ψ	scale parameter, Eq. (3)
Beta-LT	fit a linear polynomial to the tail data Inverse normal cumulative distribution function applied to the CDF of S_r
Beta-QH	fit a quadratic polynomial to half of the data. Inverse normal cumulative distribution function applied to the CDF of S_r
GPD	Generalized Pareto Distribution
LnBeta-QT	fit a quadratic polynomial to the tail data. Logarithmic transformation applied to the beta transformed CDF
ML	maximum likelihood
MTM	multiple tail median
Reg	regression

transformed CDF using a linear polynomial fit to about top 10% of the data. The second technique approximates the upper half of the transformed CDF by a quadratic polynomial. The third technique applies a logarithmic transformation to the already transformed CDF and approximates the tail with a quadratic polynomial. It is to be noted that all five techniques do not approximate the functional expression of the model output; rather they approximate the tail of CDF. Thus, they do not need to be tailored to any functional form of the output.

Goel et al. [3] developed a method to extend the utility of an ensemble of surrogates. When faced with multiple surrogates, they explored the possibility of using the best surrogate or a weighted average surrogate model instead of individual surrogate models. In a similar fashion, in order to take advantage of both the classical tail modeling techniques and alternative extrapolation schemes for achieving the best prediction, we propose to apply all the techniques simultaneously and use the median of the five estimates as the compromise best estimate. Using all the techniques simultaneously is referred to multiple tail median (MTM).

The paper is structured as follows. Section 2 illustrates the five tail modeling techniques. Classical tail modeling concepts and alternative extrapolation schemes developed in this work are presented in Section 3. Section 4 discusses the proposed multiple tail median (MTM) approach. The MTM approach is demonstrated on standard statistical distributions followed by two engineering examples in Section 5.

2. Tail extrapolation illustration

It would be beneficial to outline the proposed MTM before explaining detailed extrapolation schemes. For that purpose, a log-normally distributed random variable G with a data mean of 10 and standard deviation of 3 is taken as an example. Although the proposed MTM approach does not require any assumptions on distribution type, we use a particular distribution in order to illustrate the accuracy of the models. We assume that failure is defined when G is large so that we are interested in extrapolating the right tail of G. First, a set of 500 samples is generated according to its distribution parameters. It is assumed that the tail of the CDF starts at 0.9 so that 50 samples are there beyond the threshold. This tail distribution is estimated using two classical tail modeling techniques (described in Section 3), the maximum likelihood and regression approaches. These are plotted in Fig. 1. In order to compare the behavior at the tail, 1 – CDF is plotted in logarithmic scale. For reference, the empirical CDF using 500 and 100,000 samples are also plotted. It is clear that 500 samples (50 samples in tail) are too few to predict the tail at the probability level of 0.001, while the classical tail models are able to predict it with only moderate errors.

Since the GPD tail models are a form of extrapolation, polynomial-based extrapolations can also be applied. The first proposed extrapolation scheme is to perform a nonlinear transformation of the CDF into the reliability index β and fit the tail portion (top 10% of the data) with a linear polynomial. This method is referred to Linear Tail (Beta-LT). Reliability index and failure probability P_f are related as:

$$P_f = \Phi(-\beta) \tag{1}$$

where $\Phi(\cdot)$ is the CDF of the standard normal random variable.

We also propose to fit half of the samples with a quadratic polynomial and refer to this approach as Quadratic Half (Beta-QH). Fig. 2 shows these two extrapolation schemes along with empirical CDFs (failure probabilities values are presented in the right axis). Note that these two extrapolations can also predict tail with some

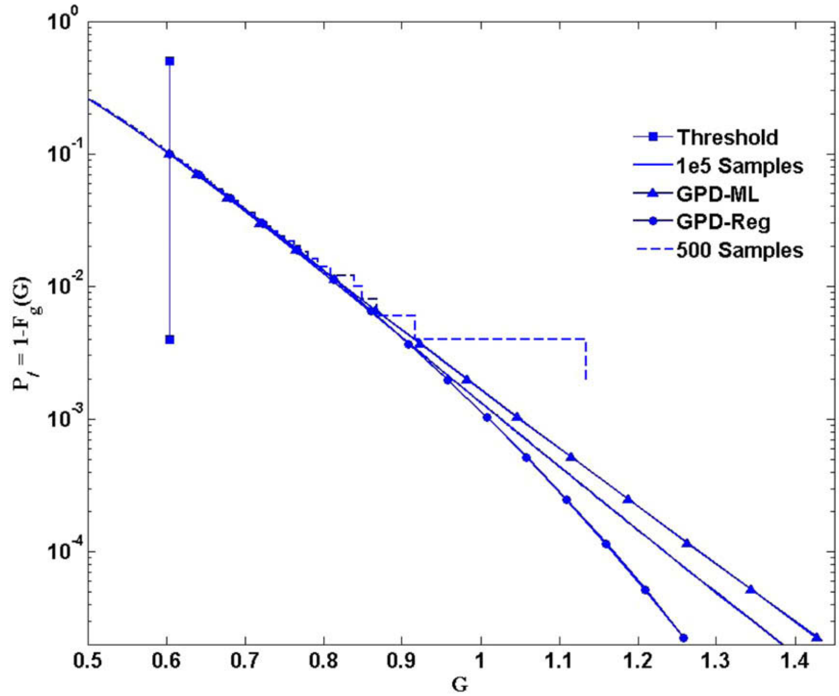

Fig. 1. Lognormal distribution. Classical tail modeling techniques – GP-ML (the tail model using maximum likelihood) and GPD-Reg, (using regression).

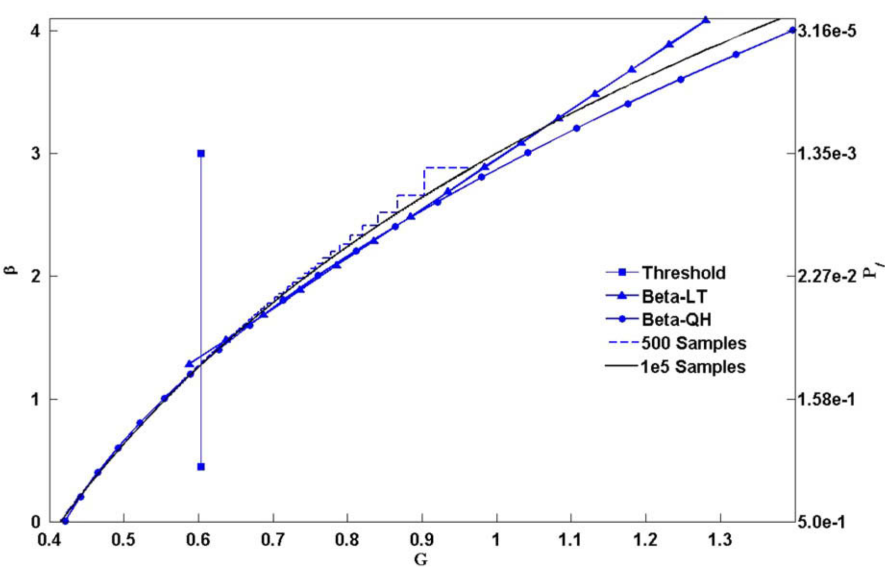

Fig. 2. Lognormal distribution. Linear fit to tail (Beta-LT) and Quadratic fit to half of the data (Beta-QH).

errors. In addition, Beta-LT is better at reliability index of 3.5, while Beta-QH is better at 2.0. The third method we propose further applies a logarithmic transformation to the transformed CDF, and the tail is approximated by a quadratic fit as shown in Fig. 3. This method is referred to Quadratic Tail (LnBeta-QT).

It can be concluded from Figs. 1–3 that one cannot gauge which method performs best for extrapolation. Since all five tail models are forms of surrogates, a technique similar to the multiple surrogate model [3] can be used to improve the quality of extrapolation. The multiple tail median (MTM) approach utilizes all five models and takes the median as a compromise estimate. Fig. 4 shows the tail estimate using MTM along with the individual methods. Although the MTM is not always the best, it is usually the second best estimate as will be shown in Sections 4 and 5. In Fig. 4, the range of the five methods at three different reliability indices is marked by double headed arrows. Mean of these ranges computed over the entire extrapolation zone are capable of representing the error associated with the median estimate from MTM. A representative actual error is presented in the plot.

It is to be noted that in all the five methods discussed above, the abscissa is same while the ordinate varies. For the purpose of comparison, the error between the estimate and exact is measured in terms of the abscissa quantity, the performance measure rather than as a difference in the ordinate quantity (failure probability or reliability index).

