CH 4. Inelastic Material Behavior

- Multi-axial stress state ⇒ failure criterion.

  : initiation of inelastic response

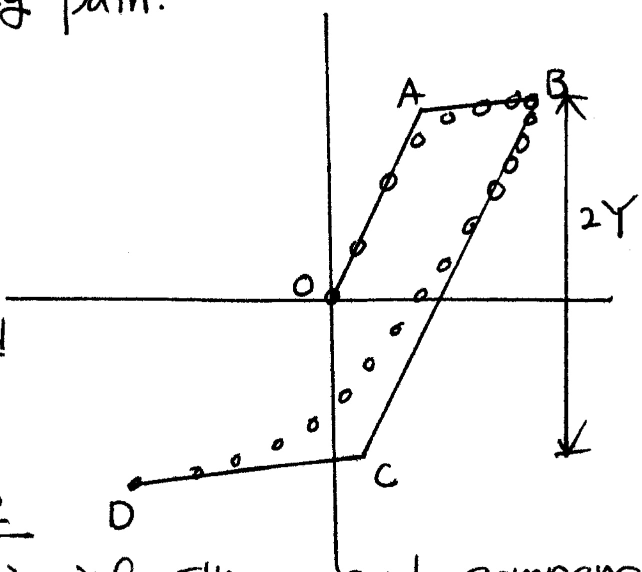
4.1. Limitation on uniaxial stress-strain data

1. Rate of Loading
   - High load rate – strain at fracture is reduced.
     material response is less ductile.
     increase yield strength and Young's modulus.

2. Low Temperature.
   - Low temperature
     - material response is brittle

3. High Temperature
   - High temperature
     - show creep : strain under constant load continues to increase until fracture occurs.

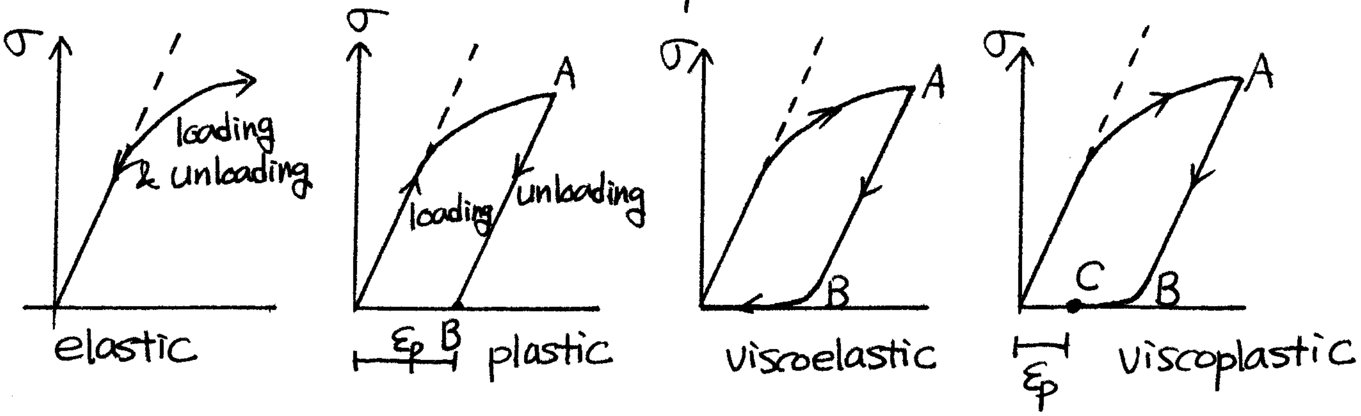
4. Loading-unloading
   - Different slope for unloading path.
   - Bauschinger effect :
     - compressive yield occurs before $-Y$.
     - Tension - compression yield stresses are $2Y$.



5. Multiaxial Stress State
   Convert to effective uniaxial stress and compare with uniaxial stress.

4.2. Nonlinear Material Response

Above linear region ~ elastic, plastic, viscoelastic viscoplastic, or fracture.



- Nonlinear elastic ~ unloading path coincides with loading path.
- Plastic ~ Remains permanent strain after the load is released.
- Viscoelastic ~ After complete unloading, the material will return to an unstrained state.
- Viscoplastic ~ After complete unloading, the response changes with time, but permanent strain remains.

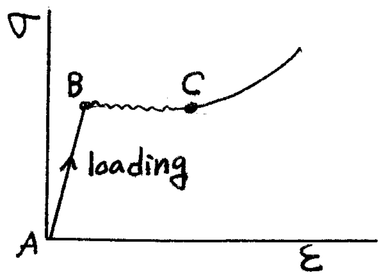
* Fracture ~ (1) linear ⇒ fracture, (2) nonlinear ⇒ fracture

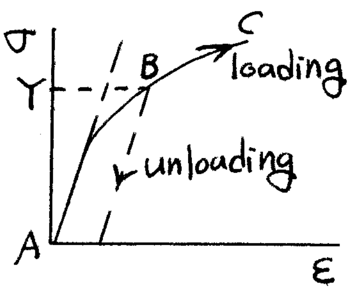
(3) crack growth with load repetition ⇒ fracture

(4) rapid crack growth ⇒ fracture.

