# STRUCTURAL DESIGN USING FINITE ELEMENTS

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# INTRODUCTION

- FEA: determining the response of a given structure for a given set of loads and boundary conditions
  - Geometry, material properties, BCs and loads are well defined
- Engineering design: a process of synthesis in which parts are put together to build a structure that will perform a given set of functions satisfactorily
- Analysis is very systematic and can be taught easily; design is an iterative process

- Creative design: creating a new structure or machine that does not exist
- **Adaptive design**: modifying an existing design (evolutionary process)

# INTRODUCTION – STRUCTURAL DESIGN

- Structural design: a procedure to improve or enhance the **performance** of a structure by changing its **parameters**
- Performance: a measurable quantity (constraint and goal)
  - the weight, stiffness or compliance; the fatigue life; noise and vibration levels; safety
- Constraint: As long as the performance satisfies the criterion, its level is not important
  - Ex: the maximum stress should be less than the allowable stress
- Goal: the performance that the engineer wants to improve as much as possible
- Design variables: system parameters that can be changed during the design process
  - Plate thickness, cross-sectional area, shape, etc

# EXAMPLE

- Design the height h of cantilevered beam with $S_F = 1.5$
  - $E = 2.9\times10^4$ ksi, $w = 2.25$ in.

1) Allowable tip deflection $D_{\text{allowable}} = 2.5$ in. (No need $S_F$)
   - FE equation after applying BCs

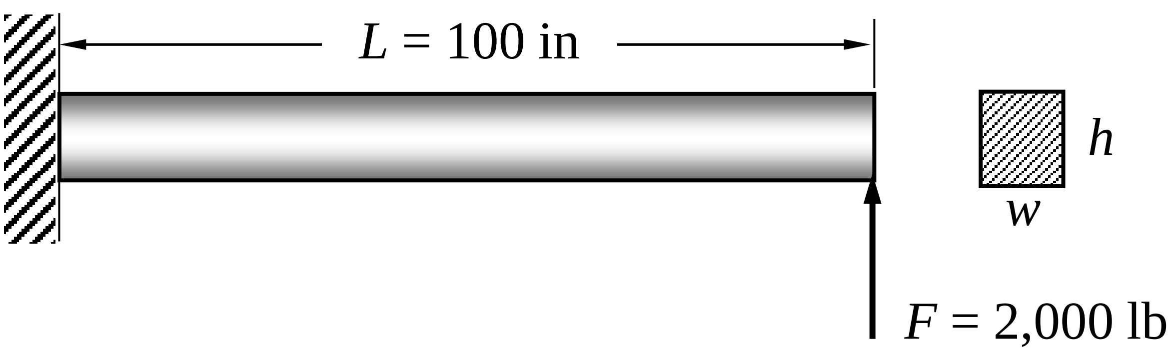
$$\frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix}$$

   - FE solution

$$v_2 = \frac{4FL^3}{Ewh^3}, \quad \theta_2 = \frac{6FL^2}{Ewh^3}$$

$$v_2 = \frac{4FL^3}{Ewh^3} = D_{\text{allowable}}$$

$$\Rightarrow \quad h = \sqrt[3]{\frac{4FL^3}{EwD_{\text{allowable}}}} = 3.66 \text{ in}$$



# EXAMPLE *cont.*

2) Failure strength = 40 ksi (Need $S_F$)

- Supporting moment at the wall

$$C_1 = \frac{EI}{L^3}\left[6Lv_1 + 4L^2\theta_1 - 6Lv_2 + 2L^2\theta_2\right] = -FL$$

- Maximum stress at the wall

$$\sigma_{\max} = \frac{M\frac{h}{2}}{I} = \frac{6FL}{wh^2}$$

- Height calculation with the factor of safety

$$\frac{6FL}{wh^2} = \frac{\sigma_F}{S_F} \quad \Rightarrow \quad h = \sqrt{\frac{6FLS_F}{w\sigma_F}} = 4.47 \text{ in}$$

# FSD EXAMPLE – CANTILEVERED BEAM

- w = 2.25, h = 3.5 in. Determine new height using FSD
- Section modulus and max. stress at the initial design

