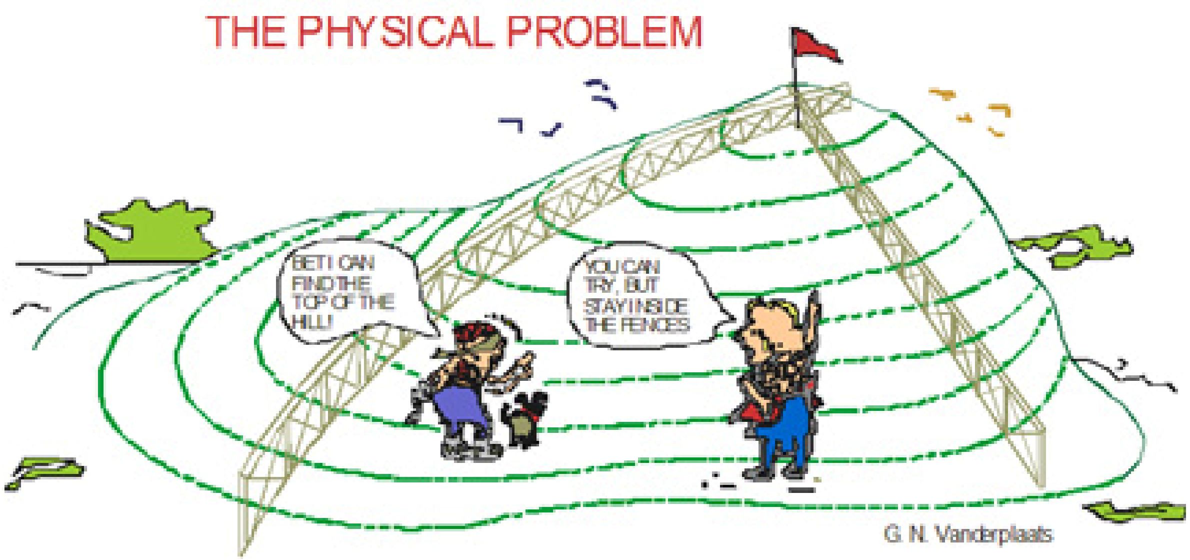
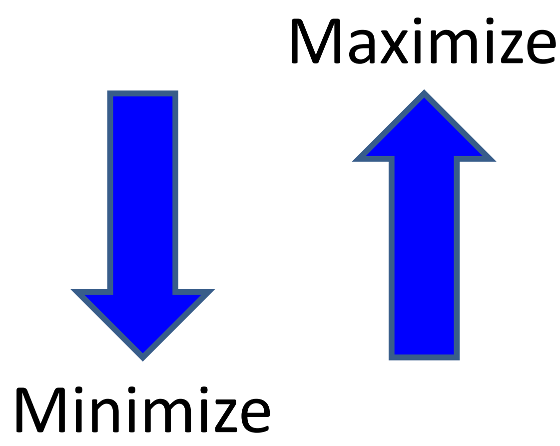
Introduction to Optimization Methods



OPTIMIZATION

- Optimization methods help us find solutions to problems where we seek to find the best of something
- Curve-fitting by minimizing error between the curve and data is an example of optimization
- This section is about how we formulate/solve the problem mathematically.
- Gradient-based methods
  - Searching for local minima
  - Function must be continuous and smooth
- Gradient-free methods
  - Direct search
  - Genetic algorithm, simulated annealing, etc.



QUANTITATIVE MEASURES

- Optimization assumes that we have choices and that we can attach numerical values to the ‘goodness’ of each alternative.

- This is not always the case. We may have problems where the only thing we can do is compare pairs of alternatives and tell which one is better, but not by how much.

STRUCTURAL DESIGN

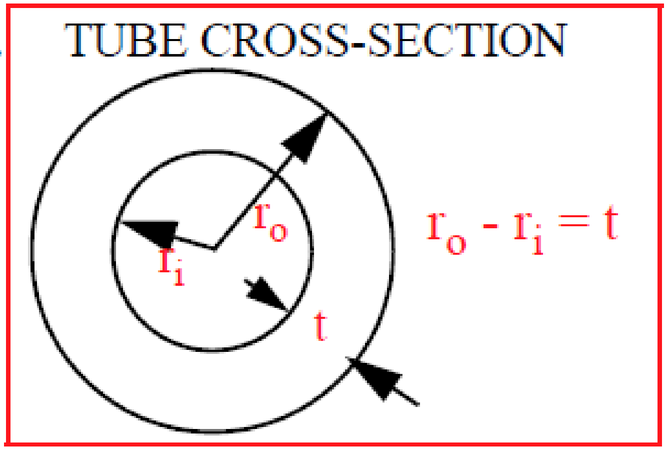
- Structural design: a procedure to improve or enhance the performance of a structure by changing its parameters
- Performance: a measurable quantity (constraint and goal)
  - weight, stiffness or compliance; the fatigue life; noise and vibration levels; safety
- Constraint: As long as the performance satisfies the criterion, its level is not important
  - Ex: the maximum stress should be less than the allowable stress
- Goal: the performance that the engineer wants to improve as much as possible (cost, objective)
- Design variables: system parameters that can be changed during the design process
  - Plate thickness, cross-sectional area, shape, etc.

DESIGN OPTIMIZATION

- What Is Design Optimization?
  - To find the best design parameters that meet the design goal and satisfies constraints.
- Design Variables: Anything the Designer Can Change
  - Thickness of a plate, Cross-sectional geometry of a beam or truss, Geometric dimensions
- Design Goal: Objective Function, Cost Function
  - Design criterion that will be minimized (or maximized)
  - Mass, Stress, Displacement, Natural Frequency, ETC
- Constraint: Conditions that the system must satisfy
  - Stress, Displacement, ETC
- Note: Design variables must affect the goal and constraints

