

Technical Notes

Uncertainty Identification of Damage Growth Parameters Using Nonlinear Regression

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DOI: 10.2514/1.J051268

I. Introduction

RECENTLY, it has been shown that structural health monitoring (SHM) systems can be used for inspection to detect damage [1]. SHM-based maintenance is effective as only those airplanes that are in danger will be sent for maintenance (condition-based maintenance). Furthermore, Coppe et al. [2] showed that in addition to damage diagnosis SHM can predict the remaining useful life by identifying damage growth parameters. They used Bayesian inference [3] to reduce uncertainty in the damage growth parameters using measured damage size information.

Bayesian inference is a powerful method of quantifying uncertainty in the model parameters. It can take into account prior knowledge on the unknown parameters and improve them using experimental observations. However, in the case of SHM, the advantage of the prior information can be overpowered by the amount of data available. In addition, when many parameters are updated simultaneously, Bayesian inference becomes computationally expensive due to multidimensional integration.

On the other hand, the traditional linear regression method [4] can be used to identify deterministic parameters when the model is a linear function of the parameters. This method is particularly powerful when many data are available, which is the case for SHM. By assuming that the noise in the experimental data is Gaussian, it is possible to estimate the uncertainty in the identified parameters.

When the physical model is a nonlinear function of model parameters, uncertainty quantification is not straightforward [5]. As will be shown in the numerical examples, the crack growth in aircraft structures nonlinearly depends on the parameters that need to be identified. In this Note, a linear perturbation concept is used to quantify uncertainty in the nonlinear regression result. First, the nonlinear optimization is used to find the model parameters that minimize error between the model and SHM data. Then the nonlinear model is linearized at the identified parameter values, from which the uncertainty quantification in the linear regression method can be used. This approach can introduce two errors into the uncertainty estimation: 1) linearization error and 2) error associated with assumption of Gaussian noise. In addition, it is assumed that noises at different experiments are uncorrelated. The objective of the Note is to examine their effect on accuracy of the uncertainty estimation.

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II. Uncertainty Quantification in Nonlinear Regression

Regression is commonly used for identifying unknown parameters of a physical model using experimental data, which normally include noise and error. Thus, if the experiment is repeated, it is likely to identify different values of the parameters. In this section, a method of calculating the uncertainty of identified parameters from nonlinear regression will be reviewed. To make the presentation more coherent, estimation of parameter uncertainty in linear regression is discussed first, followed by that of nonlinear regression.

A. Uncertainty in Linear Regression

In regression, a response function $y(t)$ is approximated by a simple function $f(t, \boldsymbol{\beta})$ with the vector $\boldsymbol{\beta}$ of parameters whose dimension is $n_\beta = \dim(\boldsymbol{\beta})$:

$$y(t) = f(t, \boldsymbol{\beta}) + \varepsilon \quad (1)$$

where ε is the approximation error. The regression model is called linear when the approximate function is a linear function of $\boldsymbol{\beta}$, as

$$f(t, \boldsymbol{\beta}) = \sum_{i=1}^{n_\beta} \beta_i \xi_i(t) \quad (2)$$

where $\xi_i(t)$ are basis functions. It is assumed that the expression of the response function, $y(t)$, is unknown, but its values can be evaluated at discrete points. Or, it is also possible that the response can be evaluated through experiments. In such a case, the experimental data may include measurement noise.

The objective of regression is to estimate the vector of parameters $\boldsymbol{\beta}$ so that the approximation error is minimized. Normally, this can be done with n_y data, which are given in the form of (t_i, y_i) , for $i = 1, \dots, n_y$, which may contain error or noise. In regression, the parameters are estimated by minimizing the sum of the squares of discrepancies between the measurements and $f(t, \boldsymbol{\beta})$. In general, the exact values of $\boldsymbol{\beta}$ can only be found when the number of experimental data is infinity. With finite n_y , the values are only estimates, which will be denoted by \mathbf{b} in this Note.