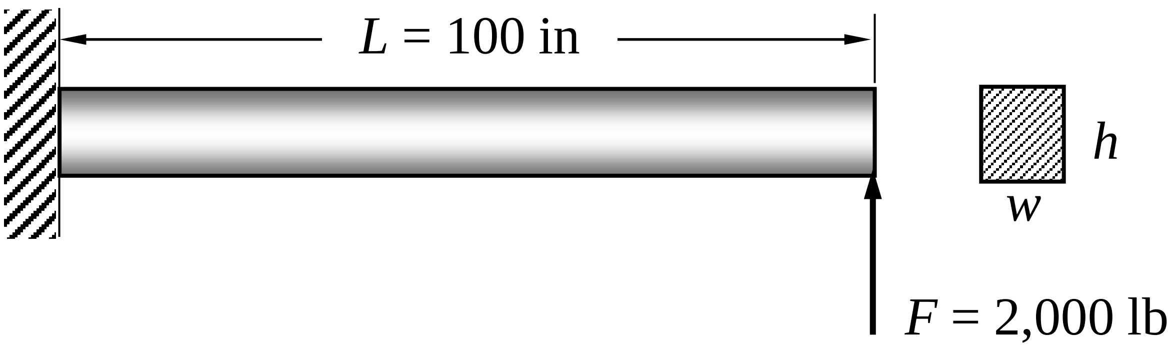
$$S_{old} = \frac{2I}{h} = \frac{wh^2}{6} = \frac{2.25 \times 3.5^2}{6} = 4.594 \text{ in}^3$$

$$\sigma_{max} = \frac{M}{S_{old}} = 43.537 \text{ ksi}$$

- New section modulus using stress ratio resizing

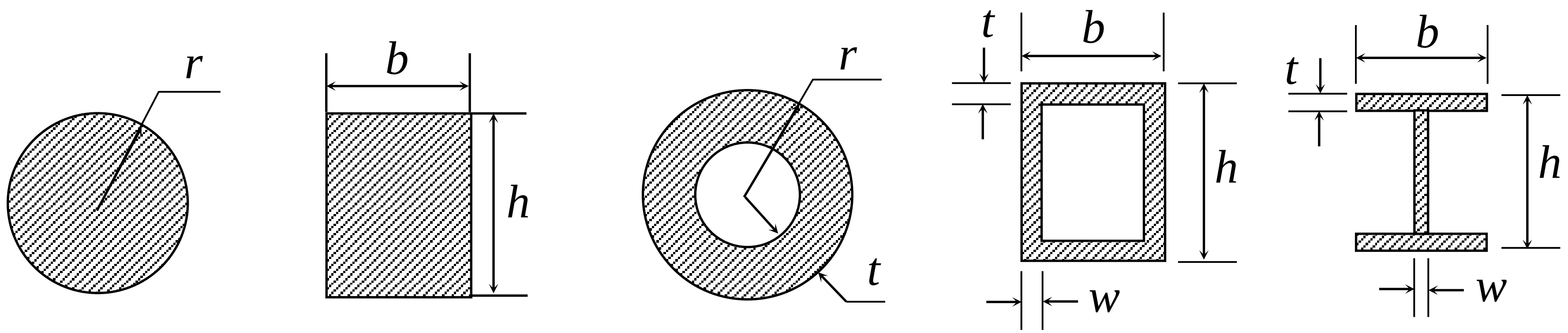
$$S_{new} = S_{old} \frac{\sigma_{max}}{\sigma_{allowable}} = 4.594 \times \frac{43.537}{26.667} = 7.5 \text{ in}^3$$

$$S_{new} = \frac{wh^2}{6} \quad \Rightarrow \quad h = \sqrt{\frac{6S_{new}}{w}} = 4.47 \text{ in}$$