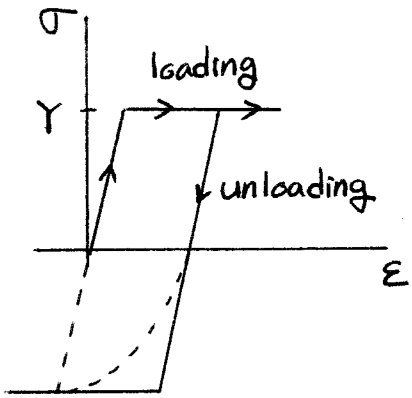
1. Models of Uniaxial Stress-Strain Curve.

- Two types of yielding (abrupt & gradual)

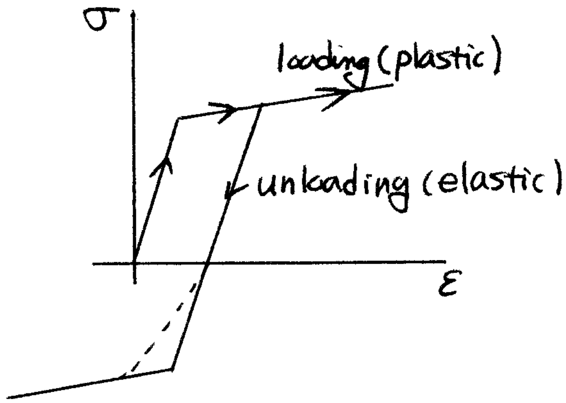




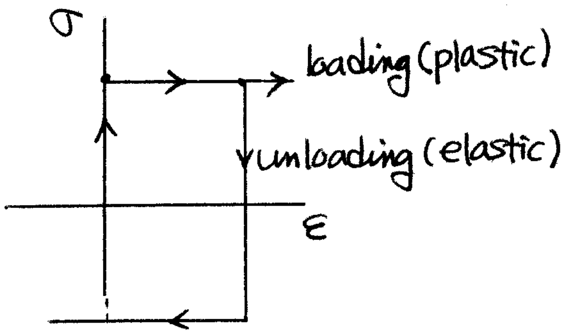
◦ Mathematical model



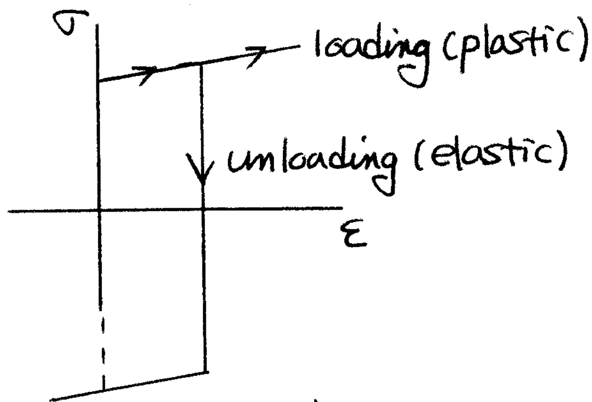
elastic-perfectly-plastic



elastic-linear-strain-hardening.



rigid perfectly plastic

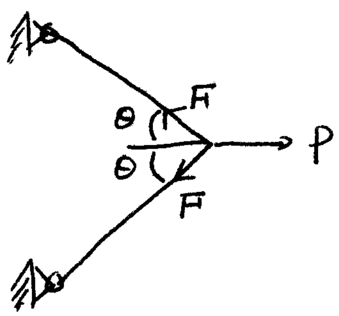


rigid strain hardening plastic

: elastic part is ignored (metal forming)

- Nonlinear elastic : unique relation b/w stress & strain (potential function). <u>path independent</u>
- Plastic : strain depends on the history of loading & unloading <u>path dependent</u>

<u>Ex 4.1</u>



elastic $\sigma =$

plastic $\sigma =$

$=$

$=$

- member force $F = \frac{P}{2\cos\theta}$
- member stress $\sigma = \frac{F}{A} = \frac{P}{2A\cos\theta}$
- elastic region ($\varepsilon < \varepsilon_Y = \frac{Y}{E}$) elongation

$\sigma =$

$e =$

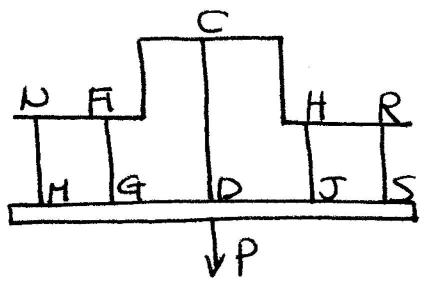
- At yield point ($\varepsilon_Y = \frac{Y}{E} = \varepsilon$)

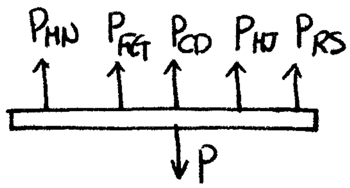
- At Plastic Region

$\sigma =$

$\therefore e =$

Example 4.2





Elastic perfectly plastic material.

