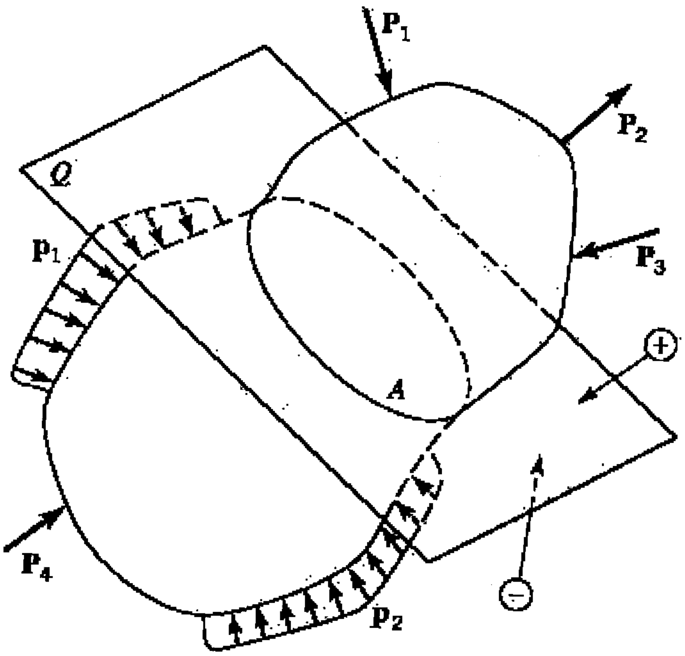
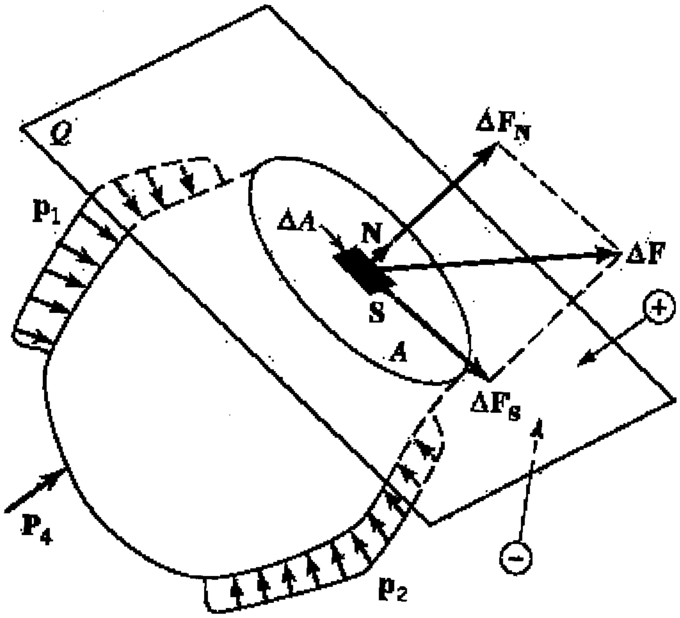
CH.2. Theories of Stress & Strain

2.1. Stress

- Internal force of area ΔA on the plane Q





- Internal force

$\Delta \underline{F} =$

normal force ↓ tangential force ↓

- stress vector

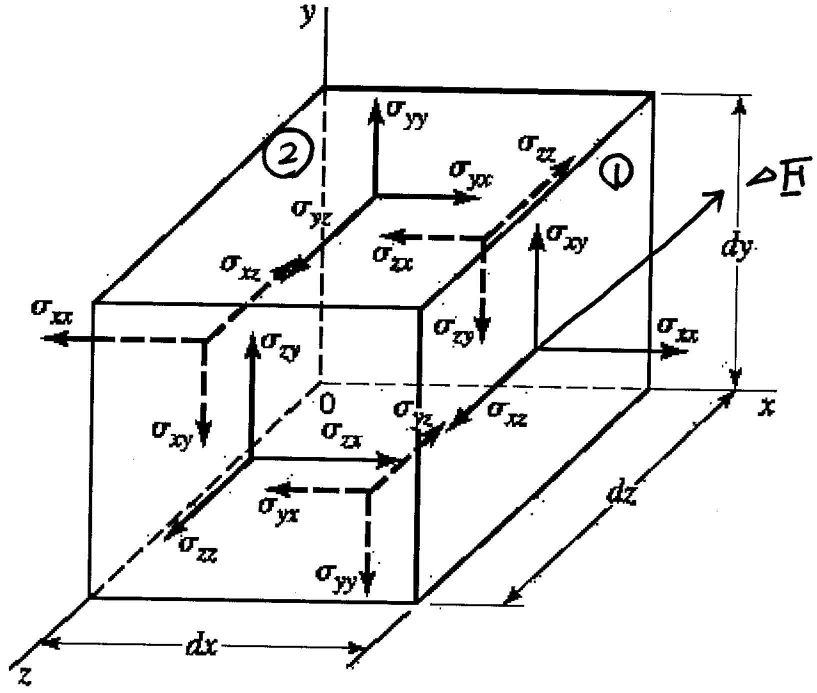
$\underline{\sigma} :=$ parallel to $\Delta \underline{F}$

- normal stress vector & shear stress vector

$\underline{\sigma}_N =$ $\underline{\sigma}_S =$

2.2. Cartesian Component of Stress

- Different plane Q has different stress.
- Use 3 planes normal to x, y, z axes.



