EAS6939 Homework #6

1. In engineering, the random wind pressure (Y) is typically modeled as a quadratic transformation of the random wind speed (X). If $X \sim N(\mu_x, \sigma_x^2)$ and $Y = X^2$, find approximation of Y based on the first-order Taylor series expansion about mean of X and equivalent linearization. For equivalent linearization, consider $Y_L = aX + b$, where a and b are optimal parameters. Calculate the mean and variance of the above approximations of Y.

Solution:

1) First-order Taylor series expansion:

$$Y_L = Y(\mu_x) + \left.\frac{dY}{dX}\right|_{x=\mu_x} (X - \mu_x) = 2\mu_x X - \mu_x^2$$

Therefore, the approximate mean and variance are

$$\mu_Y = E[Y_L] = 2\mu_x E[X] - \mu_x^2 = \mu_x^2$$

$$\sigma_Y^2 = (2\mu_x)^2 Var[X] = 2\mu_x^2 \sigma_x^2$$

2) Equivalent linearization:

The model parameters a and b can be obtained from:

$$\underset{a,b}{\text{minimize}} \;\; E[Y_L - Y]^2$$

For a general case, the minimizing conditions become

$$aE[X^2] + bE[X] = E[X^3]$$

$$aE[X] + b = E[X^2]$$

From class,

$$E[X^2] = \sigma_x^2 + \mu_x^2$$

$$E[X^3] = 2\mu_x \sigma_x^2 + \mu_x^3$$

Therefore, parameters a and b are calculated by

$$a = 2\mu_x \qquad b = \sigma_x^2 - \mu_x^2$$

And the linearized equation becomes

$$Y_L = 2\mu_x X + \sigma_x^2 - \mu_x^2$$

And the mean and variance become

$$\mu_Y = \sigma_x^2 + \mu_x^2$$

$$\sigma_Y^2 = 4\mu_x^2 \sigma_x^2$$

2. A vehicle has a deterministic mass, $m = 2$, and random velocity, V, which can take on both positive and negative values. The kinetic energy (K) of the vehicle is $K = \frac{1}{2} mV^2$. If V follows Normal (Gaussian) probability distribution with mean, $m_V = 0$, and standard deviation, $\sigma_V = 1$, determine the probability density function and cumulative probability distribution function of K. Use the method of general transformation.

Solution:

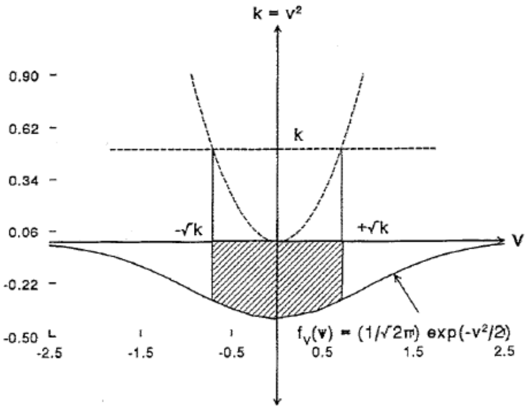
Since the relationship, $K = V^2$, is nonlinear, the method of general transformation needs to be used.

CDF of K:

$$
\begin{aligned}
F_K(k) &= P(K \le k) \\
&= P(V^2 \le k) \\
&= \int_{R=\{V^2 \le k\}} f_V(v)\mathrm{d}v \\
&= 2\int_{-\sqrt{k}}^{0} \tfrac{1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}v^2\right)\mathrm{d}v \\
&= 2\left[\tfrac{1}{2} - F_V\left(-\sqrt{k}\right)\right] \\
&= 1 - 2F_V\left(-\sqrt{k}\right) \qquad \text{for } k \ge 0
\end{aligned}
$$

Since V is Normal with zero mean, $F_V(-v) = 1 - F_V(v)$ for any v. Hence, the CDF of K can be written as

$$F_K(k) = 2F_V\left(\sqrt{k}\right) - 1 = F_V\left(\sqrt{k}\right) - F_V\left(-\sqrt{k}\right)$$



Since K is always positive, the complete PDF of K is

$$F_K(k) = \begin{cases} 0 & k < 0 \\ F_V\left(\sqrt{k}\right) - F_V\left(-\sqrt{k}\right) & k \ge 0 \end{cases}$$

PDF of K:

By differentiating PDF of K,

$$
\begin{aligned}
f_K(k) &= \frac{\mathrm{d}F_K(k)}{\mathrm{d}k} = 2\frac{\mathrm{d}F_V\left(\sqrt{k}\right)}{\mathrm{d}k} \\
&= 2 f_V\left(\sqrt{k}\right)\frac{1}{2}\frac{1}{\sqrt{k}} \\
&= \frac{1}{\sqrt{2\pi k}}\exp\left(-\tfrac{1}{2}k\right), \quad k \ge 0
\end{aligned}
$$

This is one-degree-of-freedom chi-square probability.