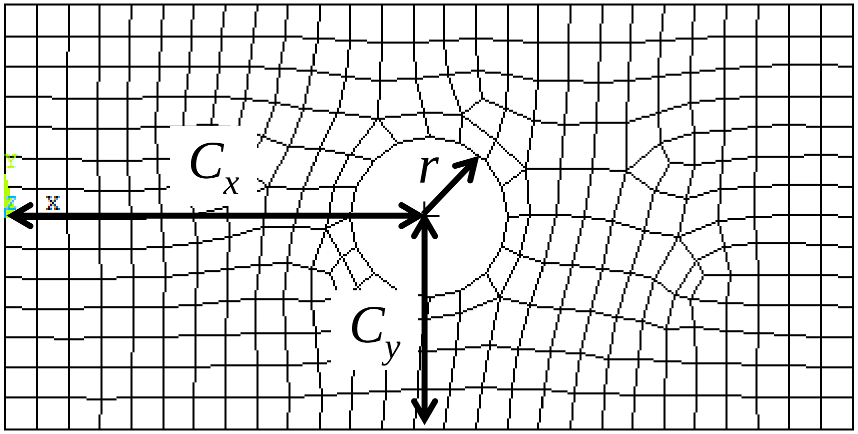
# DESIGN PARAMETERS

- Selecting design variables – easy for beam and truss, but more complicated for plane or 3D solids
- Material property design variable
  - Varying material properties to find the best material
  - Not common, but useful for designing composite materials
- Sizing design variable
  - Geometric parameters as design (parametric design variable)
  - Appears as a parameter in FEM
  - Thickness of plate/shell, cross-sectional geometry of truss/beam, etc

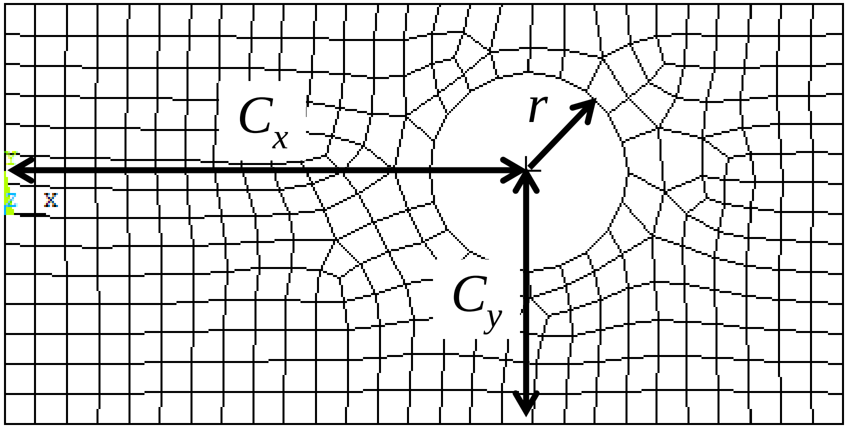


# DESIGN PARAMETERS *cont.*

- Shape design variable
  - Related to the structure's geometry, which does not appear explicitly as a parameter
  - Beam cross-section is a geometry, but it appears as a moment of inertia
  - $C_x$, $C_y$, and $r$ determine the size and location of the hole
  - Shape design variables change FE mesh
  - Design variables must be limited so that the hole remains inside of the plate



