LECTURE NOTE

EGM 5533 APPLIED ELASTICITY AND ADVANCED MECHANICS OF SOLIDS

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CH 1. INTRODUCTION

1. 1. Introduction

1. Classification

- Mechanics
  - Fluid
  - Solid
    - Statics
      - Rigid body
      - Deformable body
    - Dynamic
      - Rigid body
      - Deformable body

2. 3 Basic Concepts

a. Kinematics

b. Kinetics

c. Constitutive relation (material property)

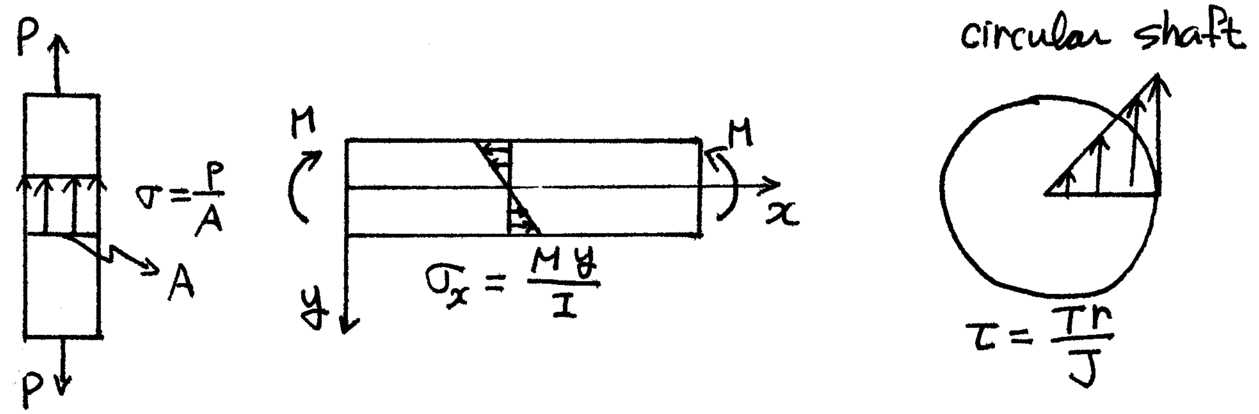
3. Problems and Solution methods

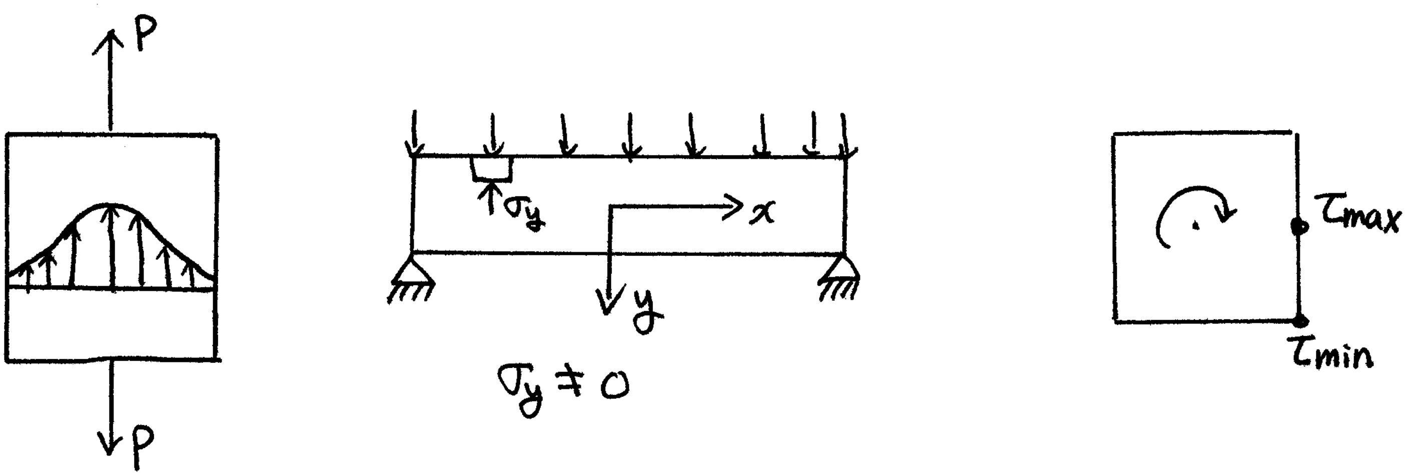
4. Elementary v.s. Advanced Mechanics of Materials