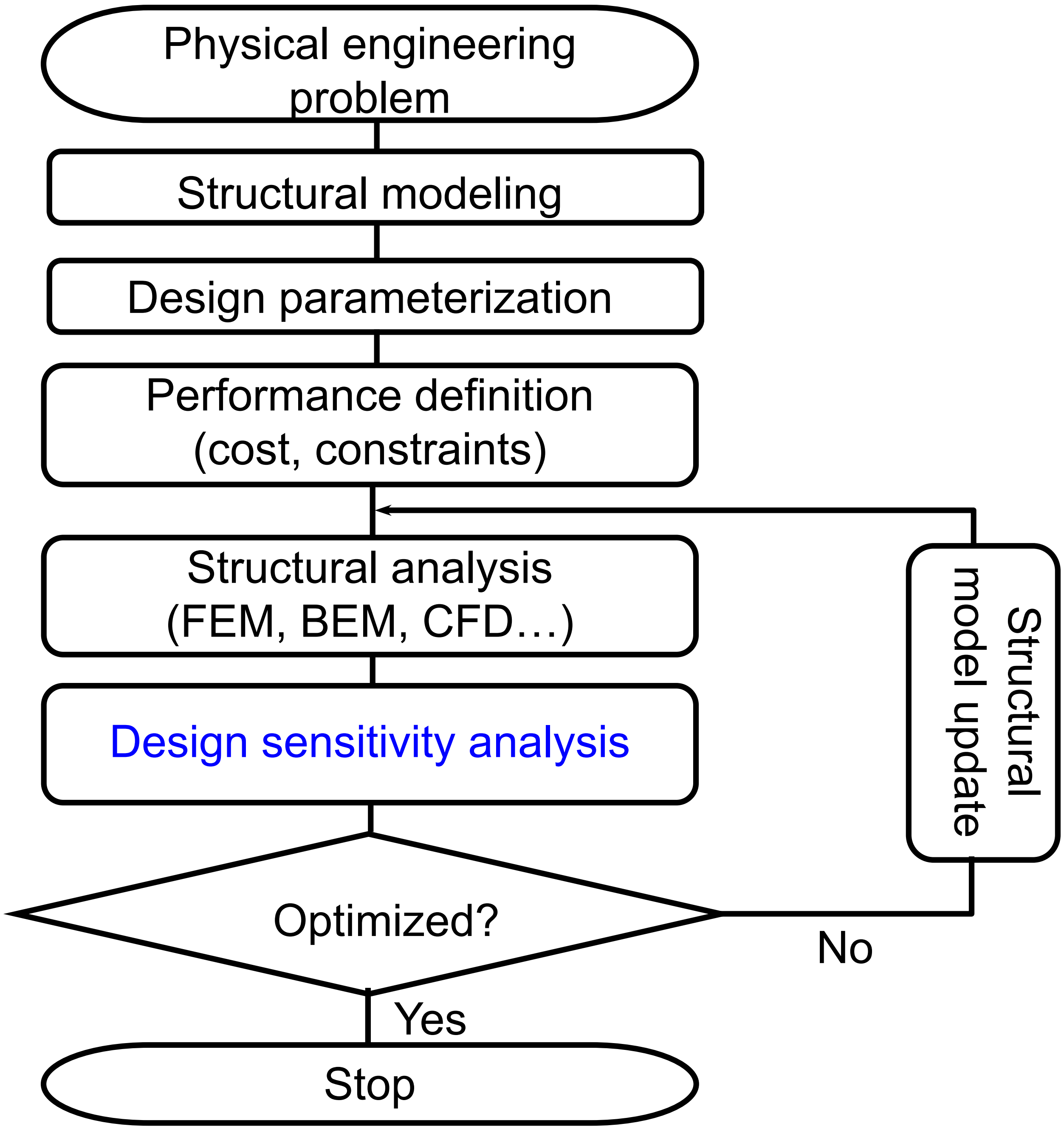
DESIGN VARIABLES

- Design variables: Parameters that change during optimization
  - Material properties
  - Sizing (parameter): thickness, cross-section
  - Shape: domain
  - Topology: birth or death
- Continuous vs. discrete
- Independent



- Feasible design $S = \{\text{feasible designs}\}$
  - A set (subdomain) of design variables that satisfies all constraints

STRUCTURAL OPTIMIZATION FLOW CHART



EXAMPLE: LEAST-SQUARES FITTING OF YOUNG MODULUS

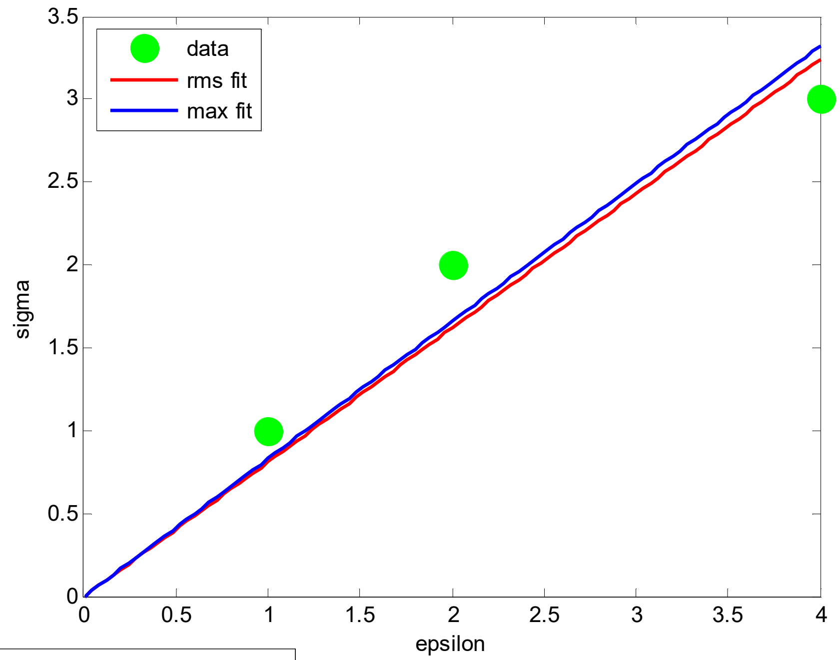
- Data

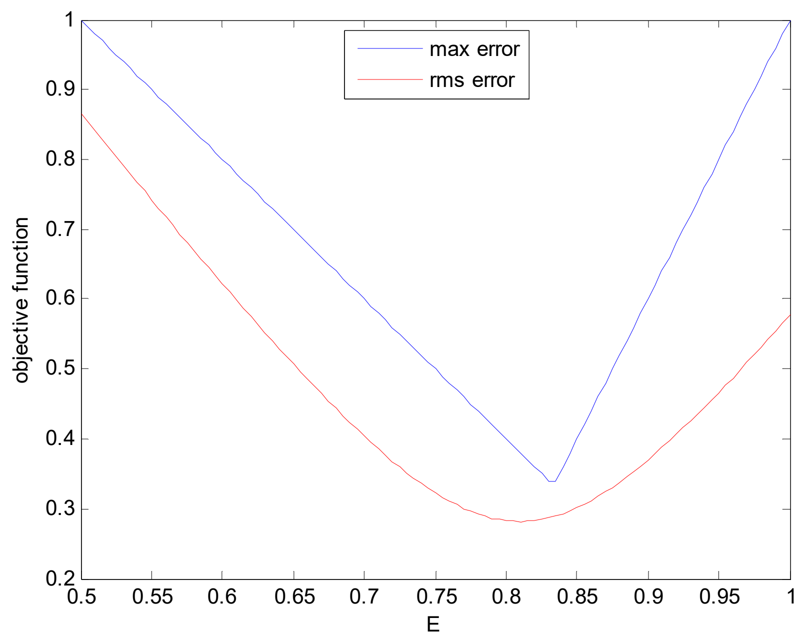
Strain ($\mu\varepsilon$)	1	2	4
Stress (ksi)	1	2	3

- Stress-strain model: $\sigma = E\varepsilon$
- Find *E* that minimizes the differences between the data and the model
- Goodness: (a) the maximum difference or (b) the root-mean-square (RMS) difference
- Design variable: *E*
- Objective: the maximum or RMS difference

UNCONSTRAINED FORMULATIONS

- Minimize max. difference: $d_{\max}(E) = \max_{i=1,2,3} |\sigma_i - E\varepsilon_i|$
- Minimize RMS of error: $d_{\text{RMS}}(E) = \sqrt{\frac{1}{3}\sum_{i=1}^{3}(\sigma_i - E\varepsilon_i)^2}$
- Different optima: $E_{\max}$ = 5/6, E_{RMS} = 17/21.
- RMS is popular because the objective function is smooth (nonlinear) in design variable





EXAMPLE: KNAPSACK PROBLEM

- Put five items in the knapsack to maximize the total value while the total weight is less than 20 lbs