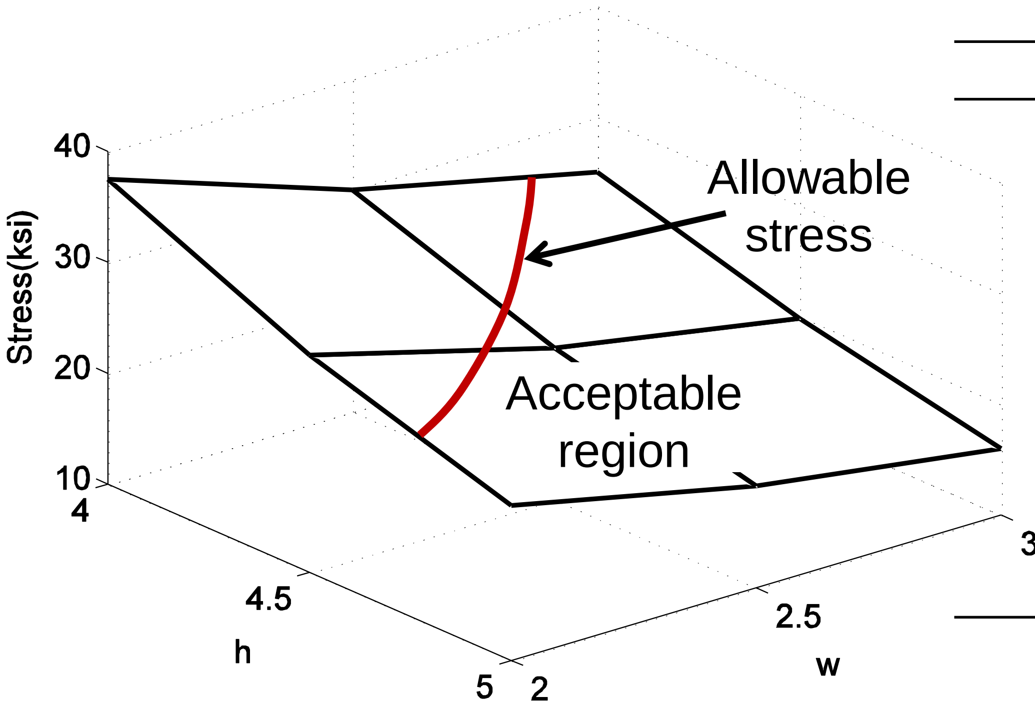

(a) Initial design




(b) Perturbed design

# PARAMETER STUDY – SENSITIVITY ANALYSIS

- Parameter study
  - Effect of a design variable on performance (gradual change of DV)
  - Cantilevered beam example:




| $w$ (in) | $h$ (in) | $\sigma_{max}$ (ksi) |
|---|---|---|
| 2.0 | 4.0 | 37.5 |
| 2.0 | 4.5 | 29.6 |
| 2.0 | 5.0 | 24.0 |
| 2.5 | 4.0 | 30.0 |
| 2.5 | 4.5 | 23.7 |
| 2.5 | 5.0 | 19.2 |
| 3.0 | 4.0 | 25.0 |
| 3.0 | 4.5 | 19.8 |
| 3.0 | 5.0 | 16.0 |

# SENSITIVITY ANALYSIS

- Parameter study becomes too expensive with many DVs
- Unable to capture rapid change in performance locally
- Design sensitivity analysis computes the rate of performance change with respect to design variables
- Sensitivity analysis calculates gradient of performance for optimization
- Explicit dependence
  - Analytical relationship exists between performance and DVs
  - Weight of circular cross-section beam

$$W(r) = \pi r^2 L$$

  - Sensitivity w.r.t. $r$: $\dfrac{dW}{dr} = 2\pi r L$

# SENSITIVITY ANALYSIS *cont.*

- Implicit dependence
  - Performance depends on DVs through state variable (displacement)
  - Sensitivity of stress:

$$\frac{d\sigma}{dr} = \frac{d\sigma}{d\mathbf{q}} \cdot \frac{d\mathbf{q}}{dr}$$

Difficult to calculate, time consuming

Easy to calculate from given expression of stress

- How to calculate displacement sensitivity?
  - Differentiate finite element equation: $[\mathbf{K}(b)]\{\mathbf{Q}\} = \{\mathbf{F}(b)\}$

$$[\mathbf{K}]\left\{\frac{d\mathbf{Q}}{db}\right\} = \left\{\frac{d\mathbf{F}}{db}\right\} - \left[\frac{d\mathbf{K}}{db}\right]\{\mathbf{Q}\}$$

  - $[d\mathbf{K}/db]$ and $\{d\mathbf{F}/db\}$ can be evaluated using either their analytical expression or numerical differentiation

# SENSITIVITY ANALYSIS *cont.*

- Sensitivity equation must be solved for each DV
- Sensitivity equation uses the same stiffness matrix with the original finite element analysis
- Consider RHS as a pseudo-force vector
- Similar to finite element analysis with multiple load cases
- Thus, solving sensitivity equation is very inexpensive using factorized stiffness matrix
- General form of performance

$$H = H\left(\mathbf{q}(b), b\right)$$

  - Sensitivity

Implicit dependent term

$$\frac{dH(\mathbf{q}(b),b)}{db} = \left.\frac{\partial H}{\partial b}\right|_{\mathbf{q}=\text{const}} + \left.\frac{\partial H}{\partial \mathbf{q}}\right|_{b=\text{const}} \cdot \frac{d\mathbf{q}}{db}$$