Plane ①

$\Delta A = dy \times dz$

$\Delta \underline{F} = \Delta F_x \hat{i} + \Delta F_y \hat{j} + \Delta F_z \hat{k}$

$\Rightarrow \underline{\sigma} =$

From Newton's law, the same stress applies in the plane ②, (negative x-plane)

· Repeat y- and z- planes

$(\sigma_{xx}, \sigma_{xy}, \sigma_{xz}), \quad (\sigma_{yy}, \sigma_{yx}, \sigma_{yz}), \quad (\sigma_{zz}, \sigma_{zx}, \sigma_{zy})$

2.3. Symmetry of stress

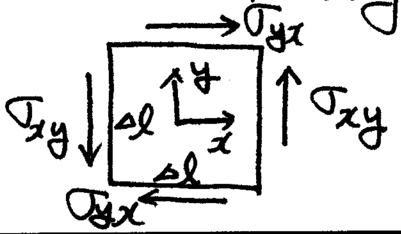
· Cartesian component of stress

$$\underline{T} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

Stress tensor

: stress state at a point

· Equilibrium of angular momentum



$$\Sigma M_z = \Delta l(\sigma_{xy} - \sigma_{yx}) = 0$$

$\therefore$

$\therefore \sigma_{xy} = \sigma_{yx}, \quad \sigma_{xz} = \sigma_{zx}, \quad \sigma_{yz} = \sigma_{zy}$

$\Rightarrow$ stress tensor is symmetric

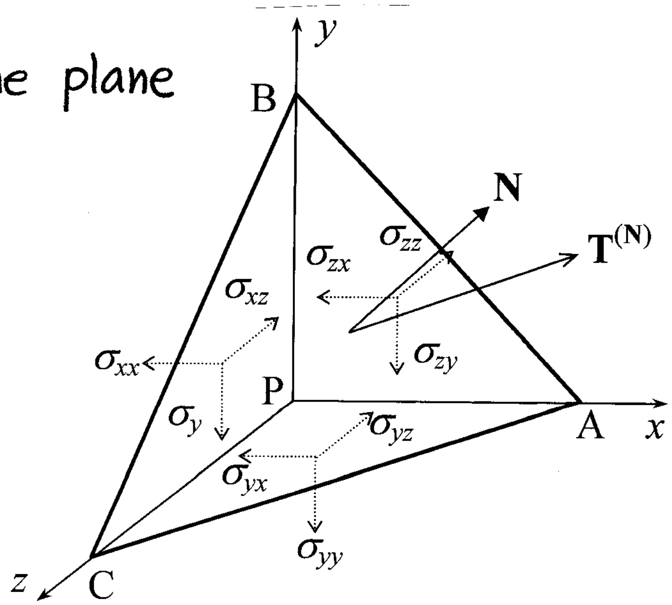
$\underline{T} =$

※ different notation : $\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$

* <u>Stress acting on arbitrary planes.</u>

stress on a plane whose normal is $\underline{N} = [l, m, n]^T$.

$\underline{N} = l\hat{i} + m\hat{j} + n\hat{k}$



· stress vector actiong on the plane

$\underline{T}^{(n)} =$

· Force equilibrium in x-dir

$\Sigma F_x = T_x^{(n)} A - \sigma_{xx} A l$

$\quad - \sigma_{yx} A m - \sigma_{zx} A n = 0$

$\therefore T_x^{(n)} =$

similarly

$T_y^{(n)} =$

$T_z^{(n)} =$

$\Rightarrow$

$\therefore$ Once stress components in xyz coord. are given, stress acting on any direction can be found.

- Normal stress

$$\sigma_n =$$

- Shear stress

$$\underline{\sigma_s} =$$

Ex) $[\sigma] = \begin{bmatrix} 3 & 7 & -7 \\ 7 & 4 & 0 \\ -7 & 0 & 2 \end{bmatrix}$, calculate normal & shear stresses on the plane given by $4x - 4y + 2z = 2$.

$\underline{N} \parallel \{4, -4, 2\}^T$ after normalization

$$\underline{N} =$$

$$\underline{T}^{(N)} = [\sigma]\cdot\underline{N} = \frac{1}{3}\begin{bmatrix} 3 & 7 & -7 \\ 7 & 4 & 0 \\ -7 & 0 & 2 \end{bmatrix}\begin{Bmatrix} 2 \\ -2 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -5 \\ 2 \\ -4 \end{Bmatrix}$$

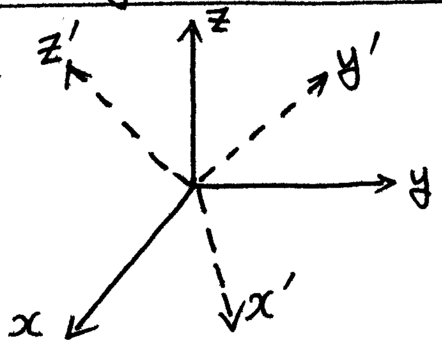
$$\sigma_n = \underline{T}^{(N)}\cdot\underline{N} = -5\times\frac{2}{3} - 2\times\frac{2}{3} - 4\times\frac{1}{3} = -6$$

$$\|T^{(N)}\|^2 = 5^2 + 2^2 + 4^2 = 45$$

$$\sigma_s = \sqrt{45 - 6^2} = 3.$$

2.4. Stress Transformation & Principal Stresses

1. Express stress $[\sigma]_{xyz}$ in xyz-coord. to stress $[\sigma]_{x'y'z'}$ in $x'y'z'$ coord.



Let the direction cosine of x' axis w.r.t xyz coord. be $[l_1 \; m_1 \; n_1]$, y' axis be $[l_2 \; m_2 \; n_2]$, and z' axis be $[l_3 \; m_3 \; n_3]$.