Item	1	2	3	4	5
Weight (lb)	4	6	7	10	3
Value ($)	12	12	12	27	5

- Design variables: binary design $\begin{cases} b_i = 1 & \text{if item } i \text{ is chosen} \\ b_i = 0 & \text{if item } i \text{ is not chosen} \end{cases}$

- Objective: Total value minus $10 if weight exceeds 20 lb

$$\underset{b_i \in \{0,1\}}{\text{Maximize}} \ value = 12(b_1 + b_2 + b_3) + 27b_4 + 5b_5$$
$$- 10 \times \text{sgn}(4b_1 + 6b_2 + 7b_3 + 10b_4 + 3b_5 - 20)$$

EXERCISES

- Formulate an optimization problem for each of the following situations, identifying the design variables and objective function.
  - Find the aspect ratio of a rectangle with the highest ratio of area to square of the perimeter
  - You need to fly to a city in Florida, rent a car, and visit Gainesville, Jacksonville and Tampa and fly back from the last city you visit. What should your itinerary be to minimize your driving distance?
  - You need to perform a task once a month, on the same day of each month (e.g., the 13th). It is more inconvenient to do on a weekend. Select the day of the month to minimize the number of times it will fall on a weekend in one given year (not a leap year).

CONSTRAINED OPTIMIZATION FORMULATION

- To avoid a non-smooth function, we can add a bound design variable α, as well as error bound constraints

$$\underset{\alpha, E}{\text{Minimize}} \quad \alpha$$
$$\text{subject to} \quad -\alpha \leq \sigma_i - E\varepsilon_i \leq \alpha, \quad i = 1,2,3$$

- The objective function is equal to one design variable, α. *E* appears only in the constraints.

- Both objective and constraints are smooth and linear

- Since we know the sign of the differences, we can rewrite

$$\underset{\alpha, E}{\text{Minimize}} \quad \alpha$$
$$\text{subject to} \quad \sigma_1 - E\varepsilon_1 \leq \alpha, \ \sigma_2 - E\varepsilon_2 \leq \alpha, \quad -\alpha \leq \sigma_3 - E\varepsilon_3$$

GRAPHICAL OPTIMIZATION

- Graphical methods can be used to solve the optimization problem when it has only one or two design variables
- Graphical methods are expensive but help to visualize the design space and to understand the nature of design problem
- Procedure
  - Draw the design space (lower- & upper-bounds of DVs)
  - Plot constraints on the graph (feasible set)
  - Plot contour lines of objective function
  - Find the optimum point (the objective function has the lowest value within the feasible set)

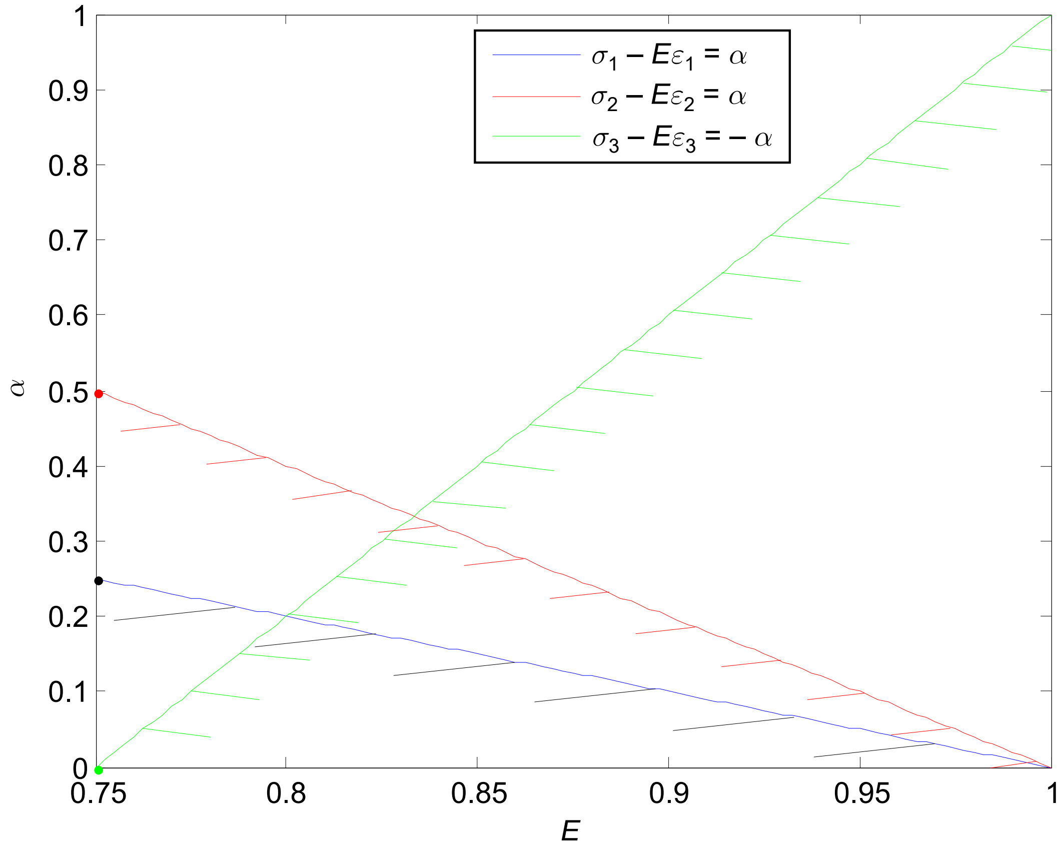
GRAPHICAL OPTIMIZATION *cont.*

$$\underset{\alpha, E}{\text{Minimize}} \quad \alpha$$

$$\text{subject to} \quad \sigma_1 - E\varepsilon_1 \le \alpha$$

$$\sigma_2 - E\varepsilon_2 \le \alpha$$

$$-\alpha \le \sigma_3 - E\varepsilon_3$$



EXAMPLE: GRAPHICAL OPTIMIZATION

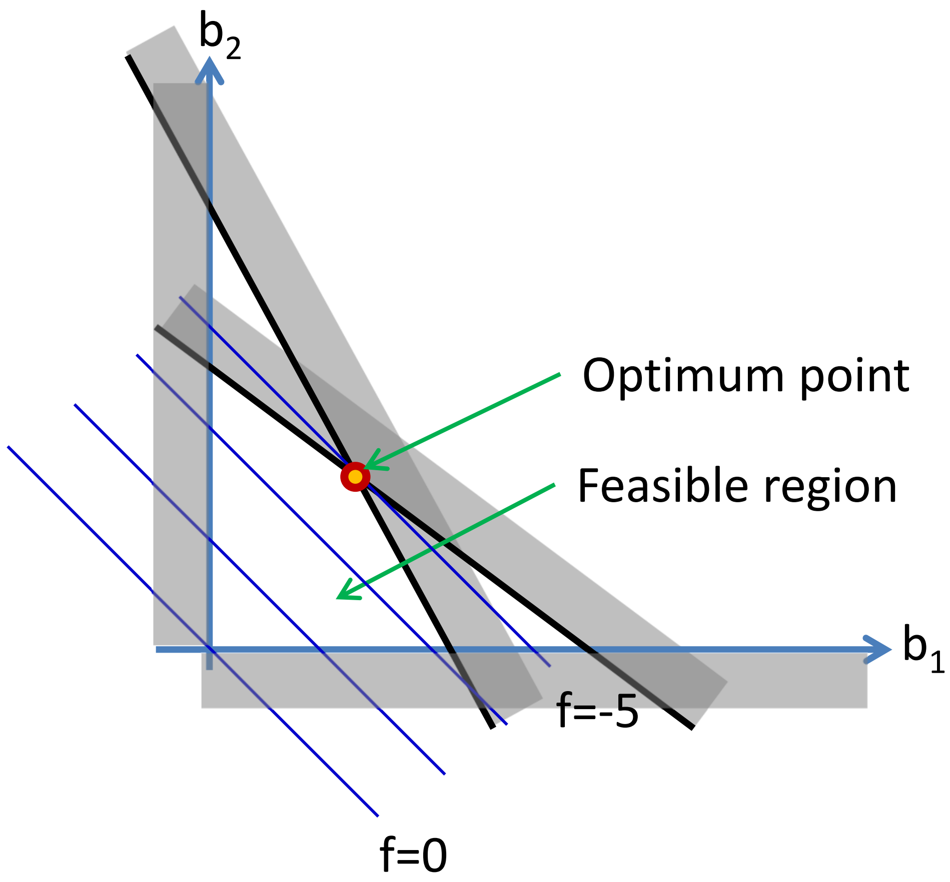
$$\underset{b_1, b_2}{\text{Minimize}} \quad f = -b_1 - b_2$$