Explicit dependent term

# FINITE DIFFERENCE SENSITIVITY

- Easiest way to compute sensitivity information of the performance
- Calculate performance at two different designs
- Forward difference method

$$\frac{\mathrm{d}H}{\mathrm{d}b} \approx \frac{H(b+\Delta b) - H(b)}{\Delta b}$$

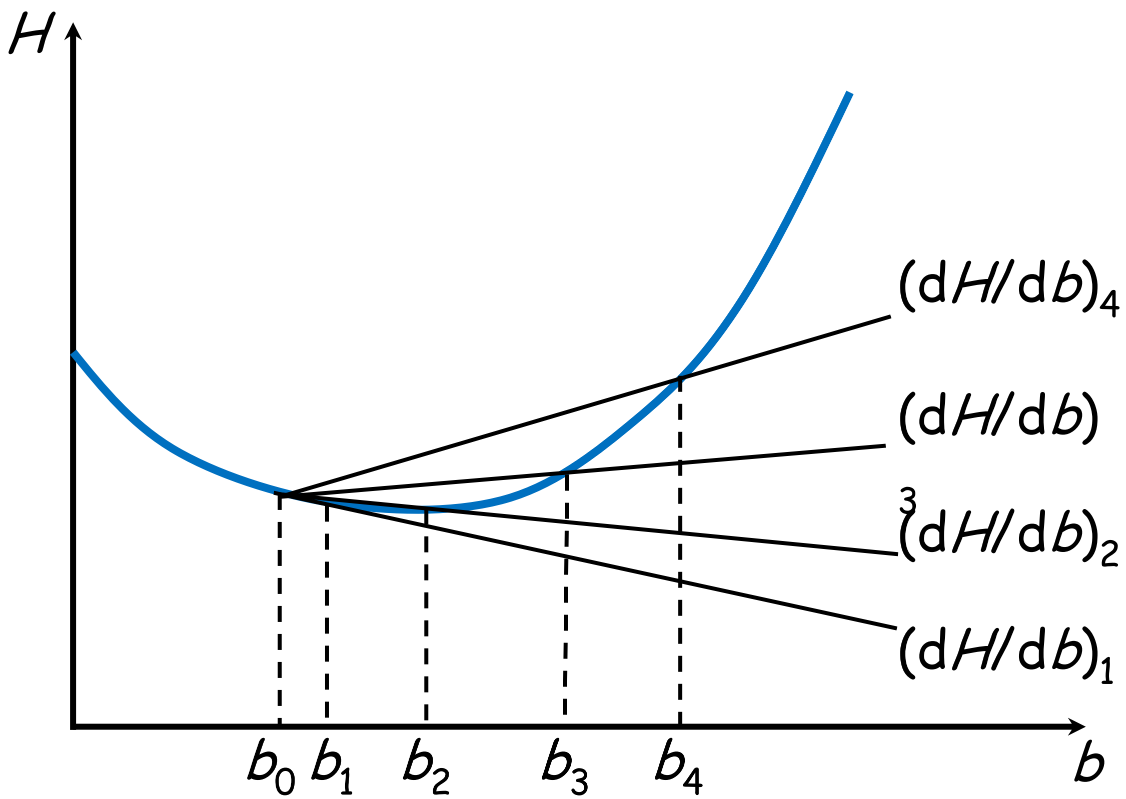
- Central difference method

$$\frac{\mathrm{d}H}{\mathrm{d}b} \approx \frac{H(b+\Delta b) - H(b-\Delta b)}{2\Delta b}$$

- Consider FEA as a black-box
- Sensitivity computation cost becomes high for many design variables
  - N+1 analyses for forward FDM
  - 2N+1 analyses for central FDM

# FINITE DIFFERENCE SENSITIVITY *cont.*

- Accuracy of finite difference sensitivity
  - Accurate results can be expected when Δb approaches zero
  - For nonlinear performances, a large perturbation yields completely inaccurate results
  - Numerical noise becomes dominant for a too-small perturbation size



# EXAMPLE – CANTILEVERED BEAM

- At optimum design (w=2.25 in, h=4.47 in), calculate sensitivity of tip displacement w.r.t. *h*
- Exact sensitivity:

$$\left.\frac{dv_2}{dh}\right|_{exact} = -\frac{12FL^3}{Ewh^4} = -\frac{12\times 2,000\times 100^3}{2.9\times 10^7\times 2.25\times 4.47^3} = -4.118$$