Define transformation tensor

$$[Q] = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$$

Then, stress vectors in x', y', z' axes will be

$$\underbrace{[\underline{T}^{(x')} \; \underline{T}^{(y')} \; \underline{T}^{(z')}]}_{\text{written in } xyz \text{ coord.}} = [\underline{\sigma}]_{xyz} [Q]$$

written in xyz coord. $\Rightarrow$ transform in $x'y'z'$ coord.

$$\Rightarrow \boxed{\qquad\qquad\qquad\qquad}$$

2. Principal Stresses.

- Find a direction $\underline{N}$ such that no shear stess exists.
  - $\Rightarrow$ $\underline{N}$: principal direction
  - σ_N : principal stress.
- In the principal direction, $\underline{T}^{(N)} \parallel \underline{N}$.

$\hookrightarrow$ principal stress.

$$\Rightarrow [\underline{\sigma}] \cdot \underline{N} = \lambda \underline{N}$$

$\Rightarrow \boxed{\qquad\qquad\qquad}$ eigenvalue problem.

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

- How can we solve it?
  ① $\underline{N} = 0$: trivial solution X.
  ② $\det[\underline{\sigma} - \lambda \underline{I}] = 0$: non-trivial solution (also the solution is not unique).
- $\det[\underline{\sigma} - \lambda \underline{I}] = 0$

$$\Rightarrow \begin{vmatrix} \sigma_{xx}-\lambda & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy}-\lambda & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}-\lambda \end{vmatrix} = 0$$ cubic eq. w.r.t. λ.

$$\Rightarrow \boxed{\quad\quad\quad} \qquad \text{———} \quad (a)$$

$\underline{I_1 =}$

$\underline{I_2 =}$

$\underline{I_3 =}$

$=$

- <u>3 roots of Eq. (a) ($\sigma_1 \geq \sigma_2 \geq \sigma_3$) are principal stresses.</u>
- I_1, I_2, I_3 : invariants. independent of coord. system.

  $I_1 =$

  $I_2 =$

  $I_3 =$

3. Principal direction.

Substitute σ_i into λ in the eigenvalue problem and solve for $\underline{N} = \{l, m, n\}$. with constraint of

$\Rightarrow \underline{N}^{(1)}$ Repeat for σ_2 & σ_3

Ex) $[\sigma] = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ Find 3 principal stresses and principal direction corresponding to σ_3.

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & -\lambda & 2 \\ 1 & 2 & -\lambda \end{vmatrix} = \quad =$$

$\therefore$

For $\sigma_3 = -2$, eigenvalue problem is

$$\begin{bmatrix} 5 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow ($$

$l = -2(m+n)$

$\Rightarrow -9(m+n) = 0 \qquad \therefore m = -n \qquad$ and $l^2 + m^2 + n^2 = 1$

$l = 0$

$$\Rightarrow \underline{N}^{(3)} = \frac{1}{\sqrt{2}} \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix} \quad \text{or} \quad \underline{N}^{(3)} = \frac{1}{\sqrt{2}} \begin{Bmatrix} 0 \\ -1 \\ 1 \end{Bmatrix}$$

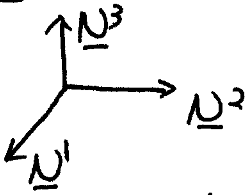
- $\underline{N}^{(1)}, \underline{N}^{(2)}, \underline{N}^{(3)}$ are mutually orthogonal.

$[\sigma] \cdot \underline{N}^i = \sigma_i \underline{N}^i \qquad \underline{N}^j \cdot [\sigma] \cdot \underline{N}^i = \sigma_i \underline{N}^j \cdot \underline{N}^i$

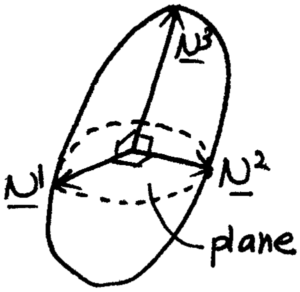
$[\sigma] \cdot \underline{N}^j = \sigma_j \underline{N}^j \qquad -) \; \underline{N}^i \cdot [\sigma] \cdot \underline{N}^j = \sigma_j \underline{N}^i \cdot \underline{N}^j$

- Special Cases

1) $\Rightarrow$ unique $\underline{N}^1, \underline{N}^2, \underline{N}^3$ mutually ortho.

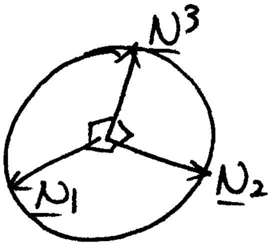


2) $\Rightarrow$ $\underline{N}^1$ and $\underline{N}^2$ are not unique, but orthogonal with $\underline{N}^3$



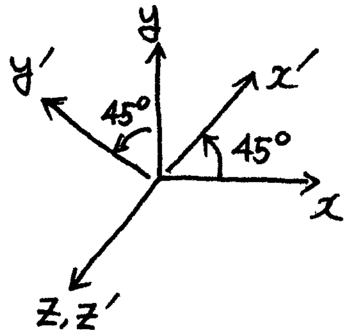
Any two perpendicular vectors on plane P are principal dirs.