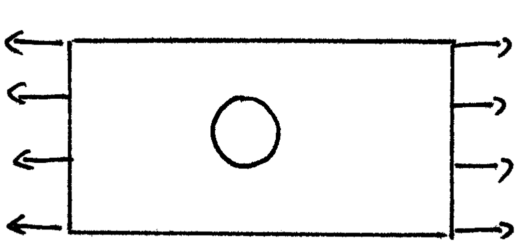
a) Elementary

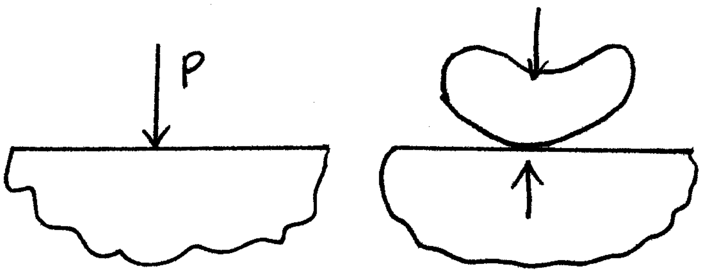
- one dimensional treatment
- simplifying basic assumptions – ex) plane cross-section
- simplest stress distribution

b) Advanced









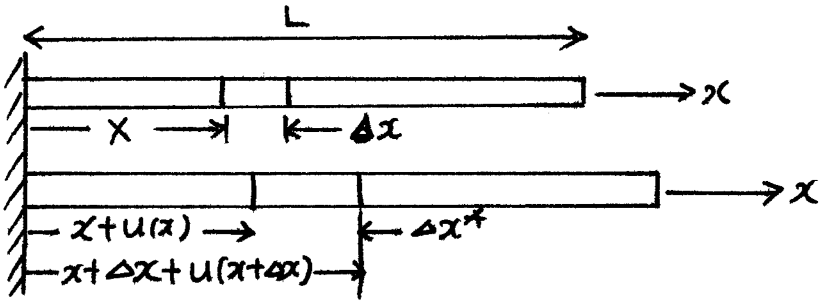
5. Assumptions on Material

1.2. Review

1. Axially Loaded Member (Bar)

a. Kinematic analysis

- Cross-section remains plane.
- $u = u(x)$

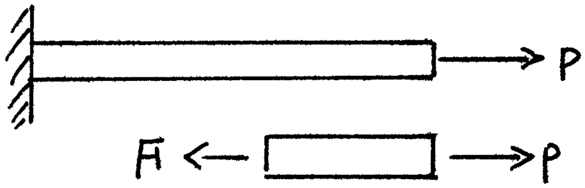


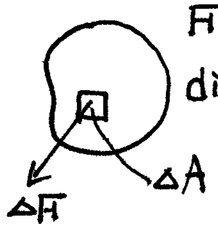
$\varepsilon_{xx} =$

$=$

$\therefore$

b. Kinetic analysis





F is equally distributed on A

$\Sigma F_x = 0$; $F = P$

c. Mechanical Property

1-D Hooke's Law

$P = F =$

$=$

$$u(x) = \int_0^x \varepsilon_{xx} dA = \int_0^x \frac{\sigma_{xx}}{E} dA = \int_0^x \frac{P}{EA} dx = \frac{Px}{EA}$$

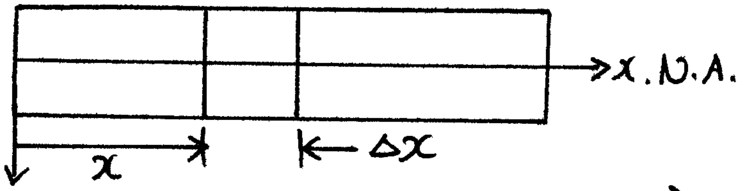
$\therefore$

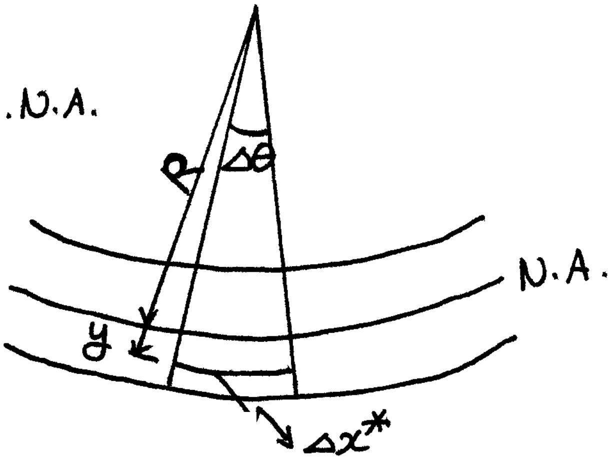
2. Beam

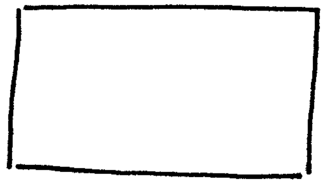
a. Kinematic analysis

1) There exists a neutral plane (N.A.) (no strain)