$$\text{subject to } 2b_1 + 3b_2 \leq 12$$

$$2b_1 + b_2 \leq 8$$

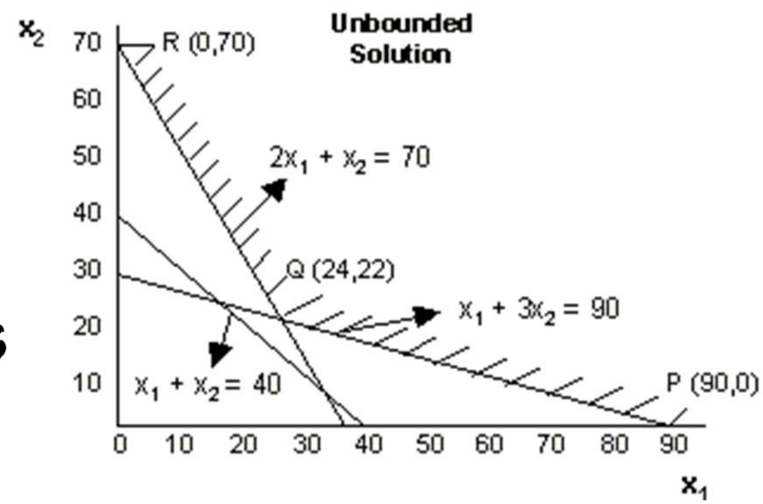
$$b_1, b_2 \geq 0$$

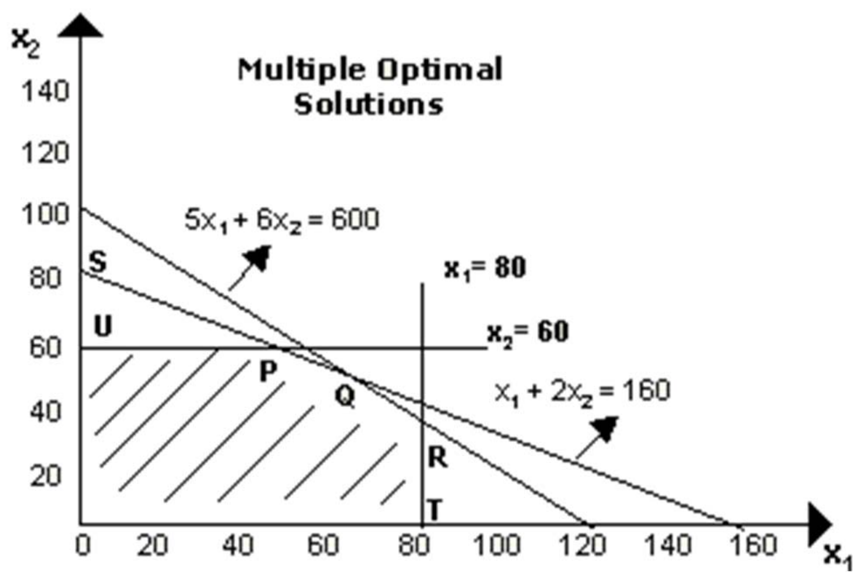
Minimum at (3,2) with f = -5



POSSIBLE DIFFICULTIES

- Unbounded problems
- Problems with multiple optima
- Problems with zero feasible regions
- Problems with no active constraints





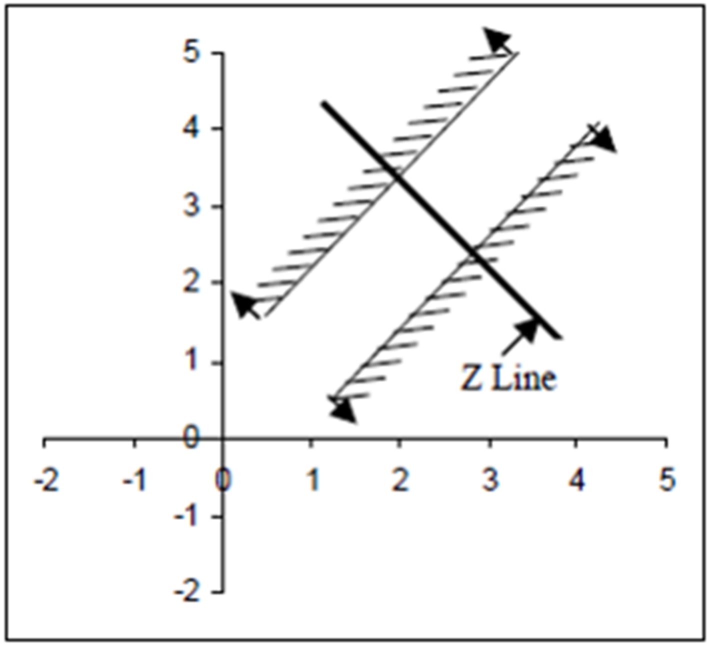


Fig: Infeasible Solution

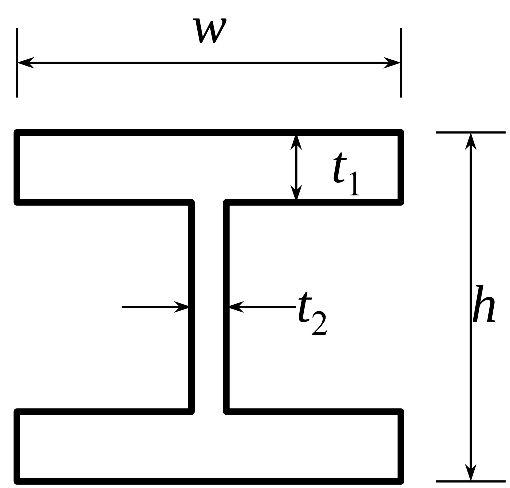
Optimization Problem Formulation

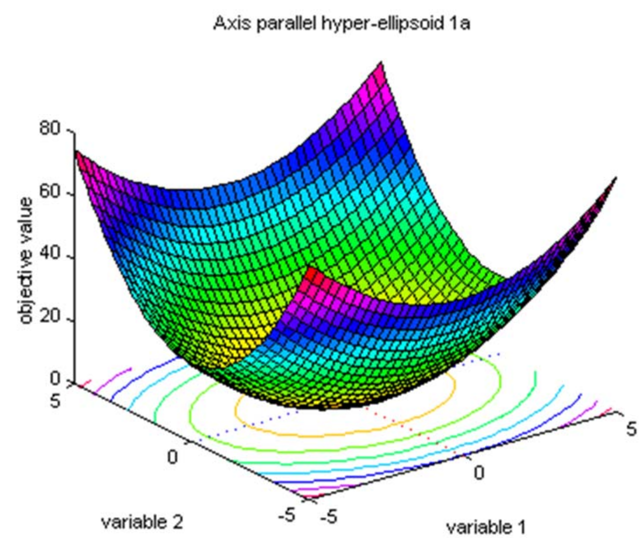
How to deal with different optimization problems?