3) $\Rightarrow$ hydrostatic state.



Any mutually perpendicular 3 vectors are principal dirs.

Ex) $[\sigma] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$



stress in x′y′z′ coord.?

Transformation matrix

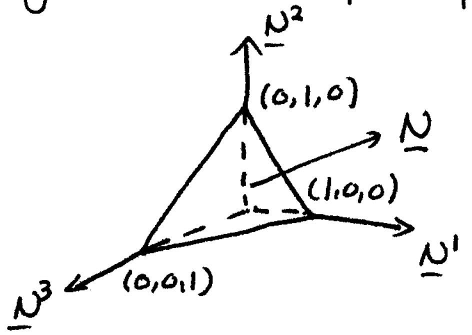
$[Q] =$

$\therefore$ x', y', z' are principal directions. !!

4. Octahedral stress.

- mean stress doesn't contribute to material failure.
  ⇒ separate mean stress from stress tensor.

∘ Octahedral stress
  $\underset{\sim}{N}$: equal angle with 3 principal directions.



5. Mean and Deviator Stresses.

- Since I_1 is invariant, σ_{oct} is same for any coord.

  Mean stress :

- Decompose stress

$$\underset{\sim}{\sigma} = \sigma_m \underset{\sim}{I} + \underset{\sim}{\sigma_d} \qquad \sigma_m \underset{\sim}{I} = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}$$

$$\underset{\sim}{\sigma_d} =$$

: deviatoric stress.

: a state of pure shear

- 3 invariants of $\underline{\sigma_d}$

$J_1 =$

$J_2 =$

$J_3 =$

$=$

- $\underline{\sigma_d}$ has 3 principal stresses.

$$\left.\begin{aligned} S_1 &= \\ S_2 &= \\ S_3 &= \end{aligned}\right\} \quad \begin{aligned} &S_1 + S_2 + S_3 = 0 \\ &\therefore \text{only two are independent.} \end{aligned}$$

6. Plane Stress

3D problem $\xrightarrow[\text{approx.}]{}$ 2D or 1D problem

• Plane stress (small thickness – z) (no force in z-dir)

$\Rightarrow$

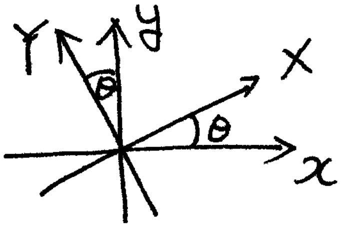
$\Rightarrow$ σ_{xx}, σ_{yy}, σ_{xy} : function of x & y.

$$\underline{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \quad : \text{plane stress tensor.}$$

- Invariants $I_1 = \sigma_{xx} + \sigma_{yy}$, $I_2 = \det(\underline{\sigma})$

※ Plane strain (no strain in z-dir)

Coord Transformation (2-D)



$l = \cos\theta$

$m = \sin\theta \Leftarrow \cos(\frac{\pi}{2} - \theta)$

- Transformation tensor $[Q] =$

$$\Rightarrow \sigma_{XX} = \sigma_{xx} l^2 + \sigma_{yy} m^2 + 2\sigma_{xy} lm$$

$$\sigma_{YY} = \sigma_{xx} m^2 + \sigma_{yy} l^2 - 2\sigma_{xy} lm$$

$$\sigma_{XY} = -(\sigma_{xx} - \sigma_{yy}) lm + \sigma_{xy}(l^2 - m^2)$$

※ trigonometry relation

$$\begin{cases} \cos^2\theta = \frac{1}{2}(1+\cos 2\theta) = l^2 \\ \sin^2\theta = \frac{1}{2}(1-\cos 2\theta) = m^2 \\ \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta = lm \end{cases}$$

$$\Rightarrow \sigma_{XX} =$$

(a) — $\left\{ \begin{array}{l} \sigma_{XX} = \\ \\ \sigma_{YY} = \\ \\ \sigma_{XY} = \end{array} \right.$

Note: $\sigma_{XX} + \sigma_{YY} = \sigma_{xx} + \sigma_{yy}$: 1st invariant I_1

$\sigma_{XX}\sigma_{YY} - \sigma_{XY}^2 = \sigma_{xx}\sigma_{yy} - \sigma_{xy}^2$: 2nd invariant I_2

= determinant of $\underline{\sigma}$

7. Mohr's Circle

Rearrange (a)