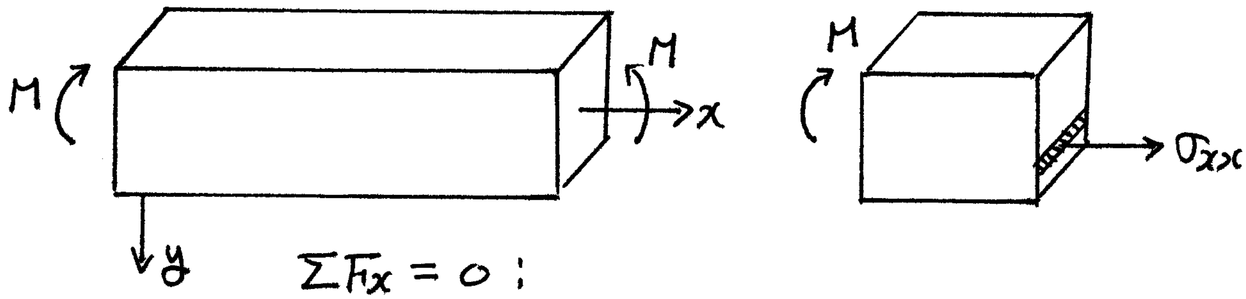
2) Cross-section perpendicular to N.A. remains plane and perpendicular to N.A.







b. Kinetic analysis



$\Sigma F_x = 0$;

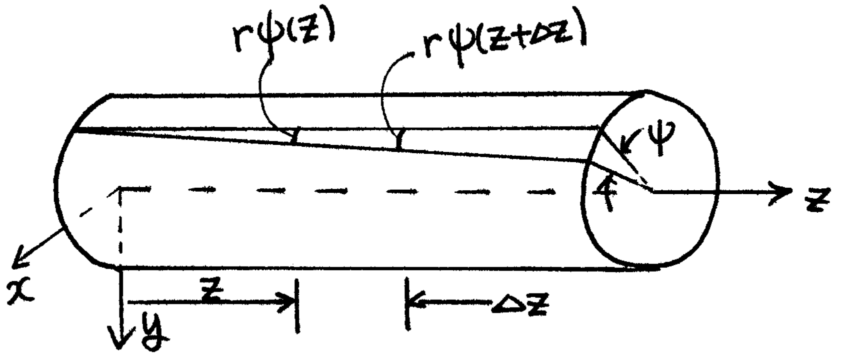
$\Sigma M = 0$;

c. Material property : $\sigma_{xx} = E\,\varepsilon_{xx}$

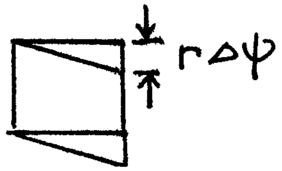
3. Circular Shaft

a: Kinematic analysis

- Plane cross-section rotates as a rigid plane

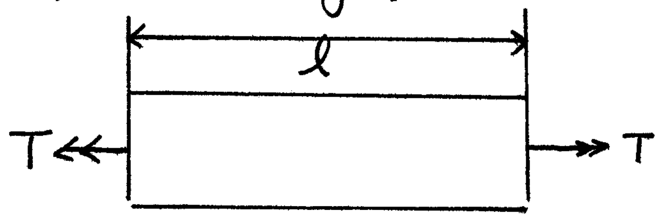


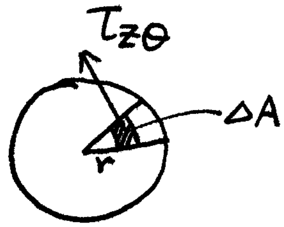
ψ : angle of twist



linear in r

b. Kinetic analysis





$$T = \int_A \tau_{z\theta} \cdot r\, dA$$

c. Constitutive relation : $\tau_{z\theta} = G\,\gamma_{z\theta}$

polar moment of inertia J

$$T = \int_A G\gamma_{z\theta} \cdot r\, dA = G\int_A r\frac{d\psi}{dz} r\, dA = G\frac{\partial \psi}{\partial z}\underbrace{\int_A r^2 dA}_{J}$$

$$\therefore \boxed{\quad}$$

4. Summary

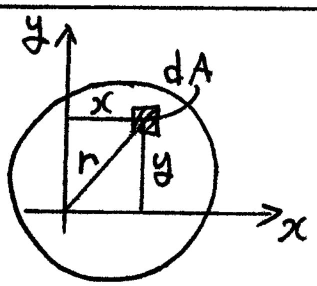
Displacement $\longleftrightarrow$ Deformation $\longleftrightarrow$ Strain