$E = 200$ GPa, $\sigma_Y(MN, RS) = 250$ MPa $\sigma_Y(FG, CD, HJ) = 500$ MPa

All members will have same displacement.

$\varepsilon = \varepsilon_{MN} = \varepsilon_{FG} = \varepsilon_{HJ} = \varepsilon_{RS}$. $\varepsilon_{CD} = \frac{1}{2}\varepsilon$

$\sigma_Y = \sigma_Y(FG, CD, HJ)$, $\sigma_Y(NM, RS) = \frac{1}{2}\sigma_Y$.

(a) which member will yield first? (MN, RS).

$\varepsilon_Y = \frac{\frac{1}{2}\sigma_Y}{E} = \frac{\sigma_Y}{2E}$ & $\varepsilon_{CD} = \frac{1}{4}\frac{\sigma_Y}{E}$

$\therefore P_Y = \sum_{i=1}^{5}\sigma_i A = A\left(4\times\frac{\sigma_Y}{2E}\cdot E + \frac{\sigma_Y}{4E}\cdot E\right) = \frac{9}{4}\sigma_Y A$

$= \frac{9}{4}\times 500\times10^6\times 100\times10^{-6} = 112.5$ kN.

(b) Fully plastic load P_P.

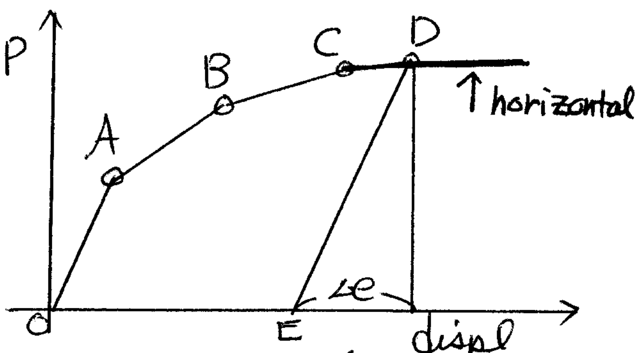
all members are in the plastic region.

$P_P = \sum_{i=1}^{5}\sigma_{Y_i} A = \left(\frac{1}{2}\sigma_Y\times2 + \sigma_Y\times3\right)A = 4\sigma_Y A$

$= 4\times500\times10^6\times100\times10^{-6} = 200$ kN.

(c) Load-displacement diagram

stage 1 : all elastic
stage 2 : MN, RS plastic
stage 3 : FG, HJ plastic
stage4 : CD plastic



① $P = 112.5$ kN, $u = \varepsilon_Y\cdot L_{MN} = \frac{\sigma_Y}{2E}\cdot L_{MN} = \frac{500\times10^6}{2\cdot200\times10^9}\cdot1 = 1.25$ mm.

② $P = A\left(2\times\frac{\sigma_Y}{2E}\cdot E + 2\times\frac{\sigma_Y}{E}\cdot E + \frac{\sigma_Y}{2E}\cdot E\right)$

$= \frac{7}{2}\sigma_Y A = 175$ kN. $u = 2.5$ mm

③ $P = 200$ kN $u = 5$ mm

(d) Residual forces in the members.

Assume all of them are in the elastic unloading

elastic slope OA // DE

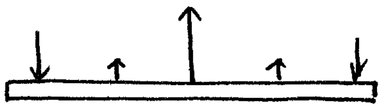
$\frac{112.5}{1.25} = \frac{200}{\Delta e}$ $\Delta e = 2.222$ mm.

$\Delta P_{MN} = \Delta P_{RS} = \Delta P_{FG} = \Delta P_{HJ} = \frac{EA\Delta e}{L_{MN}}$ $\Delta P_{CD} = \frac{EA\Delta e}{L_{CD}}$

$P_{MN} = P_{MNY} - \Delta P_{MN} = 25 - 44.44 = -19.44 = P_{RS}$

$P_{FG} = P_{FGY} - \Delta P_{FG} = 50 - 44.44 = 5.56 = P_{HJ}$

$P_{CD} = P_{CDY} - \Delta P_{CD} = 50 - 22.22 = 27.78$



4.3. Yield Criteria.

uniaxial stress $\Rightarrow$ multiaxial stress state.