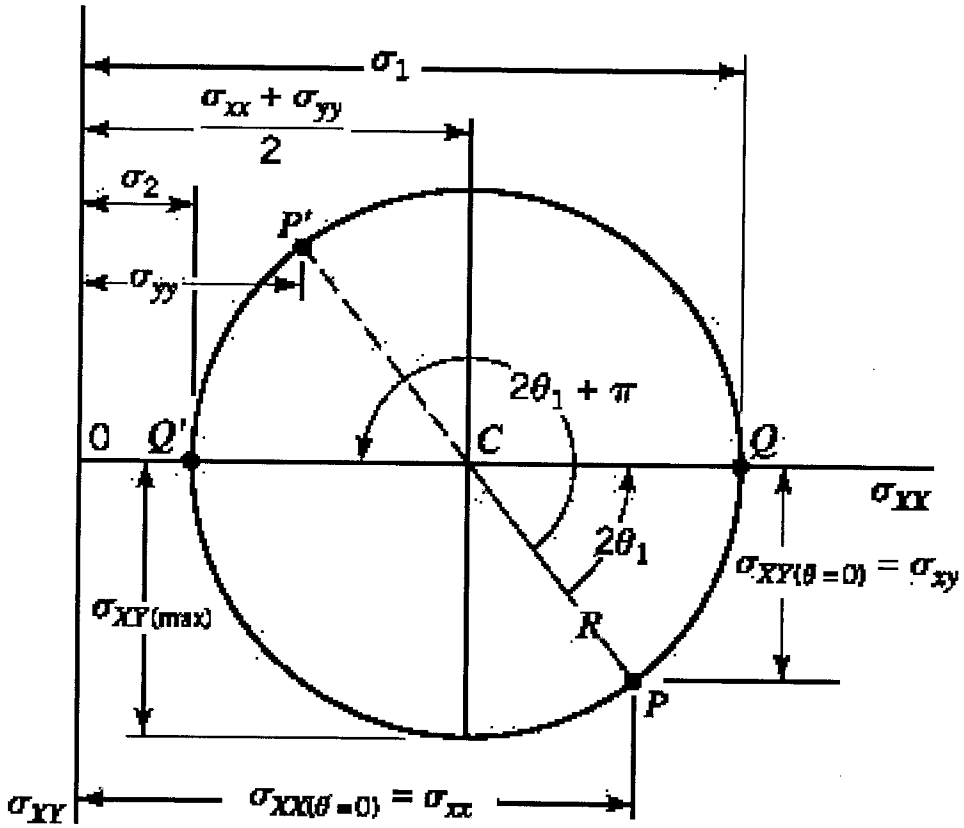
$$\left(\begin{array}{ll} \sigma_{XX} - \frac{1}{2}(\sigma_{xx}+\sigma_{yy}) & = \\ \sigma_{XY} & = \end{array}\right.$$

Square them & add.

Equation of circle with center $\left(\frac{1}{2}(\sigma_{xx}+\sigma_{yy}), 0\right)$

and radius $R = \left[\frac{1}{4}(\sigma_{xx}-\sigma_{yy})^2 + \sigma_{xy}^2\right]^{1/2}$

$\Rightarrow$ Mohr Circle.



- Principal stress : Q & Q'

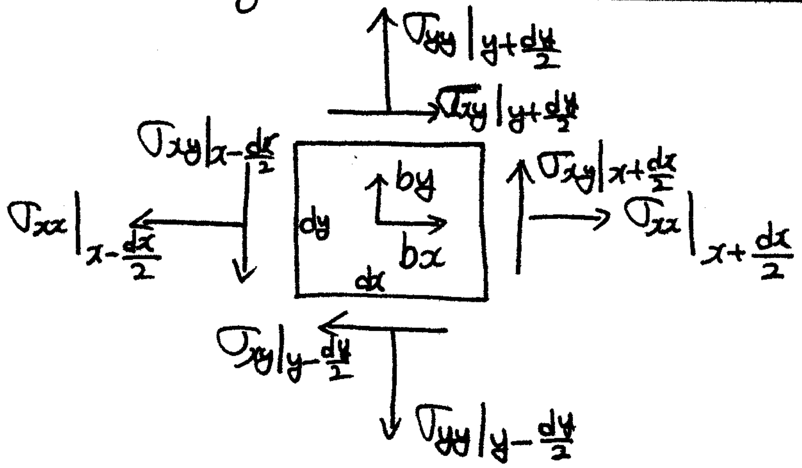
$$\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}$$

$$\sigma_1 = \frac{\sigma_{xx} + \sigma_{yy}}{2} + R$$

$$\sigma_2 = \frac{\sigma_{xx} - \sigma_{yy}}{2} - R$$

$$(\sigma_{xr})_{max} = R.$$

2.5. Equation of Motion



unit depth $dz = 1$.
b_x, b_y : body force

$$\Sigma F_x = 0 \; ; \; (\sigma_{xx}|_{x+\frac{dx}{2}})dy - (\sigma_{xx}|_{x-\frac{dx}{2}})dy + (\sigma_{xy}|_{y+\frac{dy}{2}})dx - (\sigma_{xy}|_{y-\frac{dy}{2}})dx + b_x dx dy = 0$$

1st-order Taylor series expansion

$$\cdot \; (\sigma_{xx}|_{x+\frac{dx}{2}})dy - (\sigma_{xx}|_{x-\frac{dx}{2}})dy$$

$$= \left(\sigma_{xx}|_x + \frac{dx}{2}\frac{\partial \sigma_{xx}}{\partial x} - \sigma_{xx}|_x + \frac{dx}{2}\frac{\partial \sigma_{xx}}{\partial x}\right)dy =$$

$$\cdot \; (\sigma_{xy}|_{y+\frac{dy}{2}})dx - (\sigma_{xy}|_{y-\frac{dy}{2}})dx$$

$$= \left(\sigma_{xy}|_y + \frac{dy}{2}\frac{\partial \sigma_{xy}}{\partial y}\right)dx - \left(\sigma_{xy}|_y - \frac{dy}{2}\frac{\partial \sigma_{xy}}{\partial y}\right)dx =$$

$$\therefore \Sigma F_x = 0; \quad \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + b_x = 0$$

$$\Sigma F_y = 0; \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + b_y = 0$$

$$\Sigma M_z = 0; \quad \sigma_{xy} = \sigma_{yx} \quad \text{(symmetry of stress tensor)}$$

- Generalization to 3-D

(2.45)

※ Introducing index notation

1) $x \to 1$, $y \to 2$, $z \to 3$

2) $[\sigma] \Rightarrow \sigma_{ij}$, $i, j = 1, 2, 3$.

