EAS6939 Homework #4

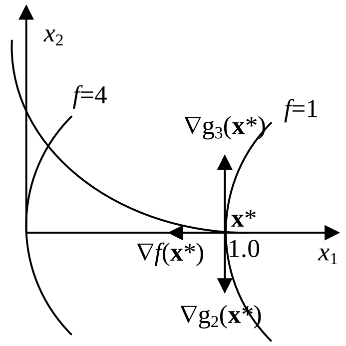
1. Consider the following design optimization problem:

$$\begin{aligned} \text{Minimize} \quad & f(\mathbf{x}) = x_1^2 + x_2^2 - 4x_1 + 4 \\ \text{Subject to} \quad & g_1(\mathbf{x}) = -x_1 \le 0 \\ & g_2(\mathbf{x}) = -x_2 \le 0 \\ & g_3(\mathbf{x}) = x_2 - (1 - x_1)^3 \le 0 \end{aligned}$$

(i) Find the optimum point graphically

(ii) Show that the optimum point does not satisfy K-T condition. Explain

(i) As shown in the figure, (1, 0) is the optimum point and f = 1 at the optimum point.



(ii) The Lagrangian function for the problem can be defined as

$$L = x_1^2 + x_2^2 - 4x_1 + 4 + \lambda_1(-x_1 + s_1^2) + \lambda_2(-x_2 + s_2^2) + \lambda_3[x_2 - (1 - x_1)^3 + s_3^2]$$

The K-T condition is

$$\begin{aligned} 2x_1 - 4 - \lambda_1 + 3\lambda_3(1 - x_1)^3 &= 0 \\ 2x_2 - \lambda_2 + \lambda_3 &= 0 \\ -x_1 + s_1^2 &= 0 \\ -x_2 + s_2^2 &= 0 \\ x_2 - (1 - x_1)^3 + s_3^2 &= 0 \\ \lambda_i s_i &= 0, \qquad i = 1, 2, 3 \end{aligned}$$

At **x** = (1, 0) since g_1 is inactive, and g_2 and g_3 are active, the slack variables should be

$$\lambda_1 = 0, \quad s_2 = 0, \quad s_3 = 0$$

The first equation in the K-T condition can't be satisfied by substituting into these values.

As is clear from the figure, the gradients of two active constraints are not independent: [0, -1] and [0, 1]. In the mathematical term, **x*** is not a **regular point** of the feasible domain. The K-T condition-based optimality conditions assume that the feasible domain is regular, which means the domain doesn't have a singular point.

2. An engineering design problem is formulated as:

Minimize $f(\mathbf{x}) = x_1^2 + 2x_2^2 - 5x_1 - 2x_2 + 10$

Subject to the constraints

$$h_1 = x_1 + 2x_2 - 3 = 0$$

$$g_1 = 3x_1 + 2x_2 - 6 \le 0$$

In all of the following questions, justify your answers.

(i) Write K-T necessary conditions

The Lagrangian function is

$$L = x_1^2 + 2x_2^2 - 5x_1 - 2x_2 + 10 + \lambda_1(x_1 + 2x_2 - 3) + \lambda_2(3x_1 + 2x_2 - 6 + s^2)$$

The K-T condition is

$$2x_1 - 5 + \lambda_1 + 3\lambda_2 = 0$$

$$4x_2 - 2 + 2\lambda_1 + 2\lambda_2 = 0$$

$$x_1 + 2x_2 - 3 = 0$$

$$3x_1 + 2x_2 - 6 + s^2 = 0$$

$$\lambda_2 s = 0$$

Since the second constraint is inequality, its Lagrange multiplier must be greater than or equal to zero.

(ii) How many cases are there to be considered? Identify those cases.

Two case.
Case 1: s = 0
Case 2: $\lambda_2 = 0$

(iii) Find the solution for the case where g_1 is active. Is this acceptable case?

Case s = 0
From third and fourth relation, we have $x_1 = 1.5$ and $x_2 = 0.75$.
From first and second relation, we have $\lambda_1 = -1.75$ and $\lambda_2 = 1.25$. Since λ_2 is greater than zero, the solution is acceptable.

(iv) Regardless of the solution you obtained in (iii), suppose the Lagrange multiplier for the constraint $h_1 = 0$ is $\lambda_1 = -2$ and the Lagrange multiplier for the constraint $g_1 \le 0$ is $\lambda_2 = 1$. If the equality and inequality constraints are simultaneously changed to $h_1 = x_1 + 2x_2 - 3.2 = 0$ and $g_1 = 3x_1 + 2x_2 - 6.2 \le 0$, what will be the new optimum cost?

For the perturbed constraint, we can write

$$h_1 = x_1 + 2x_2 - 3 = 0.2$$

$$g_1 = 3x_1 + 2x_2 - 6 \le 0.2$$

From the assumption, two constraints are active. Thus,

$$\Delta f = -0.2\lambda_1 - 0.2\lambda_2 = -0.2 \times -2 - 0.2 \times 1 = 0.2 \text{ and } f_{NEW} = f_{ORG} + \Delta f = 4.575$$

3. A design problem is formulated as an unconstrained optimization problem to minimize

$$f(x_1, x_2, x_3) = x_1^3 + 2x_2^2 + 2x_3^2 + 4x_1x_3 + 2x_2x_3$$

(i) Calculate the gradient of the cost function at (1, 1, 1)

$$\nabla f = \begin{Bmatrix} 3x_1^2 + 4x_3 \\ 4x_2 + 2x_3 \\ 4x_3 + 4x_1 + 2x_2 \end{Bmatrix} = \begin{Bmatrix} 7 \\ 6 \\ 10 \end{Bmatrix}$$

(ii) Calculate Hessian at the point (2, 1, 1)

$$\mathbf{H} = \begin{bmatrix} 6x_1 & 0 & 4 \\ 0 & 4 & 2 \\ 4 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 12 & 0 & 4 \\ 0 & 4 & 2 \\ 4 & 2 & 4 \end{bmatrix}$$

(iii) Is the cost function $f(\mathbf{x})$ a convex function? Why or Why not?

In order to be a convex function, the Hessian matrix must be positive definite. The determinants of three principal minor matrices are $6x_1$, $24x_1$, and $72x_1 - 64$. These determinants can be negative or positive depending on the value of x_1. Thus, $f(\mathbf{x})$ is not a convex function.

(iv) Is the cost function $f(\mathbf{x})$ convex for the region $x_1 > 1$? Why or Why not?

For $x_1 > 1$, the determinants of three principal minor matrices become positive. Thus, $f(\mathbf{x})$ is a convex function.

(v) Show that (0, 0, 0) is a stationary point. Is this a minimum, maximum, or inflection point? Why?

At (0, 0, 0), $\nabla f = \mathbf{0}$, but $\mathbf{H}$ is neither positive definite nor negative definite. Thus, (0, 0, 0) is a stationary point. But, this point is neither maximum nor minimum. In fact, it is an inflection point.