Applied loads $\longleftrightarrow$ Internal forces $\longleftrightarrow$ Stress

} Hooke's law

Structural element	Bar	Beam	Shaft
Deformation	Extension u	Curvature change $\frac{1}{R}$	Twist ψ
Strain	$\varepsilon_{xx} = \frac{du}{dx}$	$\varepsilon_x = \frac{y}{R}$	$\gamma_{z\theta} = r\frac{d\psi}{dz}$
Load	Axial force (P)	Bending moment (M)	Torque (T)
Stress	σ_{xx}	σ_{xx}	$\tau_{z\theta}$
Hooke's law	$\sigma_{xx} = E\,\varepsilon_{xx}$	$\sigma_{xx} = E\,\varepsilon_{xx}$	$\tau_{z\theta} = G\gamma_{z\theta}$
Stress Eq.	$\sigma_{xx} = \frac{P}{A} = E\frac{du}{dx}$	$\sigma_{xx} = \frac{My}{I} = E\frac{y}{R}$	$\tau_{z\theta} = \frac{Tr}{J} = Gr\frac{d\psi}{dz}$
Deformation	$u = \int \frac{P}{EA}dx$ $\Delta l = \frac{Pl}{EA}$	$\frac{1}{R} = \frac{M}{EI}$ $\frac{d^2y}{dx^2} = -\frac{M}{EI}$	$\psi = \int \frac{T}{GJ}dz$ $\Delta\phi = \frac{Tl}{GJ}$

* Area moment of inertia



For circular cross-section

Beam Deflection Using Maclaurin's Series

~ Representation of a continuous fn $y(x)$ by a power series of x, whose coefficients are expressed in terms of the higher derivatives of y at $x=0$.

$y(x) =$

$y'(x) =$

$y''(x) =$

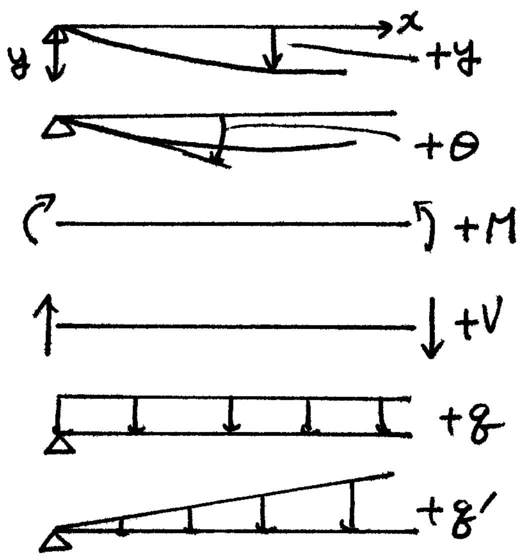
- Boundary conditions are given at $x=0$, such that

$y(0) = a_0, \quad y'(0) = a_1, \quad y''(0) = 2a_2, \quad y'''(0) = 6a_3, \cdots$

$\therefore y(x) =$

- Beam

$y = y(x)$



– Representation of beam deflection

$$y(x) = y_0 + \theta_0 x - \frac{1}{2!}\frac{M_0}{EI}x^2 - \frac{1}{3!}\frac{V_0}{EI}x^3 + \frac{1}{4!}\frac{q_0}{EI}x^4 + \frac{1}{5!}\frac{q_0'}{EI}x^5 + \cdots$$

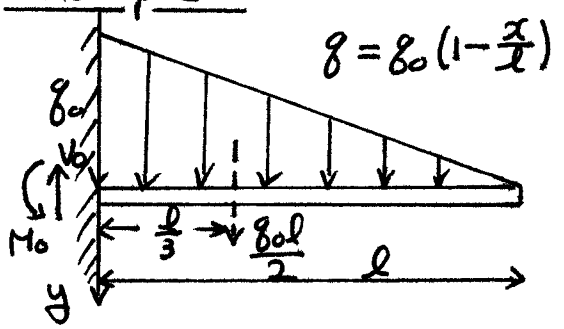
$$\theta(x) = \theta_0 - \frac{M_0}{EI}x - \frac{1}{2!}\frac{V_0}{EI}x^2 + \frac{1}{3!}\frac{q_0}{EI}x^3 + \frac{1}{4!}\frac{q_0'}{EI}x^4 \cdots$$