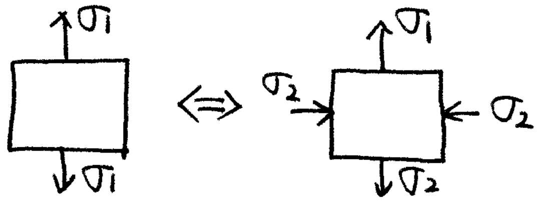
- Plasticity theory – material behavior after initial yield.
  - yield criterion – initiation of yielding
  - flow rule – plastic strain increment v.s. stress increment
  - hardening rule – change of yield surface.
- Yield criteria for multiaxial stress.
  - Define effective (equivalent) uniaxial stress.
  - Compare it with uniaxial stress.
  - Described usually by yield function $\boxed{f(\sigma_{ij}, Y).}$

  - effective stress σ_e
  - yield surface, $f = 0$, is often plotted in the principal stress space $(\sigma_1, \sigma_2, \sigma_3)$

1. Max. Principal Stress Criterion (Rankine)

- Material yields when max. principal stress σ_1 reaches tensile strength Y.

- No effect from other principal stresses. Not accurate for ductile materials, but good for brittle material.



$\sigma_1 > \sigma_2$.

$\tau_{max} = \frac{\sigma_1}{2}$ $\qquad$ $\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$.

2. Max. Principal Strain Criterion

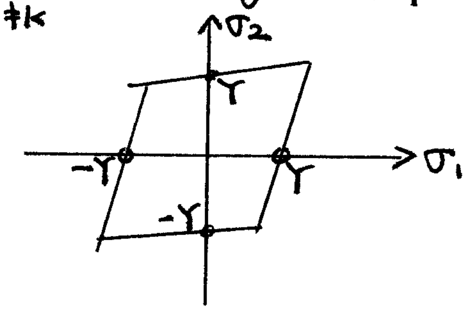
- Max. principal strain reaches yield strain of uniaxial test

$$\varepsilon_Y = \frac{Y}{E}.$$

- For biaxial stress, $\varepsilon_1 = \frac{\sigma_1}{E} - \nu \frac{\sigma_2}{E}$. Thus, yielding can begin for $\sigma_1 > Y$. Not accurate for ductile material. Relatively good for brittle material.

$$\sigma_e = \max_{i \neq j \neq k} |\sigma_i - \nu\sigma_j - \nu\sigma_k|$$: effective stress.

when $\sigma_3 = 0$



3. Strain-Energy Density Criterion

- Material yields when 3D strain-energy density = strain-energy density of uniaxial test at yield.

$$U_0 = \frac{1}{2E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-2\nu(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_1\sigma_3)\right] =$$

- biaxial stress $\sigma_1=\sigma_2=\sigma,\ \sigma_3=0$

$$U_0 = \frac{1}{2E}\left[2(1-\nu)\sigma^2\right] = \frac{1}{2E}Y^2$$

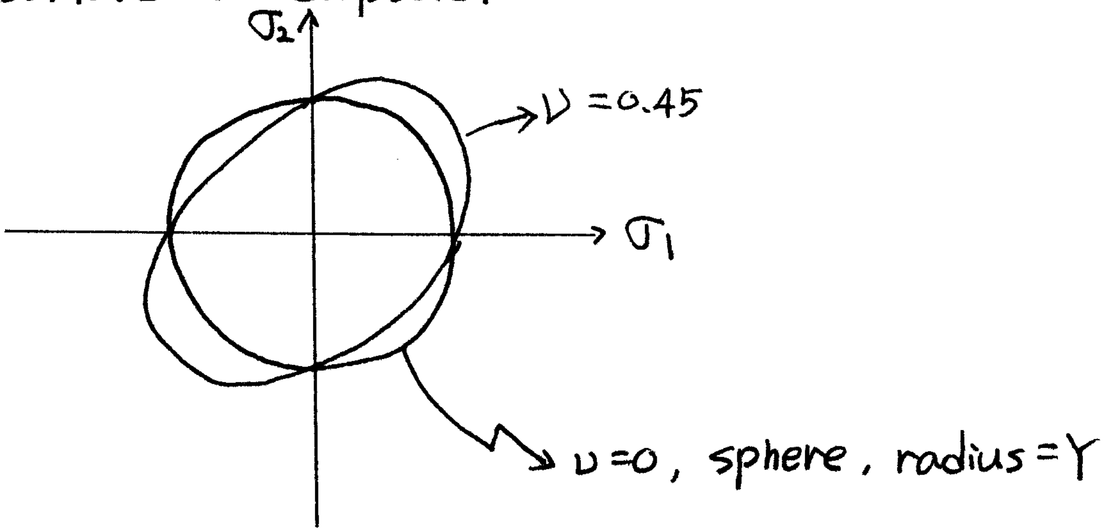
$\Rightarrow 2(1-\nu)\sigma^2 = Y^2$ when $\nu=0$ $\Rightarrow \sigma = Y/\sqrt{2}$

- uniaxial stress $\sigma_1=\sigma,\ \sigma_2=\sigma_3=0$

$\frac{1}{2E}\sigma_1^2 = \frac{1}{2E}Y^2$ $\therefore$ $\sigma = Y$ at yield.