3) Repeated indices mean summation

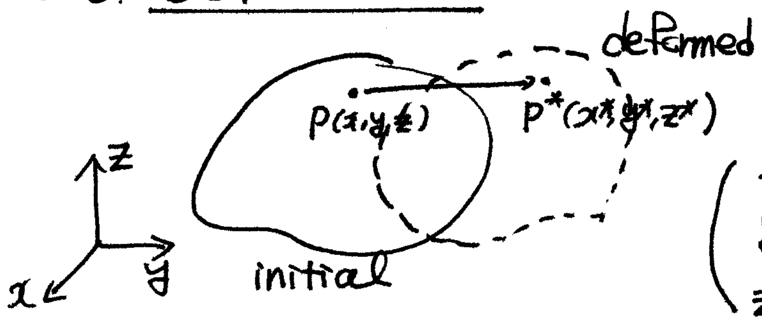
$$\sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33}.$$

Then, the equation of motion becomes

$$\frac{\partial a}{\partial x} = a_{,1}$$

or $\quad \operatorname{div} \underline{\sigma} + \underline{b} = 0 \quad$: vector notation.

2.6. Deformation

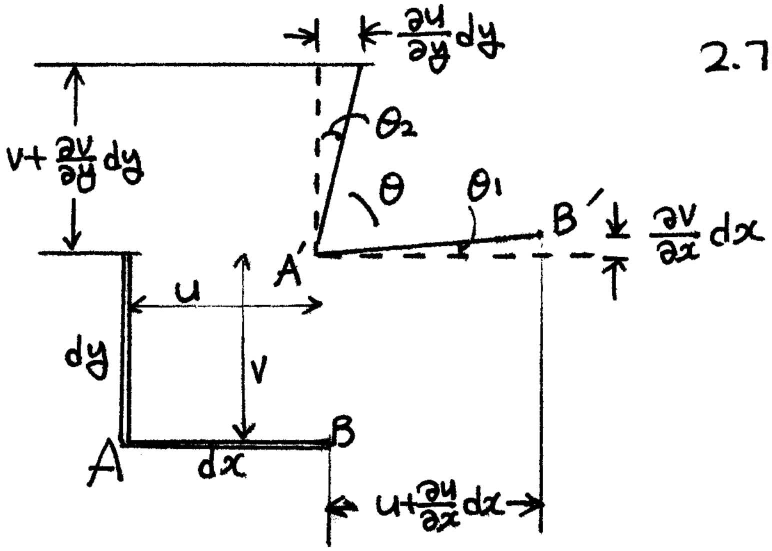


1-to-1 correspondence

$$\begin{cases} x^* = x^*(x,y,z) \\ y^* = y^*(x,y,z) \\ z^* = z^*(x,y,z) \end{cases}$$

- Displacement
  - $u = u(x, y, z)$ $\quad v = v(x, y, z)$ $\quad w = w(x, y, z)$
  - Displacement = Rigid body motion + deformation
  - Small deformation : neglect higher-order term
  - Change in size – extensional strain $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}$.
  - Change in shape – shear strain $\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$.

2.7. Strain (Small Deformation)



- Kinematic relation

$$
\begin{cases}
\varepsilon_{xx} = \\
\varepsilon_{yy} = \frac{\partial v}{\partial y} \\
\varepsilon_{zz} = \frac{\partial w}{\partial z} \\
\gamma_{xy} = \\
\quad = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 2\varepsilon_{xy} \\
\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 2\varepsilon_{xz} \\
\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 2\varepsilon_{yz}
\end{cases}
$$

when $\alpha \ll 1$, $\tan\alpha \simeq \alpha$

$\gamma_{xy} = \gamma_{yx}$, $\quad \gamma_{xz} = \gamma_{zx}$ $\quad \gamma_{yz} = \gamma_{zy}$

- Sign convention

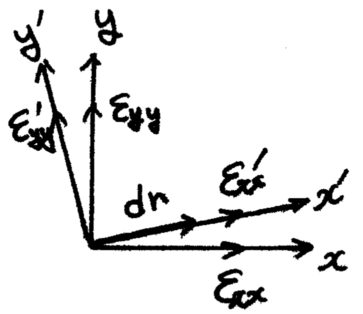
extensional strain : $+$: tensile $-$: compression

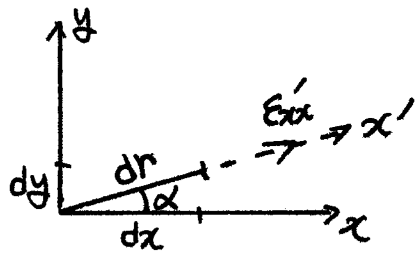
shear strain : $+$: angle $\frac{\pi}{2}$ decreases

- Coordinate Transformation

Given : ε_{xx}, ε_{yy}, ε_{zz}, γ_{xy}, γ_{xz}, γ_{yz}

Find : ε_{xx}' ε_{yy}' ε_{zz}' γ_{xy}' γ_{xz}' γ_{yz}'





$dx = dr\cos\alpha$
$\quad = dr \cdot l$
$dy = dr\cos(\frac{\pi}{2} - \alpha)$
$\quad = dr \cdot m$