$$M(x) = -EIy'' =$$

$$V(x) = \frac{dM(x)}{dx} =$$

$$q(x) = -\frac{dV(x)}{dx} =$$

Example 1



$y_0 =$

$\theta_0 =$

$M_0 =$

$V_0 =$

$q_0 =$

$q_0' =$

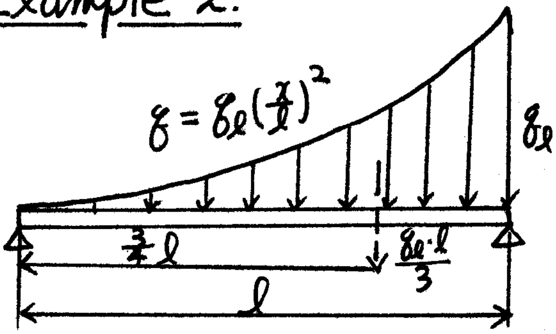
$$\therefore\ y(x) = -\frac{1}{2!}\frac{M_0}{EI}x^2 - \frac{1}{3!}\frac{V_0}{EI}x^3 + \frac{1}{4!}\frac{q_0}{EI}x^4 + \frac{1}{5!}\frac{q_0'}{EI}x^5. \quad \text{No H.O.T.}$$

$$= \frac{1}{EI}\left[\frac{1}{12}q_0 l^2 x^2 - \frac{1}{12}q_0 l x^3 + \frac{1}{24}q_0 x^4 - \frac{1}{120}\frac{q_0}{l}x^5\right]$$

$$= \frac{q_0 l^4}{120EI}\left[10\left(\frac{x}{l}\right)^2 - 10\left(\frac{x}{l}\right)^3 + 5\left(\frac{x}{l}\right)^4 - \left(\frac{x}{l}\right)^5\right]$$

$y_{max} =$

Example 2.



$y_0 =$
$\theta_0 =$
$M_0 =$
$V_0 =$
$q_0 =$
$q_0' =$
$q_0'' =$

$$y(x) = \theta_0 x - \frac{1}{3!}\frac{V_0}{EI}x^3 + \frac{1}{6!}\frac{q_0''}{EI}x^6$$

$$= \theta_0 x - \frac{1}{72}\frac{q_l l}{EI}x^3 + \frac{1}{720}\frac{2q_l}{EI l^2}x^6$$

$\theta_0 =$ unknown

Determine θ_0 from additional condition $y(l) = 0$.

$$y(l) = \theta_0 l - \frac{1}{72}\frac{q_l l^4}{EI} + \frac{1}{720}\frac{2q_l l^4}{EI} = 0$$

$$\theta_0 = \frac{q_l l^3}{90EI}$$

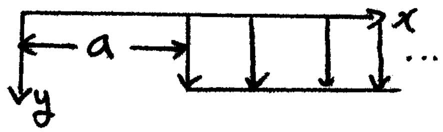
$\therefore y(x) =$

- Singularity Functions

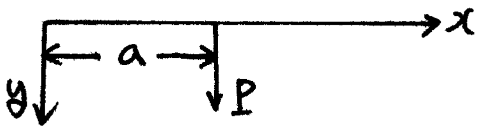
.

for $n > 1$

. $n=0$: step function

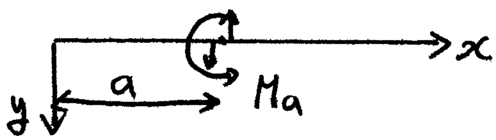


. $n=-1$: Delta function

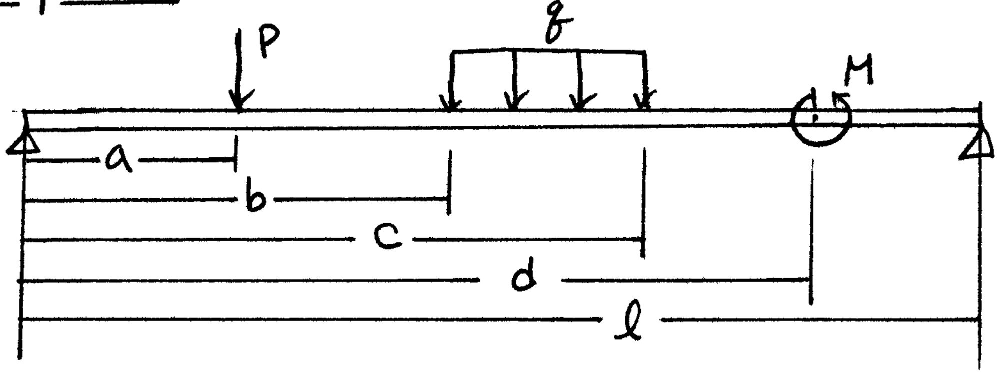


. $n=-2$: Doublet

$q(x) = M_a\langle x-a\rangle^{-2}$



Example 3



$$y(x) = \theta_0 x - \frac{V}{6EI}x^3 + \frac{P}{6EI}\langle x-a\rangle^3 + \frac{q}{24EI}\langle x-b\rangle^4 - \frac{q}{24EI}\langle x-c\rangle^4 - \frac{M}{2EI}\langle x-d\rangle^2$$