3. The resistance (or strength), R, of a mechanical component which is subject to a load, S, are modeled as random variables with the following probability density function:

$$f_R(r) = \begin{cases} 0.5 & 0 \le r \le 2 \\ 0 & \text{otherwise} \end{cases} \qquad f_S(s) = \begin{cases} 2s & 0 \le s \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Assume that R and S are statistically independent. Find the cumulative probability distribution functions, $F_Y(y)$ and $F_Z(z)$ of

(a) $Y = R - S$

(b) $Z = R/S$

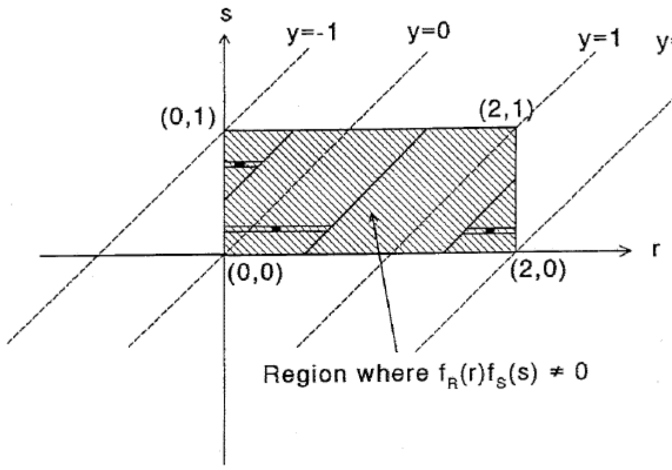
Furthermore, evaluate

(c) $F_Y(0)$

(d) $F_Z(1)$
(e) Explain why $F_Y(0) = F_Z(1)$

Solution:
(a) CDF of $Y = R - S$

$$F_Y(y) = P(Y \le y) = P(R - S \le y) = P(R - S - y \le 0) = \iint\limits_{R=\{r-s-y\le 0\}} f_R(r)\, f_S(s)\mathrm{d}r\mathrm{d}s$$



For -1<y<0

$$F_Y(y) = \int_{-y}^{1}\int_{0}^{s+y} \tfrac{1}{2}(2s)\mathrm{d}r\mathrm{d}s = \int_{-y}^{1} s(s+y)\mathrm{d}s = \left[\tfrac{s^3}{3} + y\tfrac{s^2}{2}\right]\Big|_{-y}^{1} = \tfrac{1}{3} + \tfrac{y}{2} - \tfrac{y^3}{6}$$

For 0<y<1

$$F_Y(y) = \int_{0}^{1}\int_{0}^{s+y} \tfrac{1}{2}(2s)\mathrm{d}r\mathrm{d}s = \int_{0}^{1} s(s+y)\mathrm{d}s = \left[\tfrac{s^3}{3} + y\tfrac{s^2}{2}\right]\Big|_{0}^{1} = \tfrac{1}{3} + \tfrac{y}{2}$$

For 1<y<2

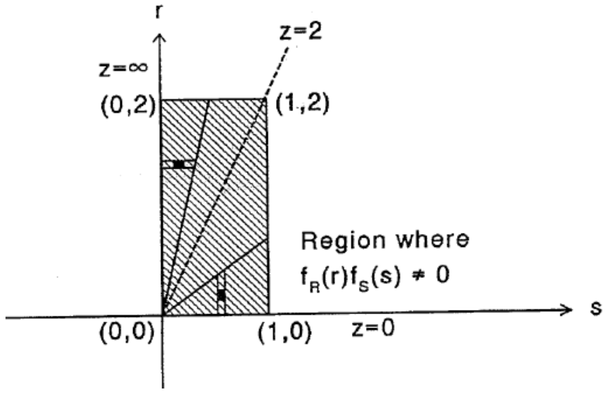
$$F_Y(y) = 1 - \int_{0}^{2-y}\int_{s+y}^{2} \tfrac{1}{2}(2s)\mathrm{d}r\mathrm{d}s = 1 - \int_{0}^{2-y} s(2-s-y)\mathrm{d}s = 1 - \left[s^2 - \tfrac{s^3}{3} - y\tfrac{s^2}{2}\right]\Big|_{0}^{2-y} = 1 - \tfrac{(2-y)^3}{6}$$

Therefore

$$F_Y(y) = \begin{cases} 0 & y \le -1 \\ \tfrac{1}{3} + \tfrac{y}{2} - \tfrac{y^3}{6} & -1 \le y \le 0 \\ \tfrac{1}{3} + \tfrac{y}{2} & 0 \le y \le 1 \\ 1 - \tfrac{(2-y)^3}{6} & 1 \le y \le 2 \\ 1 & 2 \le y \end{cases}$$

(a) CDF of $Z = R/S$

$$F_Y(y) = P(Z \le z) = P\left(\frac{R}{S} - z \le 0\right) = \iint\limits_{R=\{r/s-z\le 0\}} f_R(r)\, f_S(s)\mathrm{d}r\mathrm{d}s$$



For 0<z<2

$$F_Z(z)=\int_0^1\int_0^{sz}\tfrac{1}{2}(2s)\mathrm{d}r\mathrm{d}s=\int_0^1 s^2 z\mathrm{d}s=\left[z\tfrac{s^3}{3}\right]\Big|_0^1=\frac{z}{3}$$

For z>2

$$F_Z(z)=1-\int_0^2\int_0^{r/z}\tfrac{1}{2}(2s)\mathrm{d}s\mathrm{d}r=1-\frac{1}{2}\int_0^2\frac{r^2}{z^2}\mathrm{d}r=1-\frac{1}{2z^2}\left[\frac{z^3}{3}\right]\Big|_0^2=1-\frac{4}{3z^2}$$

Therefore

$$F_Z(z)=\begin{cases}0 & z\le 0\\ \dfrac{z}{3} & 0\le z\le 2\\ 1-\dfrac{4}{3z^2} & z\ge 2\end{cases}$$

(c) $F_Y(0)=\frac{1}{3}+\frac{0}{2}=\frac{1}{3}$

(d) $F_Z(1)=\frac{1}{3}$

(e) $F_Y(0)=P(Y\le 0)=P(R-S\le 0)=P(R/S-1\le 0)=P(Z\le 1)=F_Z(1)$

Both probabilities represent the probability of event when R < S; i.e., when the applied load exceeds the resistance of the mechanical component. Since this is a failure event for this component, this probability if called the probability of failure.