- Differentiate [K]

$$\left[\frac{d\mathbf{K}}{db}\right]\{\mathbf{Q}\} = \frac{F}{4Lh}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 4L \\ 6 \end{Bmatrix} = \frac{F}{4h}\begin{Bmatrix} -12 \\ -12L \\ 12 \\ 0 \end{Bmatrix}$$

- Pseudo load vector

$$\left\{\frac{d\mathbf{F}}{db}\right\} - \left[\frac{d\mathbf{K}}{db}\right]\{\mathbf{Q}\} = \frac{F}{4h}\begin{Bmatrix} 12 \\ 12L \\ -12 \\ 0 \end{Bmatrix}$$

# EXAMPLE – CANTILEVERED BEAM *cont.*

- Sensitivity equation:

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} dv_1/db = 0 \\ d\theta_1/db = 0 \\ dv_2/db \\ d\theta_2/db \end{Bmatrix} = \frac{F}{4h}\begin{Bmatrix} 12 \\ 12L \\ -12 \\ 0 \end{Bmatrix}$$

- Same way of applying BC

$$\frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} dv_2/db \\ d\theta_2/db \end{Bmatrix} = \frac{F}{4h}\begin{Bmatrix} -12 \\ 0 \end{Bmatrix}$$

- Sensitivity of nodal DOFs

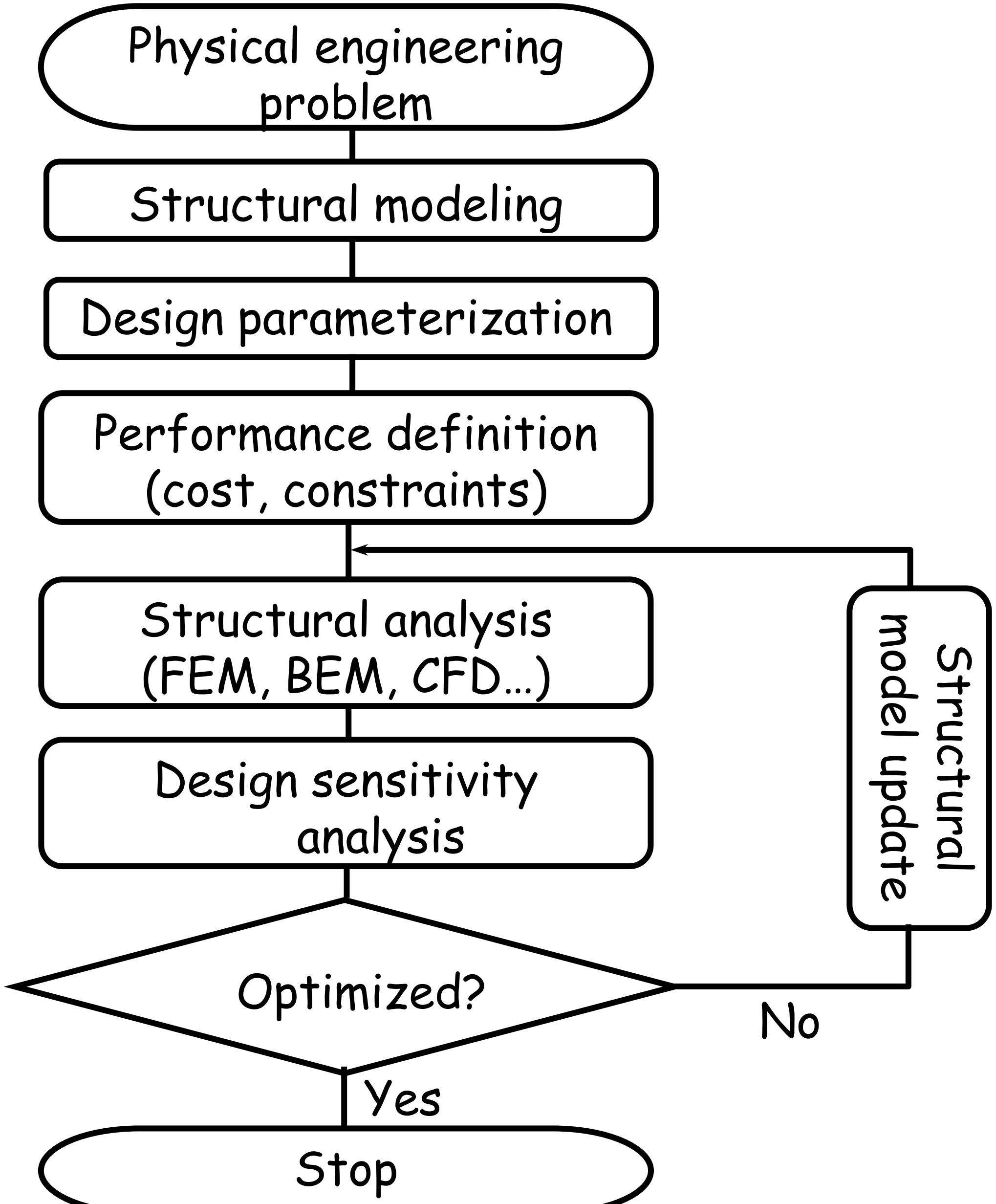
$$\frac{dv_2}{db} = -\frac{12FL^3}{Ewh^4}, \quad \frac{d\theta_2}{db} = -\frac{18FL^2}{Ewh^3}$$