Determine θ_0 from $y(l)=0$ condition.

- Alternative approach: Start from $q(x)$ and integrate.

$EIy^{(4)} = q(x)$.

$$EIy^{(4)} = P\langle x-a\rangle^{-1} + q\langle x-b\rangle^0 - q\langle x-c\rangle^0 - M\langle x-d\rangle^{-2}$$

$$EIy''' = P\langle x-a\rangle^0 + q\langle x-b\rangle^1 - q\langle x-c\rangle^1 - M\langle x-d\rangle^{-1} + C_1$$

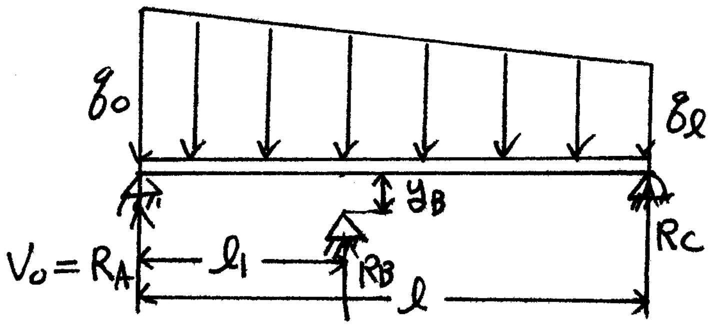
$$EIy'' = P\langle x-a\rangle^1 + \frac{q}{2}\langle x-b\rangle^2 - \frac{q}{2}\langle x-c\rangle^2 - M\langle x-d\rangle^0 + C_1 x + C_2$$

$$EIy' = \frac{P}{2}\langle x-a\rangle^2 + \frac{q}{6}\langle x-b\rangle^3 - \frac{q}{6}\langle x-c\rangle^3 - M\langle x-d\rangle^1 + \frac{C_1}{2}x^2 + C_2 x + C_3$$

$$EIy = \frac{P}{6}\langle x-a\rangle + \frac{q}{24}\langle x-b\rangle^4 - \frac{q}{24}\langle x-c\rangle^4 - \frac{M}{2}\langle x-d\rangle^2 + \frac{C_1}{6}x^3 + \frac{C_2}{2}x^2 + C_3 x + C_4$$

- Apply boundary conditions

Example 4 Statically Indeterminate Members



Find l_1 and y_B so that

$R_A = R_B = R_C = \frac{1}{3}\left[\frac{q_0+q_l}{2} l\right]$

$y_0 =$
$\theta_0 =$
$M_0 =$
$V_0 =$
$q_0 =$
$q_0' =$

$$y(x) = \theta_0 x - \frac{V_0}{6EI}x^3 + \frac{q_0}{24EI}x^4 + \frac{q_0'}{120EI}x^5 - \frac{R_B}{6EI}\langle x - l_1\rangle^3$$

$$y'(x) = \theta_0 - \frac{V_0}{2EI}x^2 + \frac{q_0}{6EI}x^3 + \frac{q_0'}{24EI}x^4 - \frac{R_B}{2EI}\langle x - l_1\rangle^2$$

$-EIy''(x) =$

Boundary conditions

$$M(l) = 0 = V_0 l - \frac{q_0}{2}l^2 + \frac{q_0 - q_l}{l} l^2 + \frac{q_0+q_l}{6} l\cdot(l - l_1) = 0$$

$$\frac{1}{12}\left(2(q_0+q_l)l^2 - 6q_0 l^2 + 2(q_0 - q_l)l^2 + 2(q_0+q_l)l^2 - 2(q_0+q_l)l l_1\right) = 0$$

$\Rightarrow \; l_1 =$

$$y(l) = 0 = \theta_0 l - \frac{V_0}{6EI}l^3 + \frac{q_0}{24EI}l^4 + \frac{q_0'}{120EI}l^5 - \frac{R_B}{6EI}(l - l_1)^3 = 0$$