- 3D stress

- Yield surface is ellipsoid.



4.4. Yielding of Ductile Metals

- Resistence to shear force along the slip planes is relatively small. ⇒ yield criterion is based on shear stress.

1. Max. Shear Stress (Tresca) Criterion

- Material yields when τ_{max} = max. shear at yield in uniaxial test.

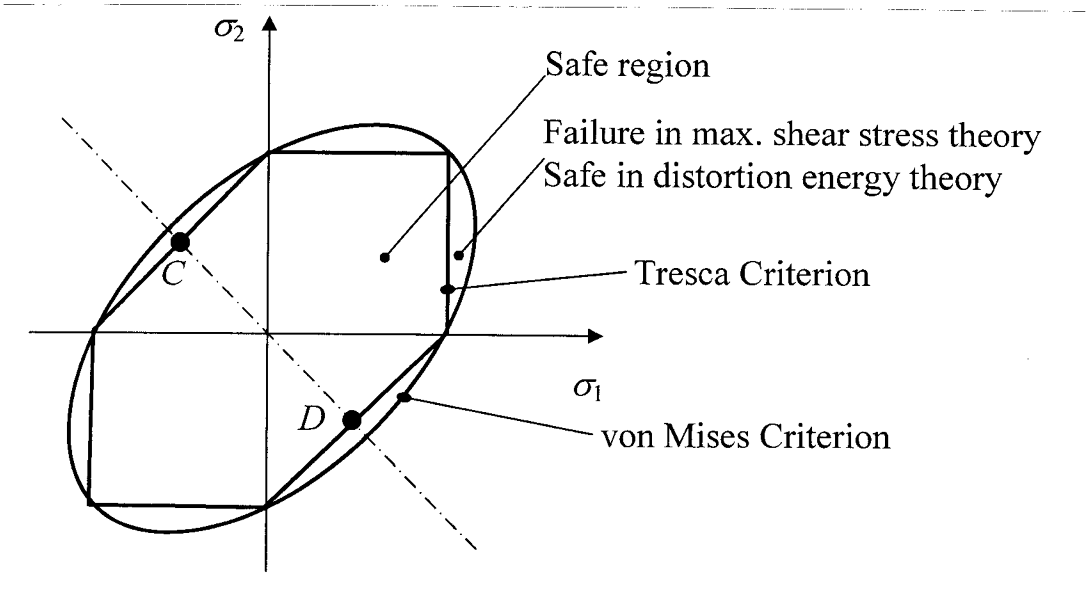
$$\tau_{max} = \frac{1}{2}(\sigma_{max} - \sigma_{min}) = \underbrace{\tau_Y = \frac{Y}{2}}_{\text{uniaxial tension test}}$$

$\sigma_1 = Y, \; \sigma_2 = \sigma_3 = 0.$

$$\boxed{f = \sigma_e - \frac{Y}{2}}$$

- f is regular hexagon in principal stress space.
- Good agreement with experimental results
- Pure shear test τ_Y is about 15% higher than Tresca theory.

⇒ conservative criterion.



2. Distortional Energy Density (von Mises) Criterion

- Material yields when distortion strain energy density = distortion strain energy density at yield in the uniaxial tensile test.

- Distortion energy = energy associated with a change in the shape of a body.

volumetric strain energy density

distortional strain energy density.

$$U_o = \frac{1}{2E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-2\nu(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_2\sigma_3)\right]$$

$\sigma_m =$

U_h = strain energy density with $\sigma_1=\sigma_2=\sigma_3=\sigma_m$.

$$= \frac{1}{2E}\left[\sigma_m^2+\sigma_m^2+\sigma_m^2-2\nu(\sigma_m\sigma_m+\sigma_m\sigma_m+\sigma_m\sigma_m)\right]$$

$$U_h = \frac{3}{2}\frac{(1-2\nu)}{E}\left[\frac{\sigma_1+\sigma_2+\sigma_3}{3}\right]^2$$

$$= \frac{(1-2\nu)}{6E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2+2(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_2\sigma_3)\right]$$

$U_d =$

$=$

$=$

- Distortion energy in 1D ($\sigma_1 = Y$, $\sigma_2 = \sigma_3 = 0$)

$U_{dY} =$ $\Leftarrow G = \frac{E}{2(1+\nu)}$

$$\therefore \quad \frac{1}{12G}\left[(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2\right] = \frac{1}{6G}Y^2$$

- Yield function

$$\boxed{}$$

↳ von Mises stress.

- second deviatoric stress invariant J_2

- Alternative yield function: $=0$
- Use octahedral stress τ_{oct} in pp. 15.

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2}$$

$\Rightarrow$

$\Rightarrow$ $\frac{\sqrt{2}}{3} \doteq 0.471$.