By denoting $y(t_i) = y_i$, the vector of errors (discrepancies) can be written as

$$\begin{Bmatrix} e_1 \\ e_2 \\ \vdots \\ e_{n_y} \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n_y} \end{Bmatrix} - \begin{bmatrix} \xi_1(t_1) & \xi_2(t_1) & \cdots & \xi_{n_\beta}(t_1) \\ \xi_1(t_2) & \xi_2(t_2) & \cdots & \xi_{n_\beta}(t_2) \\ \vdots & \vdots & \ddots & \vdots \\ \xi_1(t_{n_y}) & \xi_2(t_{n_y}) & \cdots & \xi_{n_\beta}(t_{n_y}) \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n_\beta} \end{Bmatrix} \quad (3)$$

Or, symbolically,

$$\mathbf{e} = \mathbf{y} - \mathbf{X} \cdot \mathbf{b} \quad (4)$$

The vector of parameters \mathbf{b} is estimated by minimizing the rms error defined as

$$e_{\text{rms}} = \sqrt{\frac{1}{n_y} \mathbf{e}^T \mathbf{e}} \quad (5)$$

After substituting Eq. (4) into Eq. (5) and minimizing the rms error, the following linear regression equation can be obtained:

$$\mathbf{X}^T \mathbf{X} \cdot \mathbf{b} = \mathbf{X}^T \mathbf{y} \quad (6)$$

which can be solved for \mathbf{b} , the estimate of parameters.

Because the experimental data include random noise, the estimated parameters will vary for different sets of experimental data.

Therefore, there is uncertainty in the identified parameters. The objective is to estimate the uncertainty in the estimated parameters due to the random noise in the experimental data.

When experimental data are uncorrelated with each other, random noise can be assumed normally distributed with standard deviation of σ : i.e., $v \sim N(0, \sigma^2)$. Then the unbiased estimate of the variance can be obtained from [6] (p. 27):

$$\hat{\sigma}^2 = \frac{\mathbf{e}^T \mathbf{e}}{n_y - n_\beta} \quad (7)$$

The sensitivity of estimated parameters with respect to small differences in data can be calculated using the covariance matrix of \mathbf{b} . Using Eq. (6), the covariance matrix can be obtained as

$$\Sigma_{\mathbf{b}} = \sigma^2 [\mathbf{X}^T \mathbf{X}]^{-1} \quad (8)$$

The diagonal component of $\Sigma_{\mathbf{b}}$ is the variance of \mathbf{b} , which represents a measure of the sensitivity of estimated parameters with respect to the noise. Since the standard deviation of the noise is unknown in advance, its estimate in Eq. (7) can be used. Thus, the standard error (SE) of parameter b_j can be obtained by

$$\text{SE}(b_j) = \hat{\sigma} \sqrt{[\mathbf{X}^T \mathbf{X}]_{jj}^{-1}}, \quad j = 1, \dots, n_\beta \quad (9)$$

The above standard error is indeed the estimate of the standard deviation of regression coefficient b_j due to noise in the data.

B. Uncertainty in Nonlinear Regression

Different from a linear regression problem, many physical models cannot be represented as a linear combination of unknown parameters as in Eq. (2). In such a case, instead of solving a linear regression Eq. (6), a nonlinear optimization problem is solved to minimize the rms error in Eq. (5), in which the errors are now defined as

$$e_i = y_i - f(t_i, \mathbf{b}), \quad i = 1, \dots, n_y \quad (10)$$

The nonlinear optimization problem can then be to find the optimum values of parameters \mathbf{b} such that

$$\mathbf{b}^* = \arg \min_{\mathbf{b}} e_{\text{rms}} \quad (11)$$

Once the optimum values of parameters are obtained, the uncertainties in the parameters are estimated by linearizing the nonlinear model function at the optimum point. To linearize the model function, the first-order Taylor series expansion method can be used for $f(t, \mathbf{b})$ where $\mathbf{b} = \mathbf{b}^* + \Delta \mathbf{b}$. By ignoring higher-order terms, the model problem can be linearized as

$$f(t, \mathbf{b}) \simeq f(t, \mathbf{b}^*) + \sum_i \frac{\partial f(t, \mathbf{b}^*)}{\partial b_i} \Delta b_i \quad (12)$$