Use a standard form

THREE-STEP PROBLEM FORMULATION

- Design Parameterization
  - – Clear identification
  - – Independence of designs
- Objective Function
  - – Must be a function of design parameters
  - – Minimization (–Maximization)
- Constraint Functions
  - – Inequality constraints
  - – Equality constraints
  - – Equality constraints must be less than the number of design parameters





- Standard form of design optimization

$$\begin{aligned} &\text{minimize} && f(\mathbf{b}) \\ &\text{subject to} && g_i(\mathbf{b}) \le 0,\ \ i = 1,\cdots,K \\ & && h_j(\mathbf{b}) = 0,\ \ j = 1,\cdots,M \\ & && b_l^L \le b_l \le b_l^U,\ \ l = 1,\cdots,N \end{aligned}$$

$\mathbf{b} = \{ b_1 \quad b_2 \quad \cdots \quad b_N \}^{\mathrm{T}}$: Design variables

$f(\mathbf{b})$: Objective function

$g_1(\mathbf{b}),\cdots,g_K(\mathbf{b})$: Inequality constraints

$h_1(\mathbf{b}),\cdots,h_M(\mathbf{b})$: Equality constraints

$\mathbf{b}^L, \mathbf{b}^U$: Lower and upper bounds

- Feasible set: the set of designs that satisfy constraints

$$S = \{ \mathbf{b} \,|\, g_i(\mathbf{b}) \le 0, i = 1,\cdots,K,\ \ h_j(\mathbf{b}) = 0, j = 1,\cdots,M \}$$

STANDARD FORM *cont.*

- Maximization problem
  - maximize $F(x)$ $\rightarrow$ minimize $f(x) = -F(x)$
- Greater than or equal to constraints
  - $g(x) \geq g_{\min}$ $\rightarrow$ $g_{\min} - g(x) \leq 0$
- The number of equality constraints must be less than that of design variables, $M \leq N$

Standard Types of Optimization Problems

- Linear programming (LP) problem
  - Both objective and constraints are linear functions of designs
  - Many specialized numerical algorithms are available
  - An optimum design is the global optimum
- Quadratic programming (QP) problem
  - The objective is a quadratic function of designs, while all constraints are linear functions of designs.
- Nonlinear programming (NLP) problems
  - Both objective and constraints are nonlinear functions of designs
  - Most general but most difficult to solve

EXAMPLE: STANDARD FORM

- Standard form of Young's modulus fit

$$\underset{\alpha, E}{\text{Minimize}} \quad \alpha$$

$$\text{subject to} \quad \sigma_1 - E\varepsilon_1 \le \alpha$$

$$\sigma_2 - E\varepsilon_2 \le \alpha$$

$$-\alpha \le \sigma_3 - E\varepsilon_3$$

⇓

$$\underset{\alpha, E}{\text{Minimize}} \quad \alpha$$

$$\text{subject to} \quad \sigma_i - E\varepsilon_i - \alpha \le 0$$

$$-\sigma_i + E\varepsilon_i - \alpha \le 0, \qquad i = 1,2,3$$

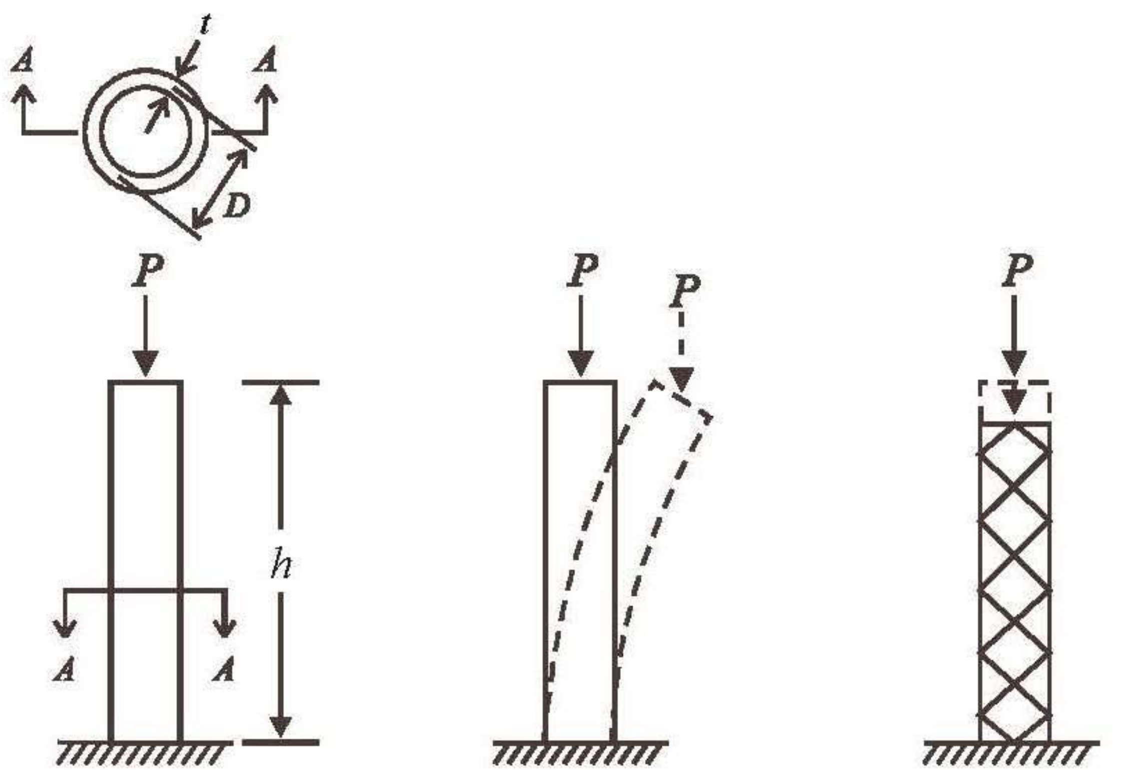
$$0.5 \le E \le 1$$

Normalization

- Different objective and constraints have different orders of magnitudes
  - Stress: Allowable stress of steel = 500 MPa = 5×10^8Pa
  - Displacement: Allowable displacement $\sim 10^{-3}$m
- Although the standard form is fine for mathematical viewpoint, it is numerically difficult to handle such a huge difference in the orders of magnitude
- Normalize the objective and constraints such that their magnitude is in the order of 1
  - $\sigma_{\text{max}} \le \sigma_{\text{allowable}} \rightarrow \dfrac{\sigma_{\text{max}}}{\sigma_{\text{allowable}}} - 1 \le 0$
  - Normalize the objective function using the initial value or target value