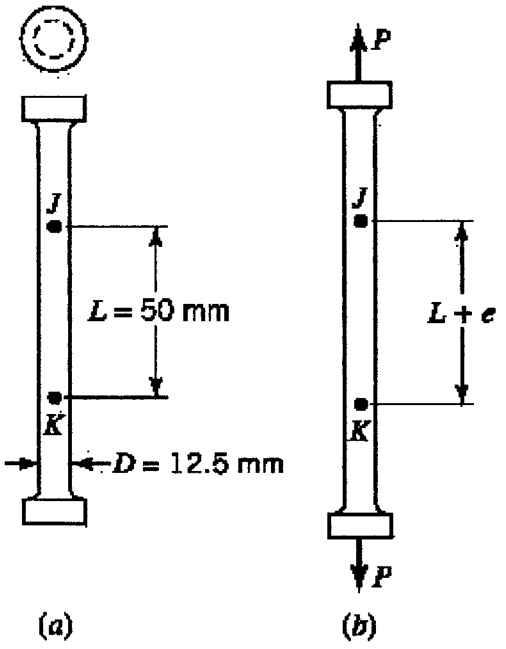
$$\Rightarrow \; \theta_0 = \frac{q_l l^3}{360EI}\frac{l}{l_1}\left\{8 - 21\left(\frac{l_1}{l}\right) + 30\left(\frac{l_1}{l}\right)^2 - 10\left(\frac{l_1}{l}\right)^3\right\}$$

$\Rightarrow y_B = y(l_1) =$

1.3. Stress - Strain Relations

1. Elastic & Inelastic Response

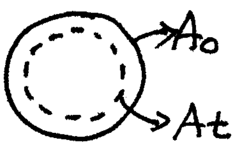
- Tension test



  - stress
  - strain
  - assumption : small deformation

engineering stress & strain

- True stress and true strain
  - True stress $\sigma_t = \dfrac{P}{A_t}$

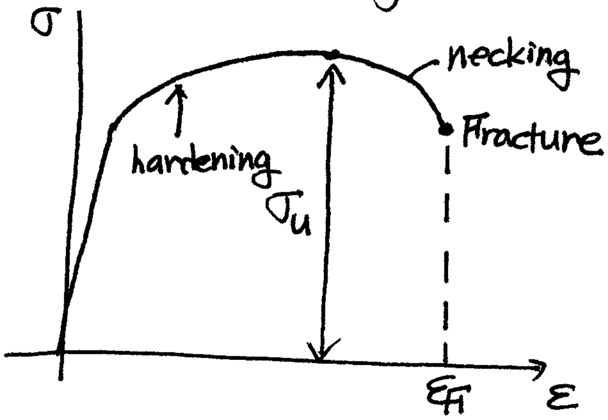


  - True strain
    - for given increment δP, the length changes by dL_t
    - infinitesimal increment of strain resulting from dP