By moving $f(t, \mathbf{b}^*)$ to the left-hand side, the expression for the residual can be obtained as

$$r = f(t, \mathbf{b}) - f(t, \mathbf{b}^*) = \sum_i \frac{\partial f(t, \mathbf{b}^*)}{\partial b_i} \Delta b_i \quad (13)$$

Equation (13) can be considered as a linear regression problem with unknown parameters $\Delta \mathbf{b}$, and the gradient $\partial f / \partial b_i$ becomes the basis vector ξ_i in Eq. (2). Therefore, the uncertainty in parameters $\Delta \mathbf{b}$ can be calculated using the same procedure described in Sec. II.A. For that purpose, the regression coefficient matrix can be written as

$$\mathbf{X} = \begin{bmatrix} \frac{\partial f(t_1, \mathbf{b}^*)}{\partial b_1} & \frac{\partial f(t_1, \mathbf{b}^*)}{\partial b_2} & \dots & \frac{\partial f(t_1, \mathbf{b}^*)}{\partial b_{n_\beta}} \\ \frac{\partial f(t_2, \mathbf{b}^*)}{\partial b_1} & \frac{\partial f(t_2, \mathbf{b}^*)}{\partial b_2} & \dots & \frac{\partial f(t_2, \mathbf{b}^*)}{\partial b_{n_\beta}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f(t_{n_y}, \mathbf{b}^*)}{\partial b_1} & \frac{\partial f(t_{n_y}, \mathbf{b}^*)}{\partial b_2} & \dots & \frac{\partial f(t_{n_y}, \mathbf{b}^*)}{\partial b_{n_\beta}} \end{bmatrix} \quad (14)$$

Then Eq. (9) can be used to estimate the standard error of $\Delta \mathbf{b}$, which can also be considered as the standard error of \mathbf{b}^* if the problem is linear. Because of nonlinearity, the standard error of $\Delta \mathbf{b}$ will be different from that of \mathbf{b}^* . However, if the nonlinearity is small, or if the uncertainty in \mathbf{b}^* is small, then the difference between them will be small.

III. Uncertainty Identification of Crack Growth Parameters

The problem of interest is to identify crack growth parameters in the Paris model [7] using measured crack sizes at different cycles:

$$\frac{da}{dN} = C(\Delta K)^m \quad (15)$$

where a is half the crack size, N is the number of cycles, m and C are two Paris model parameters, and ΔK is the range of stress intensity factor. In this model, the number of cycles is considered as time: i.e., $t_i = N_i$. For an infinite panel under mode I loading condition, the crack size at cycle N_i can be calculated from the initial crack size a_0 as

$$a_i = \left[N_i C \left(1 - \frac{m}{2} \right) (\Delta \sigma \sqrt{\pi})^m + a_0^{1-\frac{m}{2}} \right]^{\frac{2}{m}} \quad (16)$$

In the above crack size model, model parameters (such as C , m , and a_0) need to be identified. Coppe et al. [2] used Bayesian inference to identify the unknown parameters with measured crack sizes. They showed that although Bayesian inference can accurately identify uncertainty of unknown model parameters, it is a computationally-intensive process, especially when many parameters need to be identified.

In this section, the nonlinear regression method is used to identify the model parameters as well as their uncertainty. The measured data a_i^{meas} are actually simulated in this Note by applying an error model to the modeled crack size a_i in Eq. (16). The error model includes the effect of bias b and noise v of the sensor measurement. The former is deterministic and represents a calibration error, while the latter is random and represents white noise. It is first assumed that the true values of model parameters (C , m , and a_0) are known. Then, using Eq. (16), the true crack sizes are generated at given cycles N_i . The measured crack size can then be defined as

$$a_i^{\text{meas}} = a_i + b + v \quad (17)$$

In this Note, the random noise is assumed to be uniformly distributed between the lower- and upper bounds as $v \sim \text{uniform}(-z, z)$. This is an efficient way of validating the uncertainty in the model parameters because different sets of measured crack sizes can be generated.