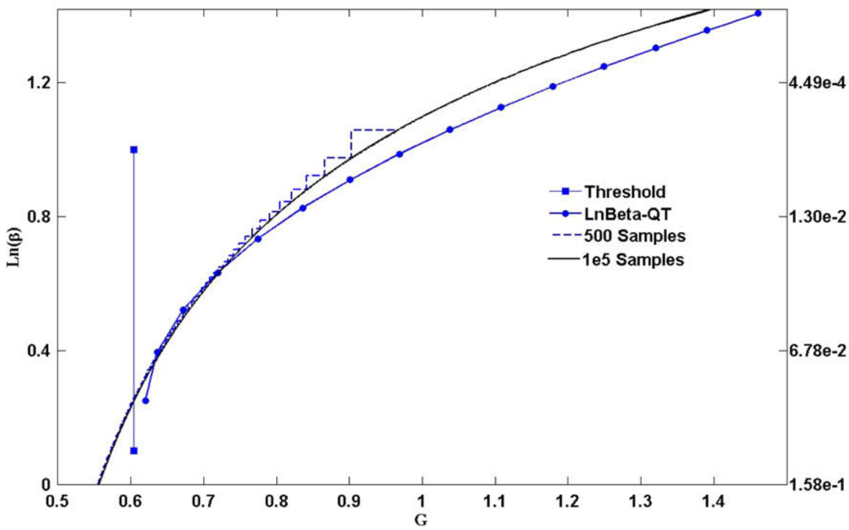

Fig. 3. Lognormal distribution. Quadratic fit to the tail (LnBeta-QT).

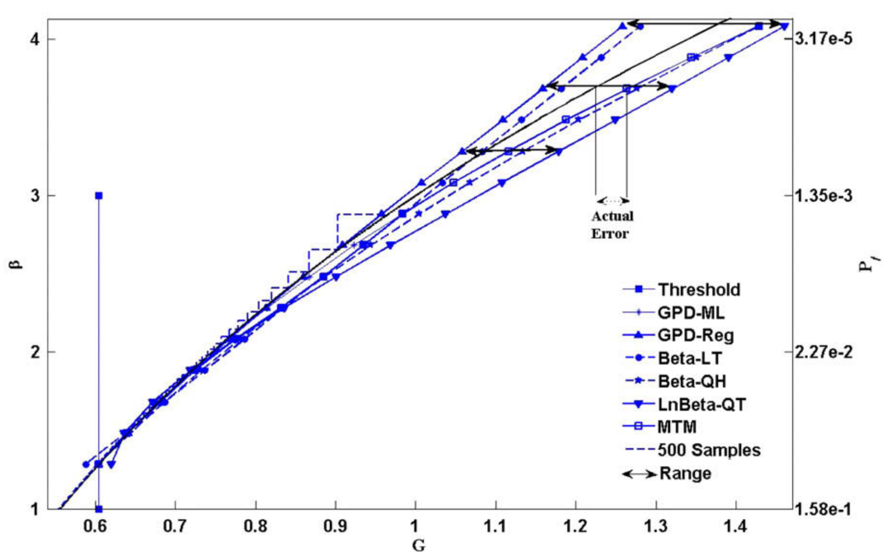

Fig. 4. Lognormal distribution. Multiple tail median-extrapolation region.

3. Classical (GPD-based) tail modeling and alternative tail extrapolation schemes

Tail modeling techniques are based on extreme value theory to predict the probability of extreme events. The theory comprises a principle for model extrapolation based on the implementation of mathematical limits as finite level approximations. Since several advantages are reported by working in performance measure space [4], it is logical to attempt to perform tail modeling in the performance measure space to estimate quantities at unobserved levels. This section discusses the tail modeling technique and how to apply it to extrapolate a performance measure. The extrapolation idea is based on the assumption of the continuity of the tail beyond existing data which from a strict mathematical point of view cannot be proved [1], p. 6.

In tail modeling, the interest is to address the excesses over a threshold. In these situations, the Generalized Pareto Distribution (GPD) arises as the limiting distribution. The concept of GPD is presented in Fig. 5. Let G be a performance measure which is random and u be a large threshold of G. The observations of G that exceed u are called exceedance, z, which is expressed as:

$$z = G - u \tag{2}$$

The conditional CDF $F_u(z)$ of the exceedance given that the data G is greater than the threshold u, is modeled fairly well by the GPD. Let approximation of $F_u(z)$ using GPD be denoted by $\hat{F}_{\xi,\psi}(z)$ where ξ *and* ψ are shape and scale parameters, respectively. For a large enough u, the distribution function of $(G-u)$, conditional on $G > u$, is approximately written as [5]:

$$\hat{F}_{\xi,\psi}(z) = \begin{cases} 1 - \left\langle 1 + \frac{\xi}{\psi} z \right\rangle_+^{-\frac{1}{\xi}} & \text{if} \quad \xi \neq 0 \\ 1 - \exp(-\frac{z}{\psi}) & \text{if} \quad \xi = 0 \end{cases} \tag{3}$$

In Eq. (3), $\langle A \rangle_+ = \max(0, A)$ and $z > 0$. In GPD, ξ plays a key role in assessing the weight of the tail. When ξ is closer to 1, the distribution is classified as heavy tailed; when $\xi < 1$, the distribution is light tailed; medium tail exponential distribution has $\xi \approx 0$; and no

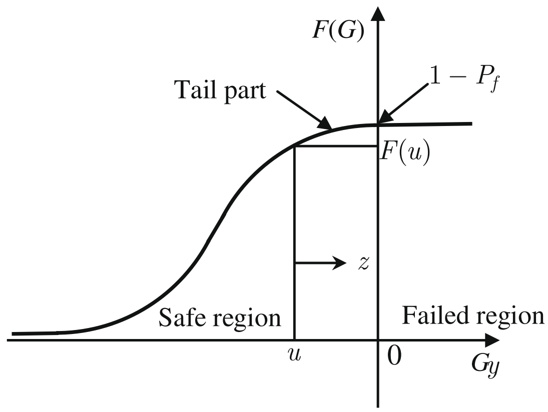

Fig. 5. Tail modeling of $F(u)$ using the threshold u. The region of $G > 0$ is failure.

tail uniform distribution has $\xi \approx -1$. Eq. (3) can be examined by changing its parameters and plotting the distribution above the threshold. Fig. 6 shows the different cumulative distributions that are generated from the GPD when the scale parameter, σ, is fixed to one, and the threshold, u, is selected such that $F(u) = 0.90$. The standard distributions discussed in this paper are selected in such a way that tails with different heaviness are covered.

It is noted that conditional excess CDF $F_u(z)$ is related to the CDF of interest $F_G(g)$ through the following expression:

$$F_u(z) = \frac{F_G(g) - F_G(u)}{1 - F_G(u)} \tag{4}$$

From Eq. (4), the CDF of G can be expressed as:

$$F_G(g) = (1 - F_G(u))F_u(z) + F_G \tag{5}$$

Substituting $F_u(z)$ from Eq. (2), Eq. (5) becomes:

$$F_G(g) = 1 - (1 - F_G(u))\left\langle 1 + \frac{\xi}{\psi}(G - u)\right\rangle_+^{-\frac{1}{\xi}} \tag{6}$$

For simplicity of presentation, only the case of $\xi \neq 0$ is considered here. The shape and scale parameters can be estimated using either the maximum likelihood estimation or least-square regression method. Let N be the total number of samples and p be a probability level. Once we obtain estimates of the parameters as $\hat{\xi}$ and $\hat{\psi}$, it is possible to estimate the $(p \times N)^{\text{th}}$ quantile of G denoted as G_p by inverting Eq. (6):

$$G_p = F_G^{-1}(p) = u + \frac{\psi}{\xi}\left(\left(\frac{1-p}{1-F_G(u)}\right)^{-\xi} - 1\right) \tag{7}$$

In structural applications, the performance measure is often defined as a difference between the capacity of a system g_c (e.g., allowable strength) and the response g_r (e.g., maximum stress). For the convenience of the following developments, we normalize the performance measure using the capacity. Thus, we have

$$G = \frac{g_c - g_r}{g_c} = 1 - S_r \tag{8}$$

where S_r is the reciprocal of the conventional safety factor. Failure occurs when $G > 0$, while the system is safe when $G < 0$. For the performance measure in the form of Eq. (8), we need to approximate the upper tail distribution.