EXAMPLE: COLUMN DESIGN

- Height is fixed, design variables are *R* and *t*
- Objective function is cross-sectional area
- Three failure modes



(a) Column (b) Euler buckling (c) Local buckling

EXAMPLE: COLUMN DESIGN FORMULATION 1

- Let $\mathbf{b} = [R, t]$: radius and thickness $R \gg t$
  - $A = 2\pi Rt, I = \pi R^3 t$
- $mass = \rho(hA) = 2\rho h\pi Rt$
- Stress constraint $\sigma = \frac{P}{2\pi Rt} \le \sigma_a$
- Buckling load $P_b = \frac{\pi^3 E R^3 t}{4h^2} \ge P$
- Local buckling failure $\sigma_s = \frac{2Et}{2\mathrm{R}\sqrt{3(1-\nu^2)}} \le \sigma_a$
- Side constraints $R_{\min} \le R \le R_{\max}, t_{\min} \le t \le t_{\max}$

EXAMPLE: COLUMN DESIGN FORMULATION 2

- Let $\mathbf{b} = [R_o, R_i]$: outer and inner radii
  - $A = \pi(R_o^2 - R_i^2), I = \frac{\pi}{4}(R_o^4 - R_i^4)$
- $mass = \rho(hA) = \pi\rho h(R_o^2 - R_i^2)$
- Stress constraint $\sigma = \frac{P}{\pi(R_o^2 - R_i^2)} \le \sigma_a$
- Buckling load $P_b = \frac{\pi^3 E}{16h^2}(R_o^2 - R_i^2) \ge P$
- Local buckling failure $\sigma_s = \frac{2E(R_o - R_i)}{(R_o + R_i)\sqrt{3(1-\nu^2)}} \le \sigma_a$
- Side constraints $R_{o_{\min}} \le R_o \le R_{o_{\max}}, R_{i_{\min}} \le R_i \le R_{i_{\max}}$

EXAMPLE: COLUMN DESIGN STANDARD FORM

- Dimensionless form

$$\underset{R,t}{\text{Minimize}} \quad 2\rho h \pi R t$$

$$\text{subject to} \quad g_1(R,t) = \frac{\sigma}{\sigma_a} - 1 \le 0$$

$$g_2(R,t) = \frac{P}{P_b} - 1 \le 0$$

$$g_3(R,t) = \frac{\sigma_a}{\sigma_s} - 1 \le 0$$

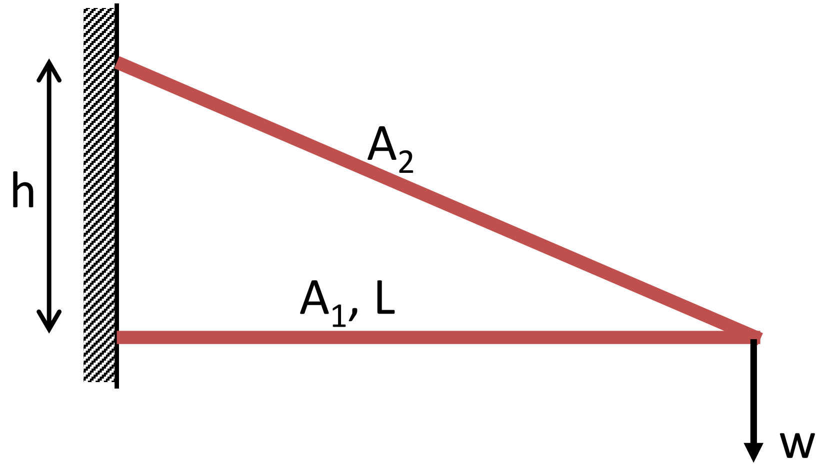
$$R_{\min} \le R \le R_{\max}$$

$$t_{\min} \le t \le t_{\max}$$

- Normalized constraints are better both numerically and for communicating degree of satisfaction.

EXAMPLE

- Design: A_1, A_2, L, h
- Objective: weight
- Constraints:
  - $\sigma_i \le \sigma_Y$, $u_i \le u_{igiven}$, $\sigma_C \le \sigma_{buckling}$
- Standard form



$$\text{Minimize} \quad \rho(A_1 L + A_2\sqrt{L^2 + h^2})$$

$$\text{subject to} \quad \frac{\sigma_i}{\sigma_Y} - 1 \le 0$$

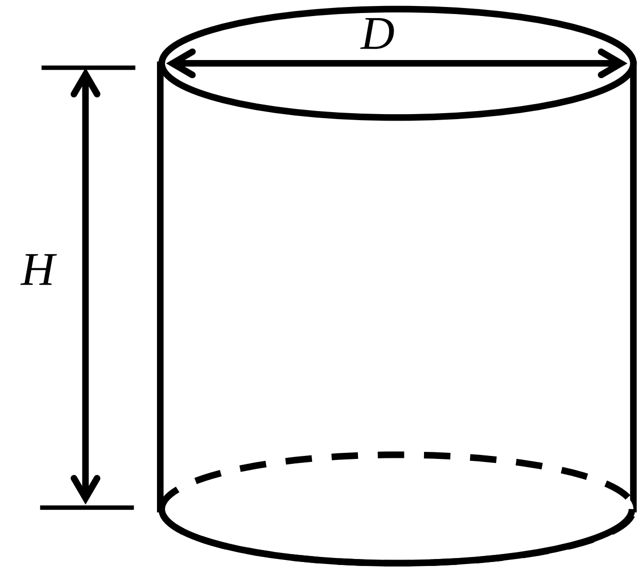
$$\frac{u_i}{u_{i\,given}} - 1 \le 0$$

$$\frac{\sigma_C}{\sigma_{buckling}} - 1 \le 0$$

$$A^L \le A_i \le A^u, \quad L^L \le L \le L^u, \quad h^L \le h \le h^u$$

EXAMPLE: BEER CAN DESIGN

- Design the beer can size so that the minimum amount of sheet metal can be used (minimize manufacturing cost)



- Constraints
  – It is required to hold at least 400 ml of fluid.
  – The diameter of the can should be no more than 8 cm. In addition, it should not be less than 3.5 cm (shipping & handling).
  – The height of the can should be no more than 18 cm and no less than 8 cm.