Let the vector of unknown model parameters be defined as $\mathbf{b} = \{m, a_0, b\}^T$: i.e., $n_\beta = 3$. Using measured crack size a_i^{meas} at cycle N_i , the regression problem can be stated as minimizing the rms error in Eq. (5), in which the errors are defined as

$$e_i = a_i^{\text{meas}} - f(N_i, \mathbf{b}) \quad (18)$$

and

$$f(N_i, \mathbf{b}) = a_i(N_i, \mathbf{b}) + b \quad (19)$$

Using Eq. (18), the nonlinear optimization problem in Eq. (11) is solved for the optimum parameters $\mathbf{b}^* = \{m^*, a_0^*, b^*\}^T$. The `lsqnonlin` function in MATLAB is used to solve the nonlinear optimization problem in Eq. (11). Once the model parameters are identified, the uncertainty of these parameters are estimated using the standard error in Eq. (9). The derivatives of model function $f(N_i, \mathbf{b})$ with respect to model parameters \mathbf{b} are obtained using the symbolic differentiation in MATLAB.

To assess the accuracy of this uncertainty quantification approach, the Monte Carlo simulation (MCS) is used to estimate the uncertainty in the identified parameters. In MCS, it is assumed that the

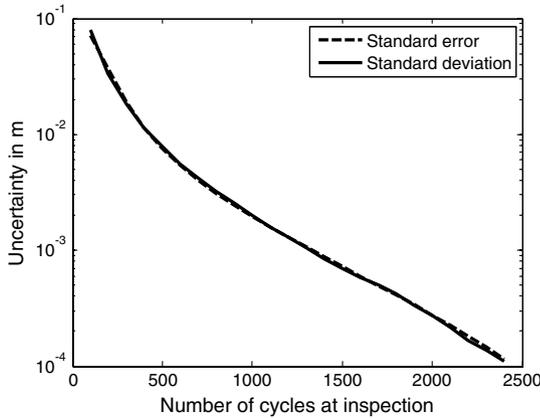


Fig. 1 Comparison of the calculated standard error with the simulated standard deviation ($a_0 = 10$ mm, $m = 3.8$, $C = 1.5E - 10$, $\Delta\sigma = 78.6$ MPa, and $b = 0$ mm).

experiments are repeated many times, and the parameters are identified for each experiment, from which the distribution of identified parameters can be estimated.

The first example assumes that all model parameters are known except for the Paris model exponent, m . It is assumed that the experimental data have random noise $v \sim \text{uniform}(-1, 1)$ mm, but no bias. Figure 1 shows the standard error estimated from the proposed linearization method in a dashed curve. The uncertainty is calculated every 100 cycles. For example, at $N_5 = 500$, five measured data are used: i.e., $n_y = 5$. It is noted that the standard error decreases significantly as more data are used. To validate the estimated standard error, the standard deviation from MCS is also plotted on a solid curve. For MCS, the nonlinear regression problem

is solved 1000 times with randomly generated data using Eq. (17), from which the standard deviation of parameter m is estimated. It can be observed that the calculated standard error (dashed curve) fits the estimated standard deviation (solid curve) very well.

The second example considers all three parameters unknown. Figure 2 shows the estimated standard error along with the standard deviation from 1000 MCS. It can be observed that in this case, the calculated standard error does not clearly match the simulated standard deviation for the first 1000 cycles. There are many explanations for this discrepancy. First, the regression method predicts a larger standard error because not many data are available at the early stage. Second, linearization error can also contribute to the discrepancy. Finally, correlation between a_0 and b can contribute to large error at the early stage, which can lead to an ill-conditioned $\mathbf{X}^T \mathbf{X}$ matrix. As the damage grows, i.e., at the later stage, the effect of a_0 and b becomes more independent and the linearization can estimate the standard deviation accurately.

IV. Conclusions

This Note presents a method of estimating the uncertainty of model parameters in nonlinear regression using a linear perturbation concept. It has been shown that the method yields a good estimate of standard deviation for the case of single-variable identification. For multiple variables, the correlation between variables can cause overestimation of standard deviation if an insufficient numbers of data are used. As the crack grows quickly, however, the proposed method is able to identify uncertainty accurately.