$\therefore \varepsilon_t =$

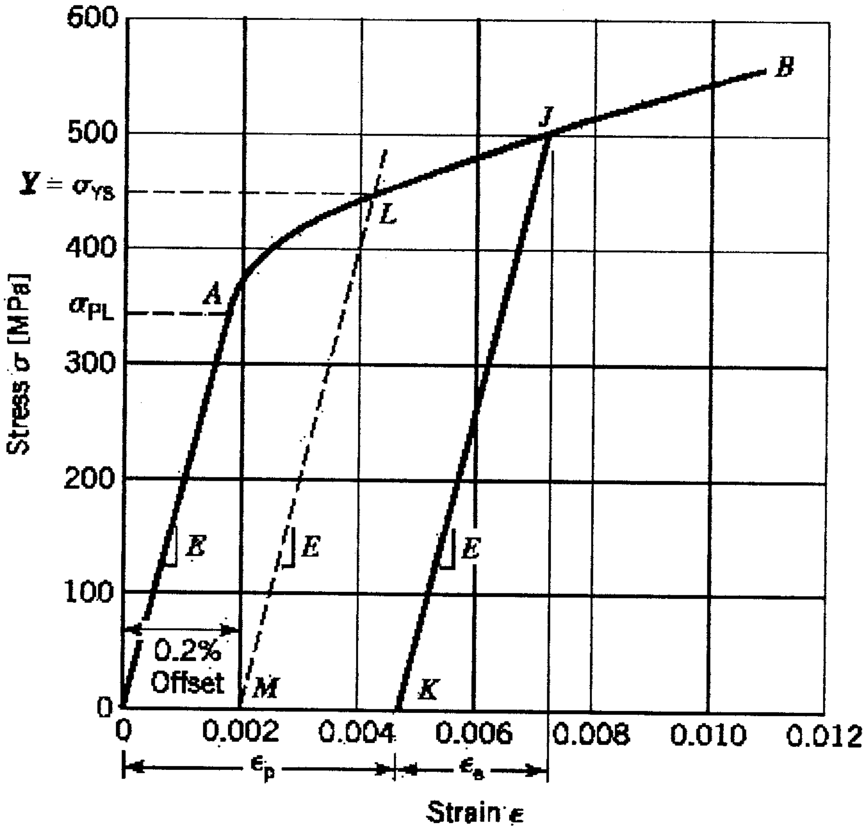
$=$

- Stress - strain Diagram



2. Material Properties

- Engineering stress-strain diagram



- σ_{EL} : elastic limit
- σ_{PL} : proportional limit

  $\sigma_{EL} \geq \sigma_{PL}$.
- ε_p : permanent strain

  $\varepsilon = \varepsilon_e + \varepsilon_p$
- σ_{YS} : yield strength (0.2% permanent strain)
- σ_u : ultimate tensile strength
- E : modulus of elasticity
- Percent elongation $= \varepsilon_{f} \times 100\%$ (ductility)
- Modulus of Resilience (strain energy density)

  $= \frac{1}{2} \frac{\sigma_{YS}^2}{E}$ (energy per unit volume)
- Modulus of Toughness : U_{F}

  strain energy density at fracture.
- Approximation of true strain (from volume conservation)

  $\div A_t L$

  from $\varepsilon_t = \ln(1+\varepsilon)$

1.4. Failure and Limits on Design

- Failure can be defined using stress, strain, displ., load, number of load cycles, etc.

- Factor of Safety

$$SF = \qquad 1.0 \sim 3.0.$$

- Design inequality

$$\sum_{i}^{N} Q_i \le \frac{R_n}{SF}$$

↑ effect of ith load.

- Design with Reliability

$$\sum_{i=1}^{N} \gamma_i Q_i \le \phi R_n$$

γ_i : statistical variance of load i

ϕ : statistical variance of the capacity.

1. Excessive elastic deflection
   - Static deformation
   - Buckling
   - Amplitude of vibration
2. Material Yielding (inelastic)
   - Permanent deformation caused by dislocation in the grain boundary
   - Strain hardening ; entanglement of dislocations
3. Fracture
   - Brittle material
   - Fatigue
4. Buckling

HW #1: Solve Problems 1.26 & 1.27 16

1.26. A steel bar and an aluminum bar are joined end to end and fixed between two rigid walls as shown in Figure P1.26. The cross-sectional area of the steel bar is A_s and that of the aluminum bar is A_a. Initially, the two bars are stress free. Derive general expressions for the deflection of point A, the stress in the steel bar, and the stress in the aluminum bar for the following conditions:

a. A load P is applied at point A.

b. The left wall is displaced an amount δ to the right.

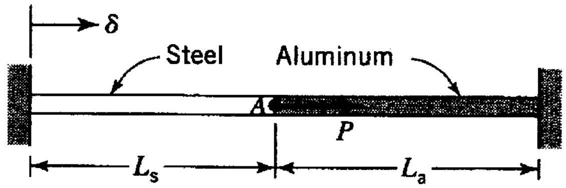


FIGURE P1.26 Bi-metallic rod.

1.27. In South African gold mines, cables are used to lower worker cages down mine shafts. Ordinarily, the cables are made of steel. To save weight, an engineer decides to use cables made of aluminum. A design requirement is that the stress in the cable resulting from self-weight must not exceed one-tenth of the ultimate strength σ_u of the cable. A steel cable has a mass density $\rho = 7.92$ Mg/m^3 and $\sigma_u = 1030$ MPa. For an aluminum cable, $\rho = 2.77$ Mg/m^3 and $\sigma_u = 570$ MPa.

a. Determine the lengths of two cables, one of steel and the other of aluminum, for which the stress resulting from the self-weight of each cable equals one-tenth of the ultimate strength of the material. Assume that the cross-sectional area A of a cable is constant over the length of the cable.

b. Assuming that A is constant, determine the elongation of each cable when the maximum stress in the cable is $0.10\sigma_u$. The steel cable has a modulus of elasticity $E = 193$ GPa and for the aluminum cable $E = 72$ GPa.

c. The cables are used to lower a cage to a mine depth of 1 km. Each cable has a cross section with diameter $D = 75$ mm. Determine the maximum allowable weight of the cage (including workers and equipment), if the stress in a cable is not to exceed $0.20\sigma_u$.