The accuracy of this approach depends on the choice of the threshold value u. Selection of threshold is a tradeoff between bias and variance. If the threshold selected is too low, then some data points belong to the central part of the distribution and do not provide a good approximation to the tails. On the other hand, if the threshold selected is too high, the data available for the tail approximation are too few and this might lead to excessive scatter in the final estimate. The proper selection of threshold has important repercussions on the estimated value of the shape factor [1,6] and hence on the final estimates such as the quantile. Boos [7] suggests that the ratio of N_{ex} (number of tail data) over N (total number data) should be 0.02 ($50 \leqslant N < 500$) and the ratio should be 0.1 for $500 \leqslant N < 1000$. Hasofer [8], suggests using $N_{ex} = 1.5\sqrt{N}$. Caers and Maes [1], propose to use a finite sample mean square error (MSE) as a criterion for estimating the threshold. They use the threshold value that minimizes the MSE. In a similar fashion, Beirlant et al. [9], find an optimal threshold by minimizing an approximate expression for asymptotic mean square error. The other methods include plotting the quantile, shape or scale factor or any quantity of interest with respect to different thresholds and look for stability in the curve [5, pp. 84–86, 10]. Examples

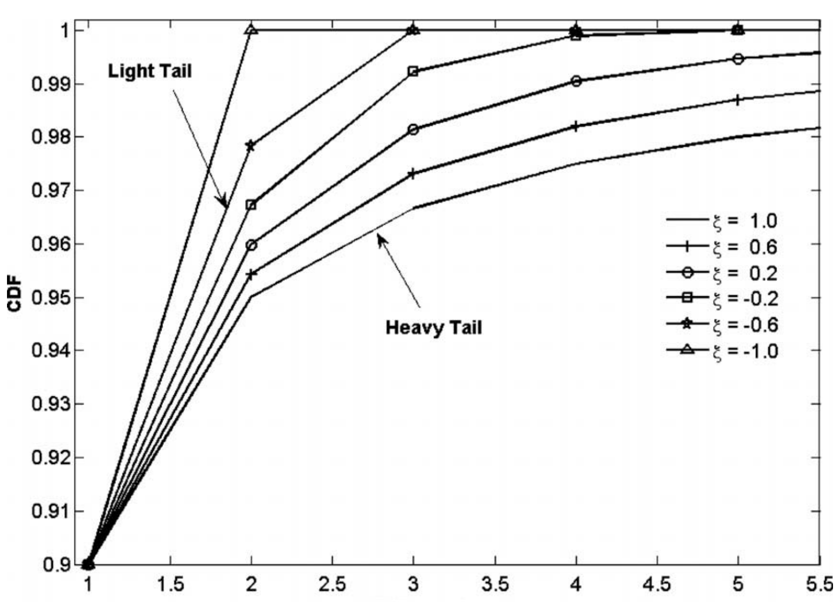

Fig. 6. Generalized Pareto Distribution for different shape parameters.

discussed in this paper use $N = 500$. Therefore, we adopt what Boos [7] suggested and use the 90% quantile as the threshold ($N_{ex} = 50$).

There are several methods such as maximum likelihood (MLE) and regression to estimate the parameters, $\hat{\xi}$ and $\hat{\psi}$. MLE is a popular statistical method that is used to make inferences about the parameters of the underlying probability distribution of a given data set. The likelihood of a set of data is the probability of obtaining that particular set of data, given the chosen probability distribution model. MLE is based on a likelihood function, which contains the unknown distribution parameters. The values of these parameters that maximize the likelihood function are the maximum likelihood estimators. The maximum likelihood method is discussed in detail in [5].

The method of least squares minimizes the sum of the deviations squared (*least square error*) from a given set of data. The parameters are obtained by solving the following minimization problem

$$\underset{\xi,\psi}{Min} \sum_{i=n}^{N} (F_G(g_i) - P_i)^2 \tag{9}$$

where P_i is the empirical CDF and $F_G(g_i)$ is the CDF of G in Eq. (6). The empirical CDF is computed as:

$$P_i = \frac{i}{N+1}, \quad i = 1, \ldots, N \tag{10}$$

where N is the total number of samples. Least-square regression requires no or minimal distributional assumptions.

Table 1
The four standard statistical distributions and their parameters.

Distribution	Parameters	
	a	b
Uniform	$\bar{x} - \frac{\sqrt{12}}{2}\sigma$	$\bar{x} - \frac{\sqrt{12}}{2}\sigma$
Normal	$\bar{x}$	σ
Exponential[a]	$\bar{x} - \sigma$	$\bar{x}$
LogNormal	$\ln(\bar{x}) - 0.5(b^2)$	$\sqrt{\ln\left[1 + \left(\frac{\sigma}{\bar{x}}\right)^2\right]}$

[a] Single parameter distribution.

Table 2
Normal distribution. Summary of errors in S_r for different techniques at different reliability indices (mean and median over 1000 repetitions of 500 samples).

Rel index		3	3.6	4.2
GP-MLE	Mean	0.038	0.096	0.184
	Median	0.037	0.096	0.191
GP-Reg	Mean	0.010	0.043	0.116
	Median	0.008	0.042	0.117
LnBeta-QT	Mean	0.071	0.113	0.150
	Median	0.056	0.085	0.105
Beta-LT	Mean	0.044	0.062	0.081
	Median	0.037	0.053	0.068
Beta-QH	Mean	0.043	0.070	0.103
	Median	0.037	0.059	0.087
MTM	Mean	0.043	0.061	0.080
	Median	0.037	0.053	0.068

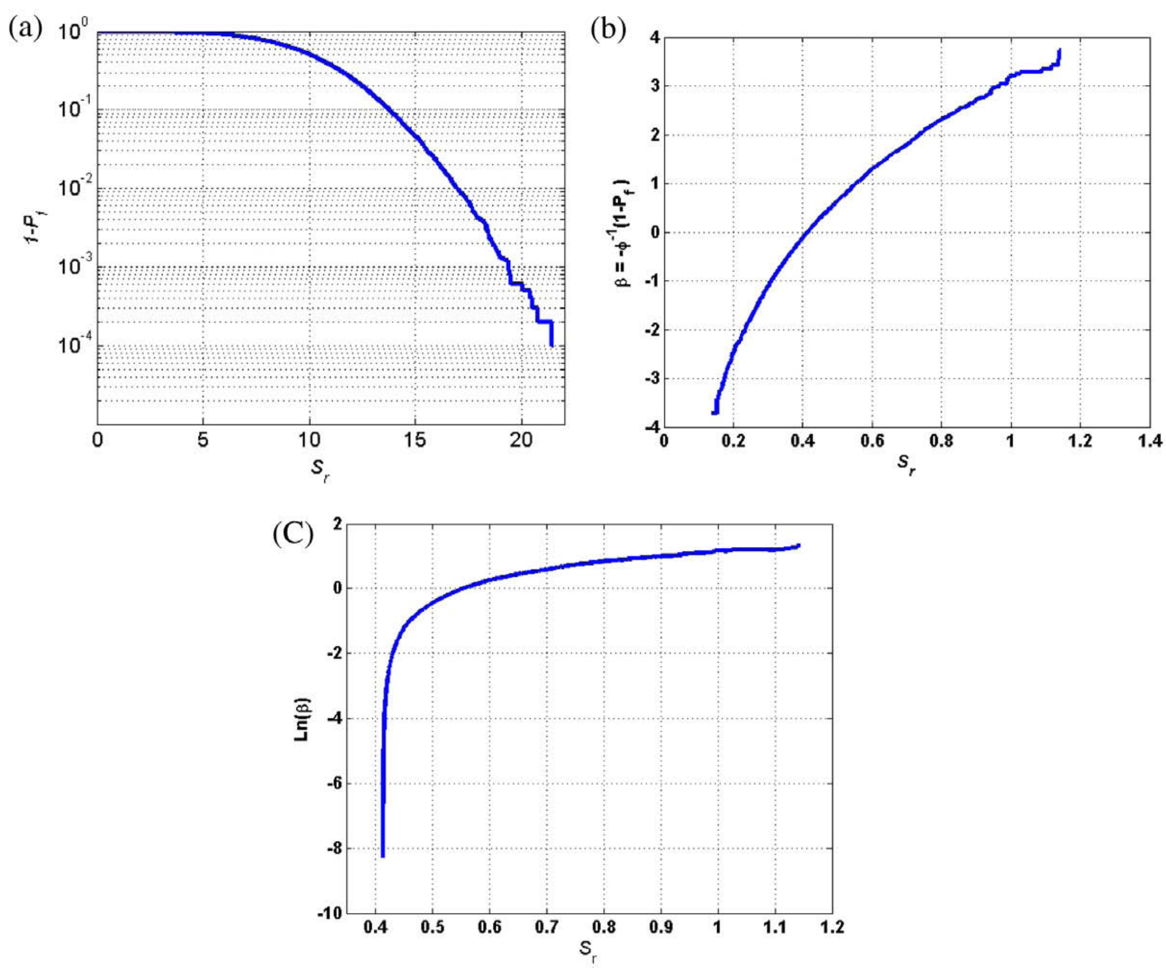

Fig. 7. Transformation of the CDF of safety factor reciprocal (S_r). (a) CDF of S_r. (b) Inverse standard normal cumulative distribution function applied to the CDF. (c) Logarithmic transformation applied to the reliability index.