Acknowledgment

This work was supported by the U.S. Air Force Office of Scientific Research under grant FA9550-07-1-0018 and by NASA under grant NNX08AC334.

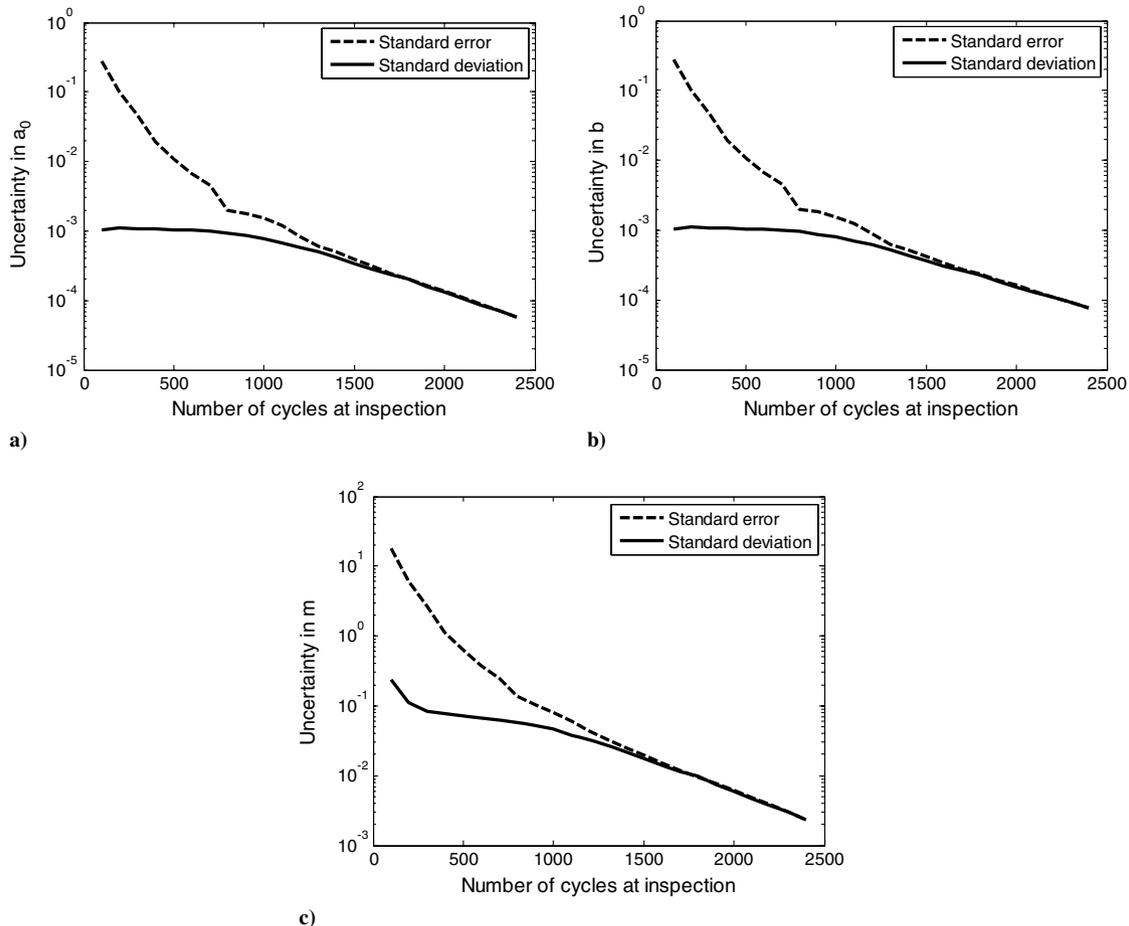


Fig. 2 Uncertainty in a) a_0 , b) b , and c) m using linearization of nonlinear regression ($a_0 = 10$ mm, $m = 3.8$, $C = 1.5E - 10$, $\Delta\sigma = 78.6$ MPa, and $b = 0$ mm).

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