- Yield surface is a cylinder that covers Tresca hexagon.
- Biaxial stress is ellipsoid in $\sigma_1 - \sigma_2$ plane.
- Accurate for ductile metal. more accurate than Tresca.

ex) $\sigma_1 = -\sigma_2 = \sigma, \quad \sigma_3 = 0.$

- von Mises yield function

$$f = 3\sigma^2 - Y^2 = 0 \qquad \sigma = \frac{Y}{\sqrt{3}}$$

max. shear stress $\tau_{max} = \frac{1}{2}|\sigma_1 - \sigma_2| = \sigma.$

Thus, material yields at $\sigma = \tau_{max} = \tau_Y.$

$$\Rightarrow \tau_Y = \frac{Y}{\sqrt{3}} = \underline{0.577Y.}$$

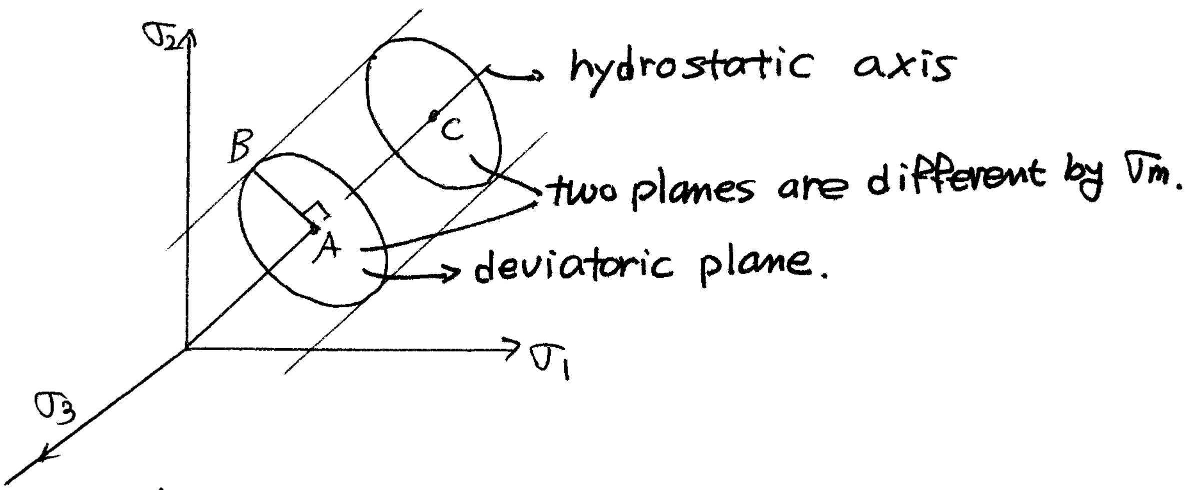
Max. shear stress criterion $\underline{\tau_Y = 0.5Y.}$

$\therefore$ Tresca criterion is more conservative.

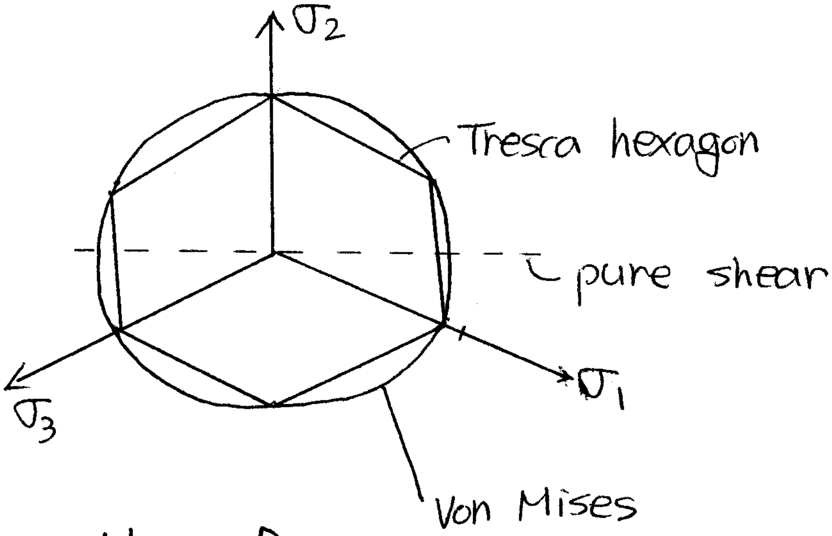
- Tresca is easier to apply.
- von Mises criterion has continuous normal vector
  $\Rightarrow$ convenient for computational plasticity.

3. Hydrostatic Stress & π-plane

- hydrostatic stress does not contribute to yield.



- π-plane : deviatoric plane with $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_m = 0.$



- von Mises yield surface is circle in π-plane.
- Tresca yield surface is regular hexagon.

4.5. Alternative Yield Criteria

1. Mohr-Coulomb Yield Criterion

- Rock & concrete depends on hydrostatic stress.