Table 3
Normal distribution. Summary of lowest and highest error to MTM error (mean and median over 1000 repetitions of 500 samples).

Rel index	Lowest error/MTM error				Highest error/MTM error			
	25 percentile	Mean	Median	75 percentile	25 percentile	Mean	Median	75 percentile
3	0.08	0.18	0.25	0.34	1.57	1.77	2.14	1.99
3.6	0.26	0.58	0.60	1.00	1.92	2.19	6.80	3.51
4.2	0.56	1.00	0.79	1.00	2.13	2.51	20.74	7.13

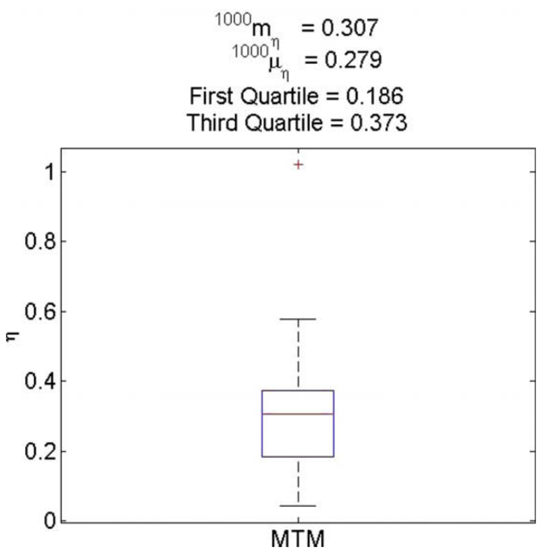

Fig. 8. Normal distribution. Boxplot of η.

Table 5
Summary of the median ratio (η) of MTM Error in S_r to range for different distributions.

Distribution	$^{1000}m_{\eta}$
Uniform	0.003
Normal	0.31
Exponential	0.23
Lognormal	0.20

Table 6
Number of samples (without extrapolation) required to achieve error levels obtained using MTM.

Distribution	Number of samples
Uniform	500,000
Normal	500,000
Exponential	500,000
Lognormal	350,000

In addition to the previous two classical tail modeling techniques, additional tail extrapolation techniques are proposed to estimate S_r, the reciprocal of the safety factor, for low failure probability with samples that is sufficient only to estimate S_r for substantially high failure probability (low reliability index). Failure probability can be transformed to reliability index by using Eq. (1). The same transformation is applied here to the CDF of S_r. The tail of the resulting transformed CDF is approximated by a linear polynomial in reliability index in order to take advantage of fact that normally distributed S_r will be linearly related to the reliability index. This transformation is denoted by Beta-LT. Since this approximation will not be accurate enough if S_r follows distributions very different from normal, the second technique approximates the relationship between S_r and reliability index from the mean to the maximum data (about half of the sample) using a quadratic polynomial and denoted as Beta-QH. The third technique further applies a logarithmic transformation to the reliability index of tail data that tends to linearize the tail of the transformed CDF. This tail is approximated using a quadratic polynomial in $\ln(\beta)$ and is denoted by LnBeta-QT. The three transformations are described with the help of Fig. 7. A data set of $N = 500$ with a mean of 10 and variance 9 following a lognormal distribution is used to illustrate the three techniques. In this paper we use least-square regression to find the coefficients. However, MLE approach can also be used to find the coefficients.

Fig. 7a shows the general relationship between S_r and $1 - P_f$. Applying the inverse standard normal CDF transformation modifies the CDF of S_r as in Fig. 7b. If $F(S_r)$ follows a normal distribution, this curve will be linear. Therefore, the first technique approximates the tail of the curve in Fig. 7b by a linear polynomial. In order to take advantage of the data other than the tail, half of the curve in Fig. 7b corresponding to $\beta > 0$ is approximated by a quadratic polynomial in the second technique. To the reliability index in the second technique, the third technique applies a logarithmic transformation and approximates the tail of the curve shown in Fig. 7c by a quadratic polynomial. In all the above three techniques, once the coefficients of the polynomial are obtained, the S_r corresponding to any higher reliability index can be found. It is to be noted that with 500 samples, when using crude Monte Carlo simulation, it is only possible to estimate S_r at reliability levels less than 3. The two classical (GPD-based) tail modeling techniques and the three alternate extrapolation techniques presented in this section allow predicting S_r at high reliability indices without using large samples.

The alternative extrapolation techniques and classical tail modeling techniques are conceptually the same. The major difference in

Table 4
Summary of the performance of individual techniques and MTM for different distributions.

Rel index	3			3.6			4.2		
Performance	1st best	2nd best	MTM	1st best	2nd best	MTM	1st best	2nd best	MTM
Distributions									
Uniform	LnBeta-QT	GP-Reg	1st	GP-Reg	GP-ML	2nd	GP-Reg	GP-MLE	2nd
Normal	GP-Reg	GP-MLE	3rd	GP-Reg	Beta-LT	2nd	Beta-LT	Beta-QH	1st
Exponential	Beta-LT	GP-Reg	3rd	GP-Reg	Beta-LT	2nd	GP-Reg	Beta-QH	3rd
LogNormal	GP-MLE	Beta-QH	2nd	GP-MLE	LnBeta-QT	1st	LnBeta-QT	Beta-QH	1st

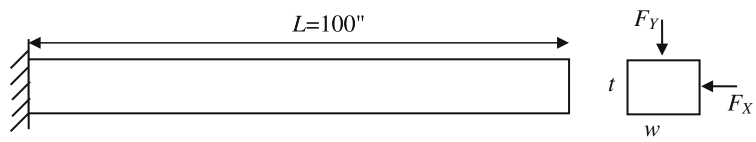

Fig. 9. Cantilever beam subjected to horizontal and vertical loads.

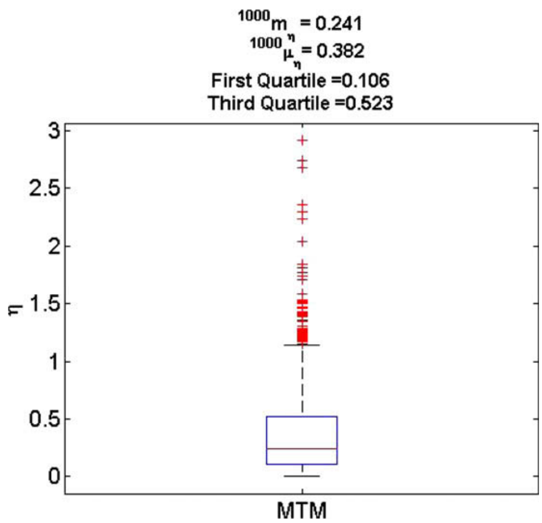

Fig. 10. Cantilever beam. Boxplot of η.

perceiving the two classes is that the classical tail modeling techniques model the CDF of S_r, whereas the extrapolation schemes approximates the trend of S_r in terms of reliability index.

4. Multiple tail median

A total of five methods to model the tails of data were discussed in the previous section. These five methods cover all possible CDF shapes [14] as discussed in Appendix C. As will be seen in the numerical examples the best technique to use may depend on the distribution underlying the sample or even on the level of reliability that is the target of the extrapolation. However, in real life situations where there is no information on the form of the output response distribution, there is no way to tell which technique performs best.