- Infinitesimal Component along x' axis

$$dr^2 = dx^2 + dy^2 + dz^2$$

$$dr^* = dr\,(1 + \varepsilon_{xx}') \qquad (dr^*)^2 = [dr(1+\varepsilon_{xx}')]^2$$

$(dr)^2 =$

$(dr^*)^2 =$

$$\Longrightarrow \underline{[dr(1+\varepsilon_{xx}')]^2 = (dx+du)^2 + (dy+dv)^2 + (dz+dw)^2} \quad \text{Solve for } \varepsilon_{xx}'$$

- 2D Case

$$\begin{cases} du = \\ dv = \end{cases}$$

$$[dr(1+\varepsilon_{xx}')]^2 = (dx + du)^2 + (dy + dv)^2$$

$$(dr)^2(1+2\varepsilon'_{xx}+\cancel{\varepsilon'^2_{xx}}) = dx^2 + \cancel{(\tfrac{\partial u}{\partial x}dx)^2} + \cancel{(\tfrac{\partial u}{\partial y}dy)^2}$$

$$+2\frac{\partial u}{\partial x}(dx)^2 + 2\frac{\partial u}{\partial y}dxdy \cancel{+2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}dxdy}$$

$$+(dy)^2 + \cancel{(\tfrac{\partial v}{\partial x}dx)^2} + \cancel{(\tfrac{\partial v}{\partial y}dy)^2} + 2\frac{\partial v}{\partial x}dxdy + 2\frac{\partial v}{\partial y}(dy)^2 + \cancel{2\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}dxdy}$$

$$= \underbrace{(dx)^2+(dy)^2}_{(dr)^2} + 2\frac{\partial u}{\partial x}(dx)^2 + 2\frac{\partial u}{\partial y}dxdy + 2\frac{\partial v}{\partial x}dxdy + 2\frac{\partial v}{\partial y}(dy)^2$$

$$\therefore\ \varepsilon'_{xx} = \frac{\partial u}{\partial x}\left(\frac{dx}{dr}\right)^2 + \frac{\partial v}{\partial y}\left(\frac{dy}{dr}\right)^2 + \left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\left(\frac{dxdy}{dr^2}\right)$$

$\underline{\varepsilon'_{xx} =}$

- Let the direction cosines of x'-axis be l_1, m_1, and the direction cosines of y'-axis be l_2, m_2

$$\begin{cases} \varepsilon'_{xx} = \varepsilon_{xx} l_1^2 + \varepsilon_{yy} m_1^2 + \gamma_{xy} l_1 m_1 \\ \varepsilon'_{yy} = \varepsilon_{xx} l_2^2 + \varepsilon_{yy} m_2^2 + \gamma_{xy} l_2 m_2 \\ \frac{1}{2}\gamma_{x'y'} = \varepsilon_{xx} l_1 l_2 + \varepsilon_{yy} m_1 m_2 + \frac{\gamma_{xy}}{2}(l_1 m_2 + l_2 m_1) \end{cases}$$

Similar to stress transformation

Let $[Q] = \begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \end{bmatrix}$ or $[Q] = \begin{bmatrix} l & -m \\ m & l \end{bmatrix}$

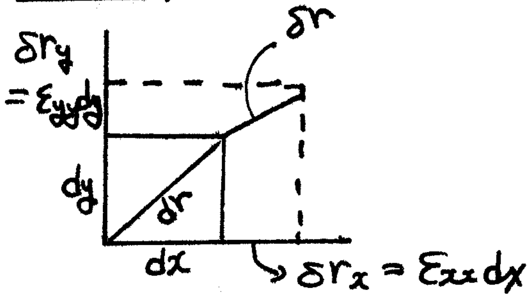
and strain tensor

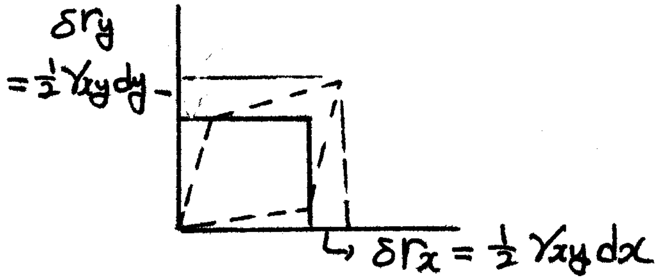
$$[\varepsilon] = \left[\begin{array}{cc|c} \varepsilon_{xx} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_{yy} & \frac{1}{2}\gamma_{yz} \\ \hline \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_{zz} \end{array}\right]$$

Then, the transformation between x-y and x'-y' coordinates

$\boxed{}$

◦ Principal Strain





$$\begin{cases} \delta r_x = \varepsilon_{xx} dx + \frac{1}{2}\gamma_{xy} dy + \frac{1}{2}\gamma_{xz} dz \\ \delta r_y = \frac{1}{2}\gamma_{xy} dx + \varepsilon_{yy} dy + \frac{1}{2}\gamma_{yz} dz \\ \delta r_z = \frac{1}{2}\gamma_{xz} dx + \frac{1}{2}\gamma_{yz} dy + \varepsilon_{zz} dz \end{cases}$$

Find a direction such that $dr \parallel \delta r$

Let $dr = r$, $dr_x = r_x$, $dr_y = r_y$, $dr_z = r_z$

Let $\delta \underline{r} = \lambda \underline{r}$

$$\Rightarrow \begin{cases} \varepsilon_{xx} r_x + \varepsilon_{xy} r_y + \varepsilon_{xz} r_z = \lambda r_x \\ \varepsilon_{xy} r_x + \varepsilon_{yy} r_y + \varepsilon_{yz} r_z = \lambda r_y \\ \varepsilon_{xz} r_x + \varepsilon_{yz} r_y + \varepsilon_{zz} r_z = \lambda r_z \end{cases}$$

$$\Rightarrow \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix} \begin{Bmatrix} r_x \\ r_y \\ r_z \end{Bmatrix} = \lambda \begin{Bmatrix} r_x \\ r_y \\ r_z \end{Bmatrix}$$

$\Rightarrow \boxed{\qquad\qquad}$ Same eigenvalue problem with stress.