EXAMPLE: BEER CAN DESIGN *cont.*

- Standard form

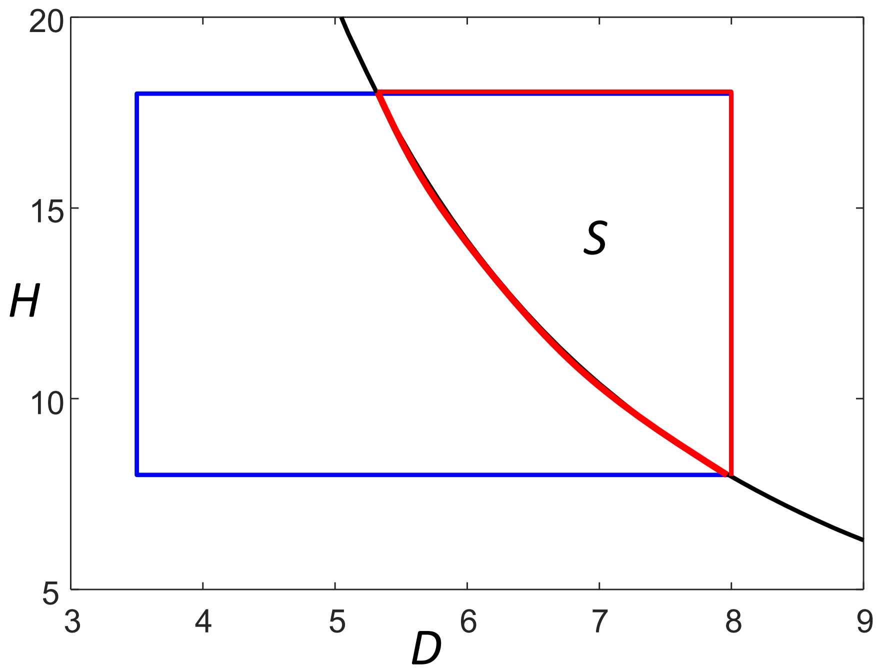
$$\underset{\mathbf{b}}{\text{Minimize}} \quad f(\mathbf{b}) = \pi DH + \frac{\pi}{2}D^2 \quad \text{cm}^2$$

$$\text{subject to} \quad 400 - \frac{\pi}{4}D^2H \le 0 \ \ \text{cm}^3$$

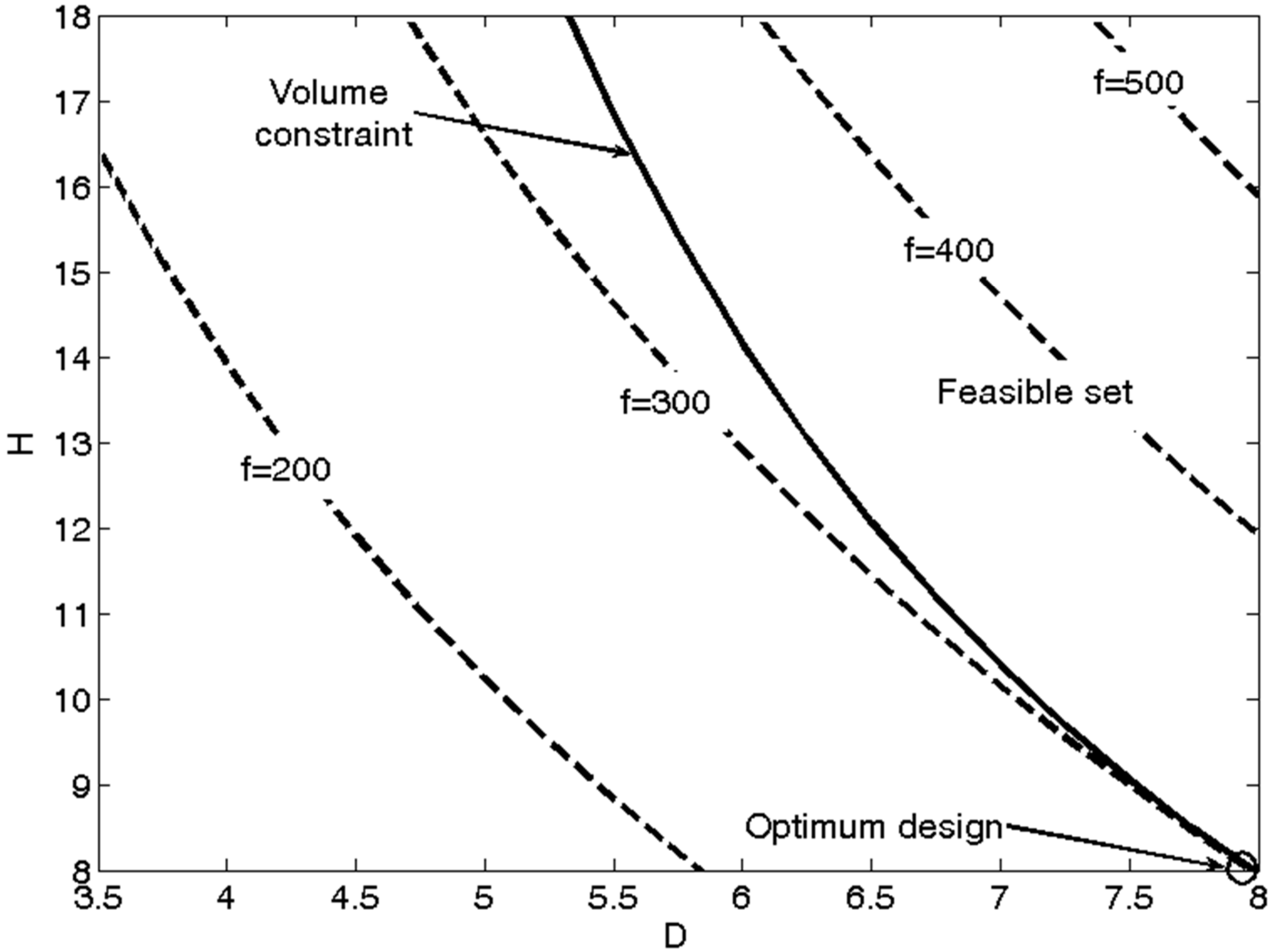
$$3.5 \le D \le 8 \quad \text{cm}$$

$$8.0 \le H \le 18 \quad \text{cm}$$

- Feasible domain



EXAMPLE: BEER CAN OPTIMIZATION



EXERCISES

- Provide two formulations in standard form for minimizing the surface area of a cylinder of a given volume when the diameter and height are the design variables. One formulation should use the volume as equality constraint, and another use it to reduce the number of design variables.

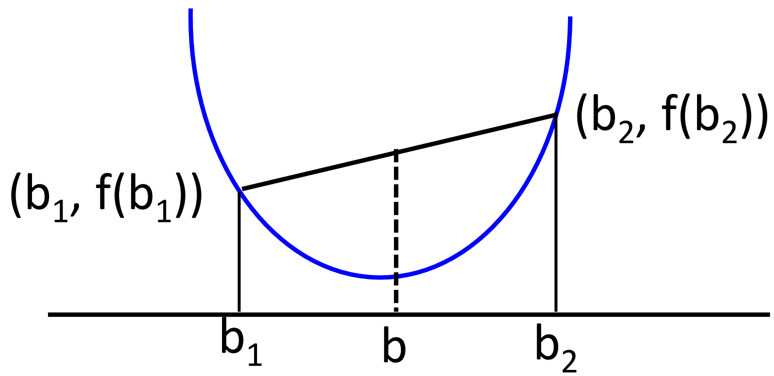
- Formulate in standard normal form the problem of finding an open-top rectangle with an area of at least 50 and minimum perimeter.

- You need to go from point A to point B in minimum time while maintaining a safe distance from point C. Formulate an optimization problem in standard normalized form to find the path with no more than three design variables when A=(0,0), B=(10,10), C=(4,4), and the minimum safe distance is 5.