The multiple tail median (MTM) approach applies the five techniques simultaneously and uses the median of the five estimates as a compromise best estimate. It is observed that the median is a more robust estimate than the mean, because the median is less sensitive to the outliers than the mean. MTM is demonstrated first on four different statistical distributions followed by two engineering examples. The GPD parameter for the four distributions is presented in Appendix B and it should be noted that all types of tails are addressed. In the examples, it is observed that the median of the five estimates is usually the second best compared to estimates from individual techniques. Moreover, the range of the five estimates provides an order of magnitude of the error in the median. Thus using multiple tail median not only ensures to choose a good estimate and buys insurance against bad predictions but also provides an estimate of the error. For illustration, we selected to use different response ($x = g_r$) distributions having the same mean $\bar{x} = 2$ and the same coefficient of variance COV = 7.5. In order to compare the performance of the proposed method on different distributions, a common failure probability $P_f = 0.00135 (\beta = 3)$ is used. With these given, one has to find the parameters a_0 and b_0 (from Table 1) of the distribution and the capacity g_c corresponding to the common failure probability. g_c is deterministic in this case. $\bar{x}$ is divided by g_c to obtain the mean of S_r, $\hat{\bar{x}}$. Now, $\hat{\bar{x}}$ and COV are used to obtain the parameters a and b. For each distribution we generate 500 Latin hypercube samples (LHS). For each set of 500 samples we obtain five different estimates of S_r, S_{ri}, $i = 1, \ldots, 5$ at each required higher reliability index. The median (^5m) and range (^5r) of S_{ri} are calculated at every target reliability index. That is

$$^5m = \text{median}(S_{r1}, \ldots, S_{r5}) \tag{11}$$

$$^5r = \max(S_{r1}, \ldots, S_{r5}) - \min(S_{r1}, \ldots, S_{r5}) \tag{12}$$

The exact values of S_r are obtained using inverse CDF functions. The error in the median is calculated as:

$$e_i = |S_{rexact} - S_{ri}| \tag{13}$$

$$e_{\text{MTM}} = |^5m - S_{rexact}| \tag{14}$$

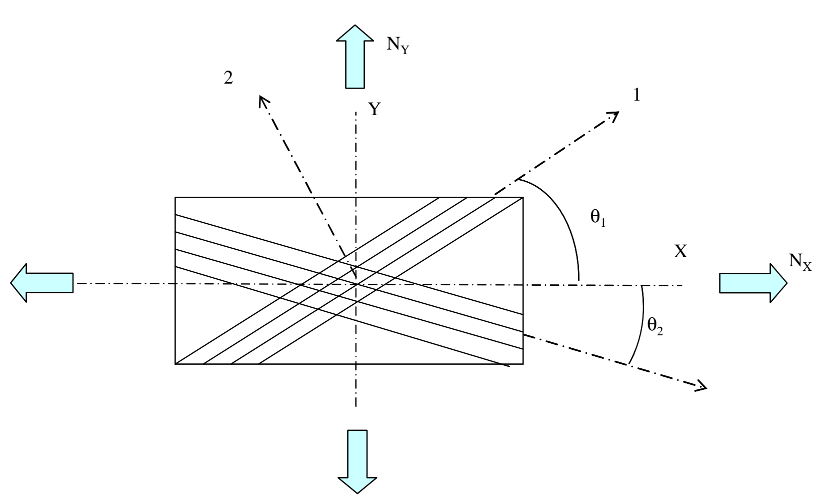

Fig. 11. Geometry and loading for the composite laminate. *X*-hoop direction, *Y*-axial direction.

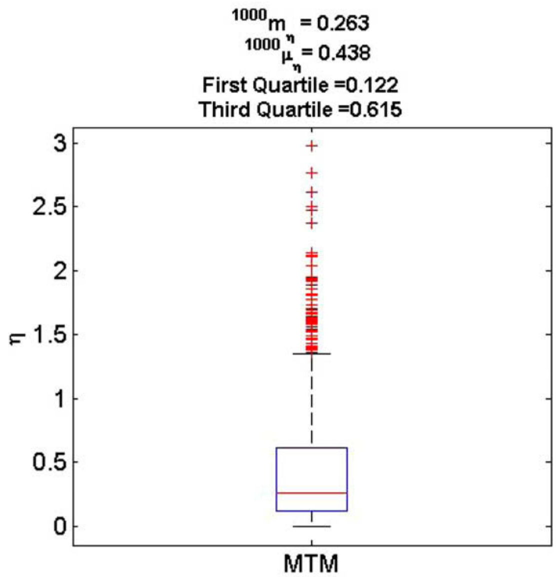

Fig. 12. Composite laminate of cryogenic tank. Boxplot of η ratio.

The average of the error in MTM over all the reliability levels checked (here usually seven though only three are presented) to the average of the range over the same reliability level is η, That is

$$\eta = \sum_{\beta_i} e_{\mathrm{MTM}}(\beta_i) \Bigg/ \sum_{\beta_i} {}^5r(\beta_i) \tag{15}$$

To average out the dependence on a random sample, the error estimation and the ratio computation are repeated 1000 times with different samples, and the mean and median values of the errors are recorded as ${}^{1000}\mu_e$ and ${}^{1000}m_e$, respectively. The implementation of this study is presented in Appendix A.

For each distribution, ${}^{1000}\mu_e$ and ${}^{1000}m_e$ of individual errors and MTM error are calculated for reliability indices between 3 and 4.2. The results for the normal distribution are presented in Table 2. It was observed that the median was less sensitive to the outliers than the mean. At $\beta = 3$, the regression (GPD-Reg) performs best followed by the linear tail (Beta-LT). The MTM performs as well as the second best. At $\beta = 3.6$, Beta-LT performs the best followed by the regression (GPD-Reg) and MTM performs as well as Beta-LT. At $\beta = 4.2$, Beta-LT performs the best followed by the quadratic tail (LnBeta-QT) and MTM's performance is as good as Beta-LT. It is noted that the individual techniques that provided the best estimate differed at different reliability indices. However, the MTM estimate was consistent and it performed at least as well as the second best estimate from the individual techniques. The statistics for the ratio of the best error and worst error to the MTM error for the normal distribution is presented in Table 3. It is clearly observed that the MTM error is closer to the best error than the worst error. Therefore, this approach serves as an insurance against bad predictions, if we were to use a single technique.

A boxplot[2] of η for the normal distribution is presented in Fig. 8. It is clearly observed that the ratio of the mean of the error to the mean of the range in the entire extrapolation region is mostly around 0.2–0.31. Since the ratio is less than one, the range always overestimates the error, and it mostly overestimates it by a factor of 3–5. The data and boxplots for the remaining distributions are presented in the Appendix B. They show that the range is a conservative estimate of the error for all the distributions tested, but for the uniform distribution it exaggerates the error by orders of magnitude. This may indicate that tail modeling does not work well when there is no tail.

Table 4 provides a summary of the performance of the individual methods and the MTM for all the distributions. Table 4 shows that no particular technique was the best for all the distributions and the technique that provided the best estimate for a particular distribution varied as the reliability indices changed. However, the MTM estimate was mostly close to the second best available estimate and was close to the best error compared to the worst error. The summary of η is presented in Table 5. It is observed that other than uniform distribution, the ratio (median) varies between 0.2 and 0.31. This means that the range averages between 3 and 5 times the error. If all the values between the first and third quartile are included, the range varies between 2.5 and 6 times the error except for the uniform distribution. In order to provide an insight as to what value one can expect for η, a simple exercise is performed. Standard normal random numbers of size 5×7 (represents 5 estimates at 7 different reliability indices) are generated and η is computed over 100,000 repetitions. The resultant median is 0.18. This lower ratio reflects the fact that in this simple experiment we model the different estimates as being random and uncorrelated, while the five extrapolation estimates are correlated.

In order to understand the computational savings associated with tail extrapolation, we calculated the number of samples required to achieve the same level of error obtained using MTM at a reliability index of 4.2 and the results are presented in Table 6. It is observed that for the same level of accuracy the number of samples required varies between 350,000 and 500,000 for different distributions compared to the 500 used for MTM.

It is to be noted that the standard distributions discussed cover all types of tails belong to parent distribution of Type I (other than uniform distribution). Therefore, the distributions discussed are not comprehensive in terms of parent distribution of different types but cover different types of tails. Although a COV of 7.5 is unlikely in most application, the distribution data parameters are engineered to cover all types of tails. Therefore, the distributions may not be necessarily realistic.

5. Engineering examples

5.1. Application of multiple tail median for reliability estimation of a cantilever beam

Consider the cantilevered beam design problem, shown in Fig. 9 [11]. The objective of the original problem is to minimize the weight or, equivalently, the cross sectional area, $A = w \cdot t$ subject to two reliability constraints, which require the reliability indices for strength and deflection constraints to be larger than three. The expressions of two performance measures are given as:

$$\text{Strength} \qquad G_s = \frac{\sigma_{comp}}{R} = \frac{\left(\frac{600}{w^2 t} F_X + \frac{600}{wt^2} F_Y\right)}{R} \tag{16}$$

$$\text{Tip Displacement} \qquad G_d = \frac{D_O}{D} = \frac{D_O}{\frac{4L^3}{Ewt}\sqrt{\left(\frac{F_Y}{t^2}\right)^2 + \left(\frac{F_X}{w^2}\right)^2}} \tag{17}$$

where R is the yield strength, F_X and F_Y are the horizontal and vertical loads and w and t are the design parameters. L is the length and E is the elastic modulus. D_O is the allowable displacement. R, F_X, F_Y, and E are random in nature and are defined in Table 7. It is noted

[2] In a box plot, the box is defined by lines at the lower quartile (25%), median (50%), and upper quartile (75%) values. Lines extend from each end of the box and outliers show the coverage of the rest of the data. Lines are plotted at a distance of 1.5 times the inter-quartile range in each direction or the limit of the data, if the limit of the data falls within 1.5 times the inter-quartile range. Outliers are data with values beyond the ends of the lines and are denoted by placing a “ + ” sign for each point.