  $\Rightarrow$ compressive σ_m increases yield resistance
  different yield stesses for tension & compression.

- Material properties : $\begin{cases} \text{cohesion } c \\ \text{internal friction } \phi. \end{cases}$

- Mohr-Coulomb yield function ($\sigma_1 > \sigma_2 > \sigma_3$)

- 1D tension

$$Y_T = \frac{2c\cos\phi}{1+\sin\phi} \qquad (\sigma_1 = Y_T,\ \sigma_2 = \sigma_3 = 0)$$

  1D compression

$$Y_C = \frac{2c\cos\phi}{1-\sin\phi} \qquad (\sigma_1 = \sigma_2 = 0,\ \sigma_3 = -Y_C)$$

- When Y_T & Y_C are available, material properties :

$$c = \frac{1}{2}\sqrt{Y_T Y_C}\ , \quad \sin\phi = \frac{Y_C - Y_T}{Y_C + Y_T}$$

- Yield surface is irregular hexagon. in π plane.

2. Drucker – Prager Yield Criterion.

- Generalization of von Mises criterion with hydrostatic stress.

- Yield function

$$\alpha = \frac{2\sin\phi}{\sqrt{3}\,(3-\sin\phi)} \qquad K = \frac{6c\cos\phi}{\sqrt{3}\,(3-\sin\phi)} \qquad \text{compression}$$

$$\alpha = \frac{2\sin\phi}{\sqrt{3}\,(3+\sin\phi)} \qquad K = \frac{6c\cos\phi}{\sqrt{3}\,(3+\sin\phi)} \qquad \text{tension.}$$

3. Hill's Criterion for Orthotropic Materials

- Generalization of von Mises criterion.

- Let X, Y, Z be 3 yield tensile strength of 1,2,3-dirs.

  and S_{12}, S_{13}, S_{23} : shear yield strength.

$$2F = \frac{1}{Z^2} + \frac{1}{Y^2} - \frac{1}{X^2}\ , \qquad 2G = \frac{1}{Z^2} + \frac{1}{X^2} - \frac{1}{Y^2}$$

$$2H = \frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2}\ , \qquad 2L = \frac{1}{S_{23}^2}\ , \ 2M = \frac{1}{S_{13}^2}\ , \ 2N = \frac{1}{S_{12}^2}\ .$$

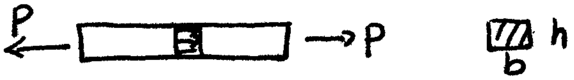
- For isotropic material, $6F = 6G = 6H = L = M = N$. $\Rightarrow$ Mises.

4.6. General Yielding

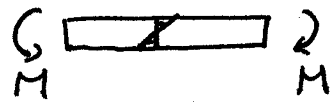
- Failure load : load for which the load-deflection curve for the member becomes nonlinear.
  - low bound load for general yielding modes.
- Fully plastic load : all cross-sections are yielded.
  - upper bound. for elastic-perfectly plastic mat.

∘ Failure Load

- tension : $P_Y =$



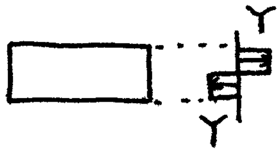
- bending : $M_Y =$



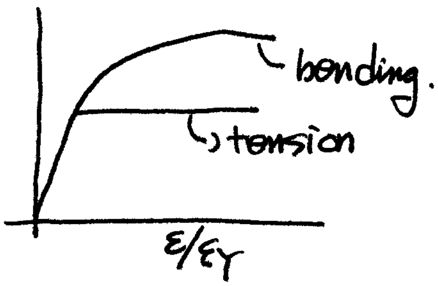
∘ Fully Plastic Load

·   ∵ elastic-perfectly plastic mat.