- A straight line connecting two points will not dip below the function graph

$$f(\mathbf{b}) \le \alpha f(\mathbf{b}^{(2)}) + (1-\alpha) f(\mathbf{b}^{(1)})$$

$$\mathbf{b} = \alpha \mathbf{b}^{(2)} + (1-\alpha)\mathbf{b}^{(1)}$$



- Convexity of a function

$$f(\mathbf{b}) \text{ convex} \quad \Leftrightarrow \quad \nabla^2 f \text{ PSD}$$

$$f(\mathbf{b}) \text{ strictly convex} \quad \Leftrightarrow \quad \nabla^2 f \text{ PD}$$

Sufficient condition: Positive semi-definite Hessian everywhere.

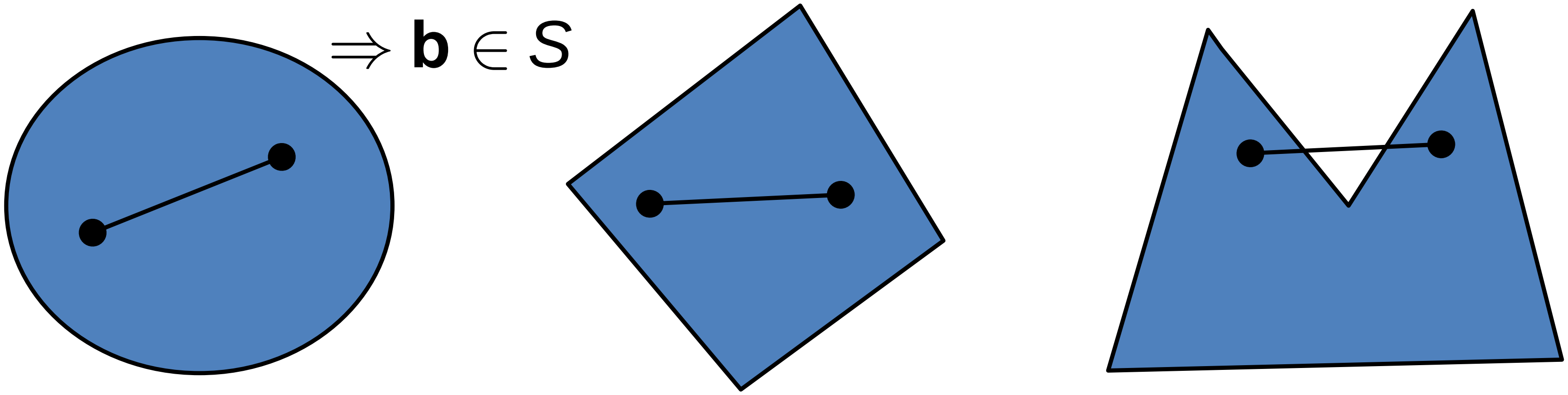
- Convex function will have a single minimum

CONVEX PROBLEM

- Convex Problem: $f(\mathbf{b})$ is a convex function over a convex feasible set S
- If a problem is convex, KKT condition is necessary as well as sufficient
- Any local minimum is also a global minimum
- Convex set: For all $\mathbf{b}^{(1)}, \mathbf{b}^{(2)} \in S$

$$\mathbf{b} = \alpha \mathbf{b}^{(2)} + (1-\alpha)\mathbf{b}^{(1)} \quad 0 \le \alpha \le 1$$

$$\Rightarrow \mathbf{b} \in S$$



Convex set

Non-convex set

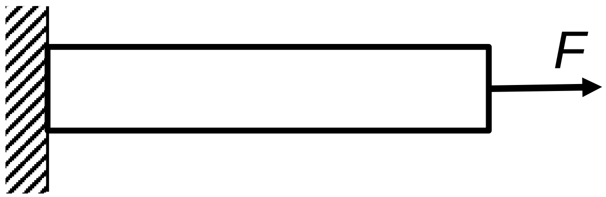
EXERCISES: PROBLEMS CONVEXITY

- Check for convexity the following functions. If the function is not convex everywhere, check its domain of convexity.

1. $b_1^3 + 2b_2^2$
2. $3b_1^2 + 2b_1b_2 + 2b_2^2 - 8$
3. $b_1^3 + 12b_1b_2^2 + 2b_2^2 + 5b_1^2$

RECIPROCAL APPROXIMATION

- Often constraint/objective is inversely proportional to design



$$\delta = \frac{FL}{EA}, \quad \sigma = \frac{F}{A}$$

- An intermediate variable can make the relationship linear

$$y = \frac{1}{A} \quad \Rightarrow \quad \delta = \frac{FL}{E}y, \quad \sigma = Fy$$

- Good for displacement and stress (statically determinate)

RECIPROCAL APPROXIMATION *cont.*

- Reciprocal approximation ($y_i = 1/b_i$) is desirable in many cases because it captures decreasing returns behavior.
- Linear approximation (convex, but inaccurate)

$$f_L(\mathbf{b}) = f(\mathbf{b}_0) + \sum_{i=1}^{n}(b_i - b_{0i})\left(\frac{\partial f}{\partial b_i}\right)_{\mathbf{b}_0}$$

- Reciprocal approximation

$$f_L(\mathbf{y}) = f(\mathbf{y}_0) + \sum_{i=1}^{m}(y_i - y_{0i})\left(\frac{\partial f}{\partial y_i}\right)_{\mathbf{y}_0} \qquad y_i = \frac{1}{b_i}$$

$$\Rightarrow \quad f_R(\mathbf{b}) = f(\mathbf{b}_0) + \sum_{i=1}^{n}(b_i - b_{0i})\boxed{\frac{b_{0i}}{b_i}}\left(\frac{\partial f}{\partial b_i}\right)_{\mathbf{b}_0}$$

CONSERVATIVE-CONVEX APPROXIMATION

- At times we benefit from conservative approximations

$$f_L(\mathbf{b}) - f_R(\mathbf{b}) = \sum_{i=1}^{n} \frac{(b_i - b_{0i})^2}{b_i} \left(\frac{\partial f}{\partial b_i} \right)_{\mathbf{b}_0}$$

- If $b_i \left(\frac{\partial f}{\partial b_i} \right)_{\mathbf{b}_0} < 0$, $f_R(\mathbf{b}) > f_L(\mathbf{b})$ ⇨ more conservative

- Conservative-convex approximation

$$f_C(\mathbf{b}) = f(\mathbf{b}_0) + \sum_{i=1}^{n} F_i (b_i - b_{0i}) \left(\frac{\partial f}{\partial b_i} \right)_{\mathbf{b}_0}$$

$$F_i = \begin{cases} b_{0i}/b_i & \text{if } b_{0i}(\partial f / \partial b_i) \le 0 \\ 1 & \text{otherwise} \end{cases}$$

CONSERVATIVE-CONVEX APPROXIMATION *cont.*

- All second derivatives of f_C are non-negative
- Called convex linearization (CONLIN), Claude Fleury

EXERCISE

- Minimize quadratic objective in a ring

$$\text{Minimize} \quad f(\mathbf{b}) = b_1^2 + 10b_2^2$$
$$\text{subject to } r_i^2 \le b_1^2 + b_2^2 \le r_o^2 \qquad r_i = 10,\ r_o = 20$$

- Is feasible domain convex?
- Solve the optimization problem with fmincon using two functions: quad2 for the objective and ring for constraints

EXERCISE: APPROXIMATIONS

- Construct the linear, reciprocal, and convex approximation at (1,1) to the function
- Plot and compare the function and the two approximations.
- Check on their properties of convexity and conservativeness.

$$f(\mathbf{b}) = 1 + \frac{3}{b_1} - \frac{1}{b_2 + b_1}$$