Table 7
Random variables for the cantilevered beam problem.

Random variable	F_X	F_Y	R	E
Distribution	Normal (500, 100) lb	Normal (1000, 100) lb	Normal (40000, 2000) psi	Normal (29E6, 1.45E6) psi

Table 8
Properties of the cantilever beam.

L	100"
w	2.6041
t	3.6746
D_0	2.145
P_{f1}	0.00099
P_{f2}	0.00117
$P_{f1} \cap P_{f2}$	0.00016

P_{f1} – failure probability in mode1. P_{f2} – failure probability in mode 2.

Table 9
Cantilever beam. Summary of errors in S_r for different techniques at different reliability indices (mean and median over 1000 repetitions of 500 samples).

Rel index		3	3.6	4.2
GP-MLE	Mean	0.026	0.056	0.106
	Median	0.022	0.051	0.093
GP-Reg	Mean	0.056	0.159	0.478
	Median	0.036	0.078	0.132
LnBeta-QT	Mean	0.027	0.044	0.062
	Median	0.022	0.036	0.052
Beta-LT	Mean	0.019	0.026	0.034
	Median	0.016	0.022	0.030
Beta-QH	Mean	0.022	0.035	0.051
	Median	0.019	0.031	0.044
MTM	Mean	0.022	0.037	0.055
	Median	0.018	0.029	0.044

that the performance measures are expressed in a fashion such that failure occurs when G_s or G_d is greater than one. In this example, we consider system failure case with both failure modes. The optimal design variables taken from Qu and Haftka [12], for a system reliability case are presented in Table 8. The value of corresponding reliability index is three. The contribution of each failure mode is also presented in Table 8. Five hundred samples are generated, and for each sample, the critical S_r (maximum of the two) is computed. The conditional CDF of S_r can be approximated by classical techniques and the relationship between S_r and reliability index can also be approximated by the three alternative extrapolation techniques and by the MTM approach. These calculations are repeated for 1,000 different samples and the errors are compared in Tables 9 and 10. The accurate estimates of S_r are calculated using MCS of sample size 1E7 and it follows a normal distribution with parameters (0.679, 0.08). From Table 9, it is observed that the Beta-LT performs the best at all three reliability indices followed by the Beta-QH as the second best. MTM consistently performed close to the second best estimate. Table 10 shows that the MTM error is closer to the best error than to the worst error.

Fig. 10 presents the box plot of η. This time there is a small percentage of cases when the range is not a conservative estimate of the error, but for most of the cases the range overestimates the error by factors of 2–10.

To show the reduction in computational requirement due to the tail extrapolations, an MCS study is performed. 100 repetitions of S_r estimates with 500,000 samples and the corresponding standard deviation are computed and presented in Table 11. At the reliability index of 4.2, the standard deviation in S_r estimate is 0.04, which is the same level with that from MTM using 500 samples. Therefore, for a same level of accuracy, the reduction in computational effort is about three orders of magnitude (500,000–500).

Table 11
S_r [a]Estimates (without tail extrapolation) and standard deviation[a] at different reliability indices.

Rel index	3	3.2	3.4	3.6	3.8	4	4.2
S_r	1.012	1.032	1.05	1.07	1.09	1.12	1.13
SD	0.003	0.004	0.01	0.01	0.01	0.02	0.04

[a] Mean of 100 repetitions of 5e5 samples each.

5.2. Application of multiple tail median for reliability estimation of a composite panel

The design of the wall of a hydrogen tank which is a composite laminate operating in cryogenic temperatures addressed by Qu et al. [13] is considered here. The geometry and loading conditions of the problem is presented in Fig. 11. The laminate is subject to mechanical loading (N_x and N_y) and thermal loading due to operating temperature −423 F where the stress-free temperature is 300 F. The objective of the actual problem was to optimize the weight of laminate with two ply angles $[\pm\theta_1, \pm\theta_2]$. The ply angles and ply thickness (t_1 and t_2) are the design variables. The material used is IM600/133 graphite epoxy of ply thickness 0.005 inch. Qu et al. [13] performed the deterministic design optimization using continuous design variables. In order to account for the uncertainties and make the deterministic optimal design comparable to probabilistic optimal design, they used a safety factor of 1.4. In this example, the reliability of the deterministic optimum design is estimated using MTM. For further details, the reader is referred to Qu et al. [13]. The mechanical properties are presented in Table 12.

Table 12
Mechanical properties of the composite laminates.

Elastic properties	E_1, E_2, G_{12} and v_{12}
Coefficients of thermal expansion	α_1 and α_2
Stress-free temperature	T_{zero}
Failure strains	ε_1^L, ε_1^U, ε_2^L, ε_2^U and γ_{12}^U
Safety factor	S_F

Table 10
Cantilever beam. Summary of ratios of lowest and highest errors to MTM error (mean and median over 1000 repetitions of 500 samples).

Rel Index	Lowest error/MTM error				Highest error/MTM error			
	25 percentile	Mean	Median	75 percentile	25 percentile	Mean	Median	75 percentile
3	0.28	0.55	0.58	0.81	1.67	10.02	2.44	5.36
3.6	0.27	0.52	0.51	0.76	1.92	16.89	3.26	7.76
4.2	0.26	0.51	0.50	0.75	2.08	26.74	3.83	9.93

Table 13
Coefficient of variation for random material properties (obtained from Qu et al. [13], 2003 – Table 4).

$E_1,\ E_2,\ G_{12},\ v_{12}$	$\alpha_1,\ \alpha_2$	T_{zero}	$\varepsilon_1^L,\ \varepsilon_1^U$	$\varepsilon_2^L,\ \varepsilon_2^U,\ \gamma_{12}^U$
0.035	0.035	0.03	0.06	0.09

Table 14
Mean of random variables (Obtained from Qu et al. [13], 2003 – Fig. A1 and Fig. A2).

E_1	21.5×10^6	T_{zero}	300
E_2[a]	$[0.3, 2.2] \times 10^6$	ε_1^L	−0.0109
G_{12}[a]	$[0.1, 1.2] \times 10^6$	ε_1^U	0.0103
v_{12}	0.359	ε_2^L	−0.013
α_1[a]	$[-0.3, 0.2] \times 10^{-6}$	ε_2^U	0.0154
α_2[a]	$[0.05, 0.45] \times 10^{-4}$	γ_{12}^U	0.0138

[a] Temperature dependent materials properties are shown by the range of the mean value.

Table 15
Deterministic optima found by Qu et al. [13].

θ_1 (deg)	θ_2 (deg)	t_1 (in)	t_2 (in)	h (in)
0.00	28.16	0.005	0.02	0.1
27.04	27.04	0.01	0.015	0.1
25.16	27.31	0.005	0.020	0.1

Table 16
Composite laminate. Summary of errors in S_r for different techniques at different reliability indices (mean and median over 1000 repetitions of 500 samples).

Rel index		3	3.6	4.2
GP-MLE	Mean	0.030	0.070	0.142
	Median	0.026	0.060	0.120
GP-Reg	Mean	0.063	0.185	0.572
	Median	0.042	0.095	0.172
LnBeta-QT	Mean	0.034	0.054	0.079
	Median	0.027	0.046	0.069
Beta-LT	Mean	0.022	0.034	0.057
	Median	0.018	0.029	0.053
Beta-QH	Mean	0.025	0.039	0.057
	Median	0.021	0.031	0.047
MTM	Mean	0.026	0.045	0.072
	Median	0.021	0.037	0.059

All material properties are assumed to be random, uncorrelated and follow normal distributions. The coefficient of variation is presented in Table 13. E_2, G_{12}, α_1, and α_2 are functions of temperature. For a feasible design, 21 different temperatures uniformly distributed between −423 F and 80 F are considered and the strain constraints are applied to these temperatures. The mean is computed at a particular temperature and random numbers generated based on coefficient of variation. The mean for other parameters are presented in Table 14. The deterministic optimal design obtained by Qu et al. [13] is presented in Table 15.

Table 18
Composite laminate. S_r estimates (without tail extrapolation) and standard deviation at different reliability indices.