$\Rightarrow$ Non trivial solution when

$$\det \left| \underline{\varepsilon} - \lambda \underline{I} \right| = 0$$

$\Rightarrow$

- Invariants

$\bar{I}_1 =$

$\bar{I}_2 =$

$\bar{I}_3 =$

2.8. Compatibility Relations.

displ. = 3

strain = 6 $\Rightarrow$ all strains cannot be independent.

Ex.) Plane strain

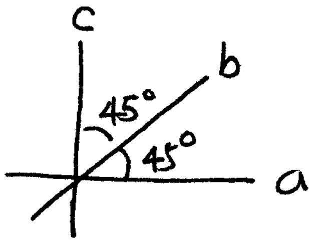
$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad 2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}.$$

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = \frac{\partial^3 u}{\partial x \partial y^2}, \quad \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^3 v}{\partial y \partial x^2}, \quad 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x^2 \partial y}$$

$$\therefore \boxed{}$$

2.9. Strain Measurement

Strain Rosettes.



measure the change of length. in 3-directions.

$\Rightarrow$ change in electric resistance

$\Rightarrow$ measure voltage change.

- From ε_a, ε_b, ε_c, use transformation eq. to calculate ε_{xx}, ε_{yy}, ε_{xy}
- Note : only measure surface strain.

HW 2: Solve Problems 2.5, 2.56, 2.67, 2.71

2.5. The stress components at a point in a plate are $\sigma_{xx} = 80$ MPa, $\sigma_{yy} = 60$ MPa, $\sigma_{zz} = \sigma_{xy} = 20$ MPa, $\sigma_{xz} = 40$ MPa, and $\sigma_{yz} = 10$ MPa.

a. Determine the stress vector on a plane normal to the vector $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

b. Determine the principal stresses $\sigma_1 \geq \sigma_2 \geq \sigma_3$.

c. Determine the maximum shear stress.

d. Determine the octahedral shear stress.

2.56. A square glass block in the side of a skyscraper (Figure P2.56) is loaded so that the block is in a state of plane strain ($\epsilon_{zz} = \epsilon_{zx} = \epsilon_{zy} = 0$).

a. Determine the displacements for the block for the deformations shown and the strain components for the (x, y) coordinate axes.

b. Determine the strain components for the (X, Y) axes.

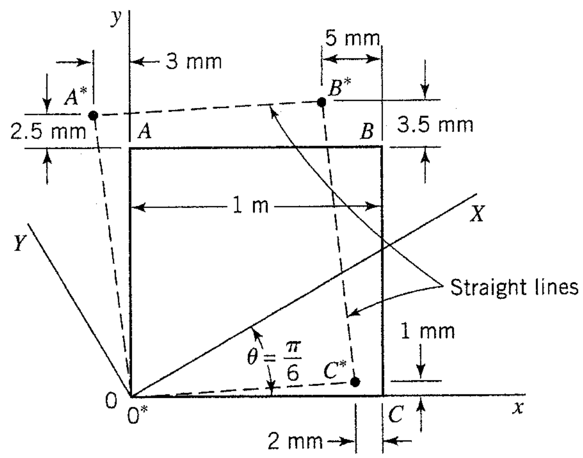


FIGURE P2.56

2.67. For a problem of small-displacement plane strain, the strain components in a machine part, relative to axes (x, y, z), are

$$\epsilon_{xx} = A(L-x), \quad \epsilon_{yy} = B(L-x), \quad \epsilon_{xy} = 0 \tag{a}$$

Determine the (x, y) displacement components (u, v), for the case where the displacement components (u, v) vanish at $x = y = 0$ and the slopes $(\partial u/\partial y, \partial v/\partial x)$ are equal at $x = y = 0$. That is,

$$u\big|_{x=y=0} = v\big|_{x=y=0} = 0 \tag{b}$$

$$\left.\frac{\partial u}{\partial y}\right|_{x=y=0} = \left.\frac{\partial v}{\partial x}\right|_{x=y=0} \tag{c}$$

2.71. Assume that the machine part shown in Figure E2.8 undergoes the (x, y, z) displacements

$$u = c_1 xz, \quad v = c_2 yz, \quad w = c_3 z$$

where the meter is the unit of length for (u, v, w) and (x, y, z). Use small-displacement theory to:

a. Determine the components of strain at point E, in terms of c_1, c_2, and c_3.

b. Determine the normal strain at E in the direction of the line EC, in terms of c_1, c_2, and c_3.

c. Determine the shear strain at E for the lines EF and ED, in terms of c_1, c_2, and c_3.

d. Obtain numerical values for parts a, b, and c, for $c_1 = 0.002$ m^{-1}, $c_2 = 0.004$ m^{-1}, and $c_3 = -0.004$ m^{-1}.