  – Same with the exact sensitivity

# STRUCTURAL OPTIMIZATION

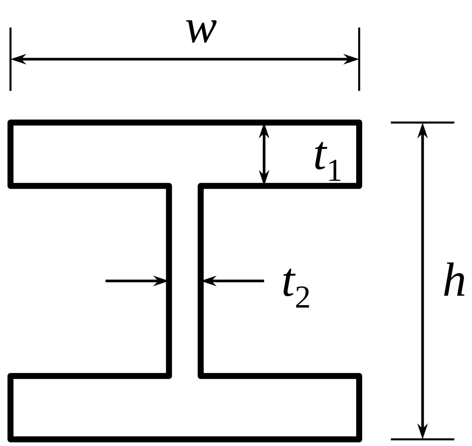
- What Is Design Optimization?
  - To find the best design parameters that meet the design goal and satisfies constraints.
- Design Parameters: Anything the Designer Can Change
  - Thickness of a plate
  - Cross-sectional geometry of a beam or truss
  - Geometric dimensions
- Design Goal: Objective Function
  - Design criterion that will be minimized (or maximized)
  - Mass, Stress, Displacement, Natural Frequency, ETC
- Constraint: Conditions that the system must satisfy
  - Stress, Displacement, ETC
- Note: Design parameters must affect the design goal and constraints.

# OPTIMIZATION FLOW CHART



# THREE-STEP PROBLEM FORMULATION

1. Design Parameterization
   - Clear identification
   - Independence of designs




2. Objective Function
   - Must be a function of design parameters
   - Minimization ( –Maximization)
3. Constraint Functions
   - Inequality constraints
   - Equality constraints
   - Equality constraints must be less than the number of design parameters

# STANDARD FORM

- **Standard form** of design optimization

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{b}) \\ \text{subject to} \quad & g_i(\mathbf{b}) \le 0, \ i = 1, \cdots, N \\ & h_j(\mathbf{b}) = 0, \ j = 1, \cdots, M \\ & b_l^L \le b_l \le b_l^U, \ l = 1, \cdots, K \end{aligned}$$

$\mathbf{b} = \{ b_1 \quad b_2 \quad \cdots \quad b_K \}^T$ : Design parameters

$f(\mathbf{b})$ : Objective function

$g_1(\mathbf{b}), \cdots, g_N(\mathbf{b})$ : Inequality constraints

$h_1(\mathbf{b}), \cdots, h_M(\mathbf{b})$ : Equality constraints

$\mathbf{b}^L, \mathbf{b}^U$ : Lower and upper bounds

- **Feasible set**: the set of designs that satisfy constraints

$$S = \left\{ \mathbf{b} \,\middle|\, g_i(\mathbf{b}) \le 0, i = 1, \cdots N, \ h_j(\mathbf{b}) = 0, j = 1, \cdots M \right\}$$