Rel index	3	3.2	3.4	3.6	3.8	4	4.2
1E + 05	1.0043	1.0304	1.0579	1.0862	1.1157	1.1480	1.1824
SD	0.007	0.009	0.015	0.021	0.025	0.046	0.063

The transverse strain in the first ply (direction 2) is the critical strain. The performance measure is defined as the ratio of the critical strain to the upper bound of the allowable strain.

$$G = \frac{\varepsilon_2}{\varepsilon_2^U} \tag{18}$$

Failure is said to occur if the performance measure is greater than one. The multiple tail median approach is used for the second optimal design in Table 15. The loads are increased by a factor of 1.1 ($N_x = -5280$ lb/inch and $N_y = -2640$ lb/inch) so that the failure probability is of the order of 0.001.

The random variables in this problem are sampled to produce five hundred critical strain ratios. The five tail extrapolation techniques are used and the results are presented in Tables 16 and 17. The accurate estimates of S_r are calculated using MCS of sample sizes 1E7 and it follows a normal distribution with parameters (0.633, 0.03). As observed in the previous examples the MTM is at least close to the second best in estimates from individual methods. In addition, the MTM error was closer to the best error than the worst error which is observed from Table 17. Fig. 12 shows that in a small percentage of cases the range is an unconservative estimate of the MTM error. However for most of the cases the range overestimates the errors by factors of 2–8.

To demonstrate the computational savings due to tail extrapolation 100,000 sample based S_r estimation and the corresponding standard deviations are presented in Table 18. At reliability index of 4.2, it is seen that the standard deviation is 0.063 and the multiple tail median provides an estimate with an error of 0.059 (from Table 16). The number of samples used for this estimation is 500, which is more than two orders of magnitude less than 500,000.

6. Conclusions

Estimating low probabilities of failure (high reliabilities) with expensive computer models pose a challenge because of the high computational expense. Extrapolating into higher reliability indices with information from lower reliability indices is one solution to this problem. Three extrapolation techniques that can complement classical tail modeling techniques were developed. This work proposes to use a multiple tail median (MTM) approach in which all the techniques are applied simultaneously to estimate the performance measure and the median is taken to be the best estimate. The median turns out to be a robust estimate compared to the mean and was usually at least the second best estimate compared to the individual techniques. Moreover, it was observed in all examples that the MTM error was closer to the best rather than the worst estimate. It is shown that the range of the five methods can be utilized to estimate the order of magnitude of the error in the MTM estimate, with that range

Table 17
Composite Laminate. Summary of ratios of lowest and highest errors to MTM error (mean and median over 1000 repetitions of 500 samples).

Rel index	Lowest error/MTM error				Highest error/MTM error			
	25 percentile	Mean	Median	75 percentile	25 percentile	Mean	Median	75 percentile
3	0.30	0.54	0.57	0.77	1.68	15.52	2.41	5.06
3.6	0.23	0.49	0.46	0.74	1.80	271.51	2.96	7.31
4.2	0.20	0.48	0.42	0.75	1.85	42.98	3.23	7.94

being a conservative estimate for the large majority of cases. The proposed method was demonstrated on seven statistical distributions and two engineering examples. It was also shown that the tail modeling allows reductions of two or three orders of magnitude in sample size.

Acknowledgement

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Appendix A. Steps for estimating errors in the individual techniques and MTM for standard distributions

1. Use $\bar{x}$ and COV to estimate distribution parameters (a_0 and b_0) for each distribution from Table 1.
2. Find capacity g_c as $(1-P_f)^{th} \times N$ quantile. Inverse transformation of CDF is used for this. In the case of single parameter distributions, add shift factor to the g_c.
3. Find normalized mean using $\hat{\bar{x}} = \bar{x}/g_c$. We are interested in the safety factor reciprocal, S_r following different statistical distributions. The normalized mean is the mean of the S_r.
4. Use $\hat{\bar{x}}$ and COV to find new parameters (a and b) for all the distributions.
5. Generate $N = 500$ Latin hypercube samples in [0, 1].
6. Find S_r at each sample using (a, b) and inverse cumulative distribution functions.
7. Use Eq. (10) to compute P_i.
8. S_r and P provides the empirical CDF of S_r. It is noted that the ordinate of the CDF is $(1-P_f)$. That is, we are interested in the upper tail. As P_f decreases, $1-P_f$ increases, reliability index increases, and S_r increase.
9. A threshold of 0.9 is used. For classical tail modeling techniques, the parameters of GPD that approximates the conditional CDF beyond the threshold are estimated using maximum likelihood and least-square regression approaches. In the alternate extrapolation schemes, the coefficients are estimated using least-square regression.
10. The transformations discussed in Section 3 can be applied to the CDF of S_r and S_r at higher reliability indices can be estimated directly from the approximated relationships.
11. There are totally five estimates (Steps 9 and 10), S_{ri} ($i = 1,\ldots,5$) in the extrapolated regions. Record the median ${}^5m$ and range ${}^5r$.
12. For the purpose of comparison, exact values of S_r can be obtained using inverse transformation of CDFs.
13. Compute error between the exact values and the estimates from individual methods as e_i and MTM as e_{MTM}.
14. Compute the ratio of mean of error from MTM to mean of range over different extrapolated reliability indices and denote it as η
15. Repeat Steps 5–14, 1000 times with different samples, for each distribution
16. Record the median, mean of as e_{MTM} over 1000 repetitions as ${}^{1000}m_e$ and ${}^{1000}\mu_e$, respectively
17. Record the median and mean of the ratio in Step 14 over 1000 repetition as ${}^{1000}m_\eta$ and ${}^{1000}\mu_\eta$.

Appendix B

In Section 4, the performance of MTM was demonstrated using normal distribution. In this section, similar comparisons are presented for different distribution types. GPD shape parameter values are presented in Table 19. For each distribution type,

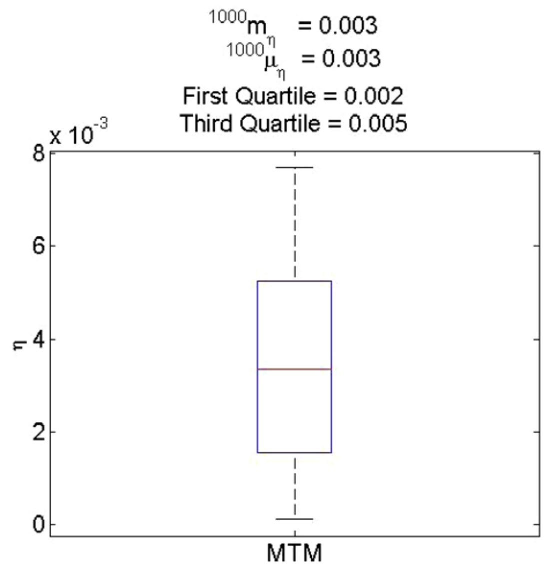

Fig. 13. Uniform distribution.

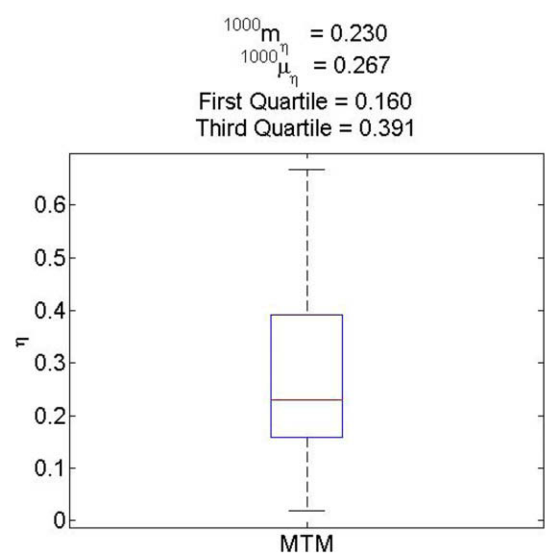

Fig. 14. Exponential distribution.

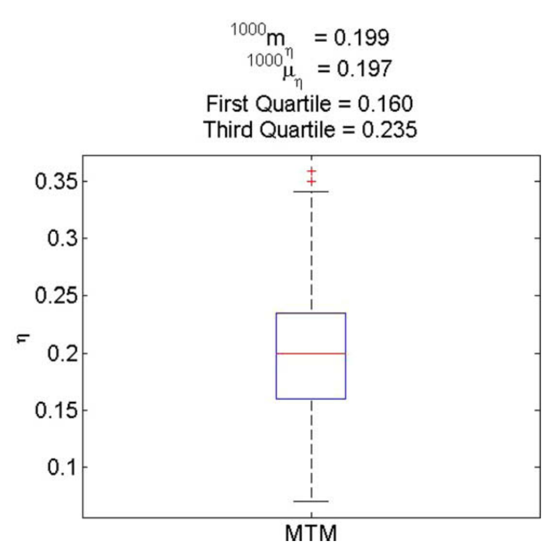

Fig. 15. LogNormal distribution.

Table 19
GPD shape parameter for different distribution.

Distribution	ξ	Tail type
Uniform	−1.05	No tail
Normal	−0.18	Light
Exponential	0.02	Medium
LogNormal	0.67	Heavy

Table 20
Uniform distribution.

Rel index		3	3.6	4.2
GP-MLE	Mean	0.001	0.002	0.002
	Median	0.001	0.002	0.002
GP-Reg	Mean	0.001	0.001	0.001
	Median	0.001	0.001	0.001
LnBeta-QT	Mean	0.001	0.016	0.042
	Median	0.001	0.016	0.042
Beta-LT	Mean	0.060	0.125	0.192
	Median	0.060	0.125	0.192
Beta-QH	Mean	0.100	0.321	0.668
	Median	0.100	0.321	0.668
MTM	Mean	0.001	0.002	0.002
	Median	0.001	0.002	0.002

$^{1000}\mu_e$ and $^{1000}m_e$ of individual errors and MTM error are calculated for reliability indices between 3 and 4.2. The results for different distribution types are presented in Tables 20–22. In addition, the statistics for the ratio of the best error and worst error to the MTM error for different distribution types are presented in Tables 23–25. Finally the boxplots of ratio η for different distributions are presented in Figs. 13–15.

Table 21
Exponential distribution.

Rel index		3	3.6	4.2
GP-MLE	Mean	3.413	9.404	20.237
	Median	3.195	8.563	17.659
GP-Reg	Mean	1.945	3.797	6.997
	Median	1.908	3.441	5.962
LnBeta-QT	Mean	6.223	8.786	11.967
	Median	4.419	4.343	7.930
Beta-LT	Mean	1.885	5.051	12.268
	Median	1.320	5.419	13.456
Beta-QH	Mean	3.503	5.238	7.157
	Median	2.725	3.911	5.138
MTM	Mean	2.520	4.587	8.345
	Median	1.876	3.117	7.399

Table 22
LogNormal distribution.

Rel index		3	3.6	4.2
GP-MLE	Mean	0.146	1.620	18.339
	Median	0.096	0.943	11.016
GP-Reg	Mean	0.357	3.703	35.995
	Median	0.354	3.666	35.398
LnBeta-QT	Mean	0.545	1.720	8.153
	Median	0.152	1.583	8.559
Beta-LT	Mean	0.359	2.233	9.627
	Median	0.310	2.340	9.885
Beta-QH	Mean	0.353	2.097	9.211
	Median	0.307	2.200	9.507
MTM	Mean	0.212	1.514	8.084
	Median	0.107	1.583	8.559

Table 23
Uniform distribution.

Rel index	Lowest error/MTM error				Highest error/MTM error			
	25 percentile	Mean	Median	75 percentile	25 percentile	Mean	Median	75 percentile
3	0.44	1.00	0.73	1.00	67.50	119.90	526.60	280.60
3.6	0.25	0.61	0.61	1.00	119.90	185.30	1316.40	397.10
4.2	0.24	0.57	0.59	1.00	242.20	365.60	1405.30	760.60

Table 24
Exponential distribution.

Rel index	Lowest error/MTM error				Highest error/MTM error			
	25 percentile	Mean	Median	75 percentile	25 percentile	Mean	Median	75 percentile
3	0.17	0.44	0.51	0.93	2.23	2.96	11.37	4.63
3.6	0.19	0.56	0.58	1.00	2.04	3.19	12.79	6.21
4.2	0.15	0.40	0.49	0.92	2.22	2.78	8.52	4.23

Table 25
LogNormal distribution.

Rel index	Lowest error/MTM error				Highest error/MTM error			
	25 percentile	Mean	Median	75 percentile	25 percentile	Mean	Median	75 percentile
3	0.51	0.84	0.72	1.00	3.05	3.96	31.03	7.66
3.6	0.17	0.51	0.55	1.00	1.95	2.51	6.92	3.64
4.2	0.63	1.00	0.81	1.00	3.64	4.39	6.96	5.43

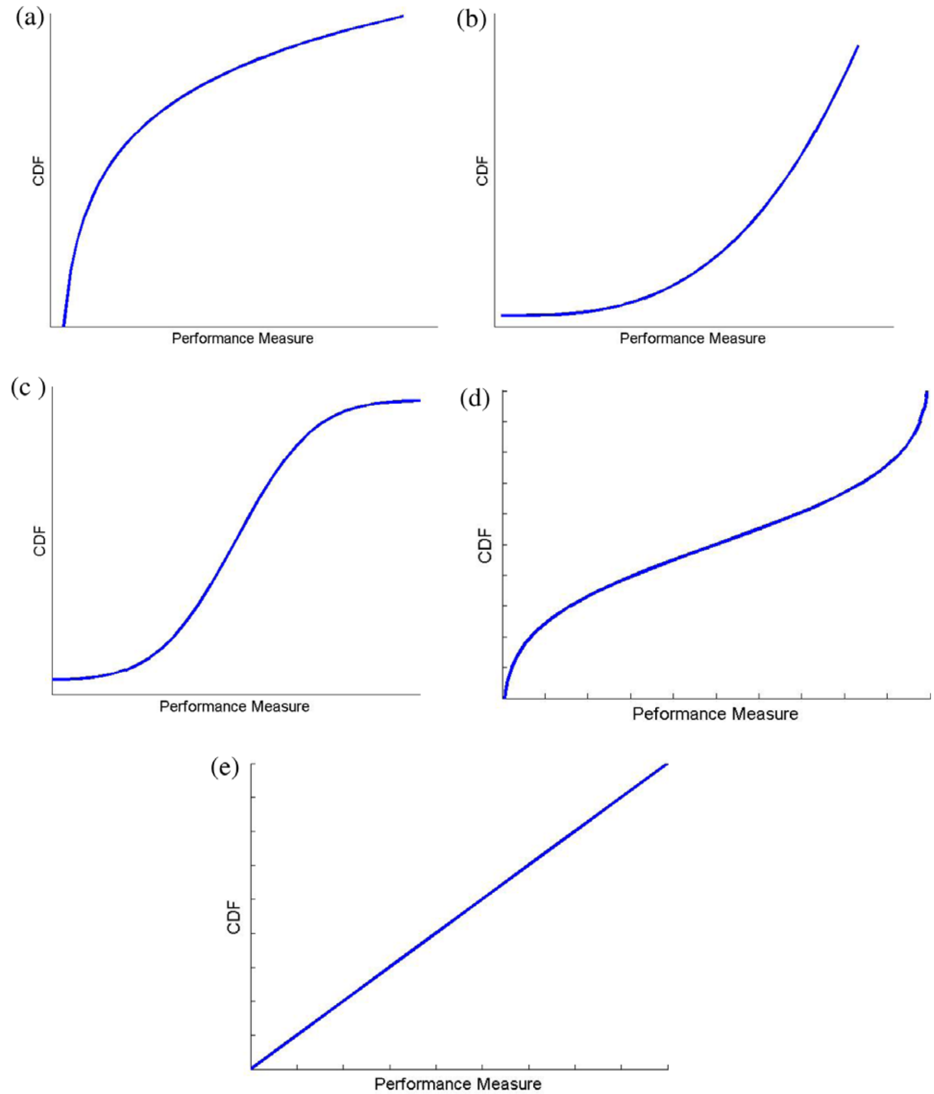

Fig. 16. Possible CDF shapes.

Appendix C

All the possible CDF shapes [14] are presented in Fig. 16. In Section 3, two classical tail modeling techniques and three alternate tail modeling techniques were discussed. The classical tail modeling techniques are capable of modeling the tails with little error irrespective of the parent distribution statistics. In addition to the classical techniques, each case in Fig. 16 can be modeled by at least one proposed alternate extrapolation technique. The Beta-LT can model the tails in all the above five cases. It models Fig. 16e very well. That is, if the performance measure is normally distributed, then reliability index, β and reciprocal of safety factor S_r are linearly related. LnBeta-QT is similar to an exponential of Beta-LT ($\beta = \exp(\ln(\beta))$). This captures case Fig. 16b well. Beta-QH can model all the five cases. Since at least one of the five methods can capture the CDF shape, using the MTM approach though might not guarantee a good estimate, buys insurance against bad predictions if one were to use a single method.

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