·



$$M = Yb\frac{h}{2}\cdot\frac{h}{4}\times 2$$



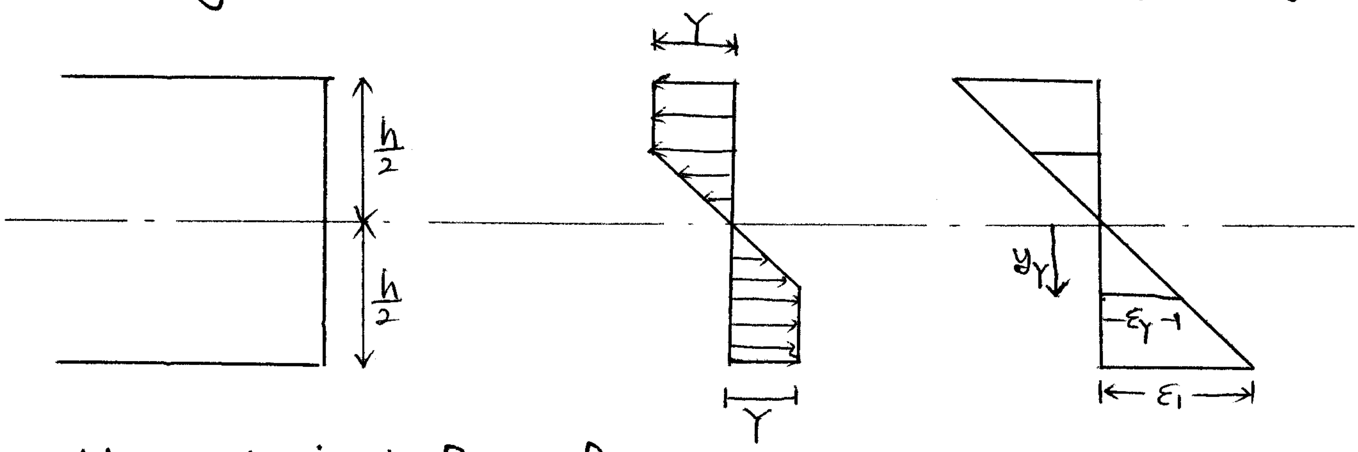
∴ For tension, $P = P_Y = P_P$

For bending $M_P = \frac{3}{2}M_Y$

- No fully plastic loading for strain hardening material.

1. Elastic - Plastic Bending

- rectangular cross-section ($b \times h$) beam (elastic-perfectly plast)



- Max. strain before fracture

$$\varepsilon_{zz} = \varepsilon_1 = k\,\varepsilon_Y \qquad k > 1.$$

$$\left.\begin{aligned} \varepsilon_{zz} &= \varepsilon_Y \quad \text{at} \quad y = y_Y \\ \varepsilon_{zz} &= \varepsilon_1 = k\,\varepsilon_Y \quad \text{at} \quad y = \frac{h}{2} \end{aligned}\right)$$

$$\therefore\ y_Y = \frac{h}{2k}$$

- Equilibrium of moment

$$\Sigma M_x = \qquad\qquad = 0$$

$$M = M_{EP} =$$

. In the elastic region

$$\sigma_{zz} = \frac{y}{y_Y} Y$$

$$M_{EP} = 2\left[\frac{Y}{y_Y}\frac{1}{3}y^3 b\right]_0^{y_Y} + 2\left[\frac{1}{2} Y y^2 b\right]_{y_Y}^{h/2}$$

$$= 2\frac{Y}{y_Y}\frac{1}{3} b y_Y^3 + Yb\left(\frac{h}{2}\right)^2 - Yb y_Y^2$$

$$= \frac{1}{4} Ybh^2 - \frac{1}{3} Yb y_Y^2$$

$$M_{EP} = \frac{1}{4}Ybh^2 - \frac{1}{3}Yb\left(\frac{h}{2k}\right)^2$$

as k becames large $\quad M_{EP} \to \frac{3}{2}M_Y = M_P$

fully plastic moment.

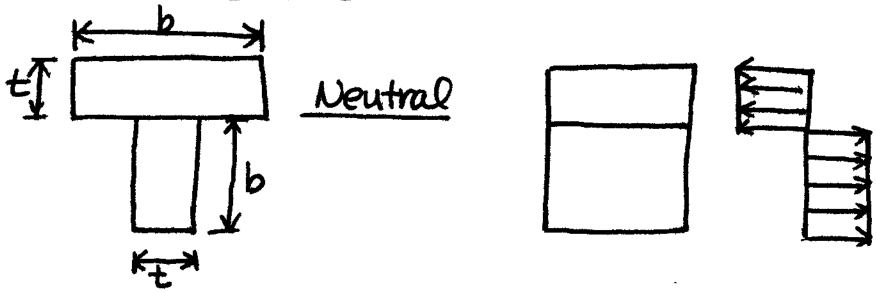
- Residual bending moment

Apply $M_x = -M_{EP}$ elastically $\Rightarrow \sigma_{el}(y)$

Residual stress $= \sigma_{EP} - \sigma_{el}$.

2. Fully Plastic Moment

~ yielding either tension or compression over the entire cross-section.



$M_P =$

3. Shear Effect for Beam Bending

$$\frac{Y}{2} = \sqrt{\left(\frac{\sigma_{zz}}{2}\right)^2 + \tau^2}$$

But, τ is small for most beam.

<u>Example 4.4.</u> When $M = 1.25 M_Y$, $y_Y = ?$

$$M_{EP} = M_Y\left[\frac{3}{2} - \frac{1}{2k^2}\right] = 1.25 M_Y = \frac{5}{4} M_Y$$

$$\frac{1}{2k^2} = \frac{3}{2} - \frac{5}{4} = \frac{1}{4} \qquad \therefore k = \sqrt{2}.$$

$$\therefore y_Y = \frac{h}{2k} = \frac{h}{2\sqrt{2}} \quad //$$

5. <u>Comparison of Failure Criteria</u>

- Tresca & von Mises criteria are acceptable for isotropic ductile metals.
- When yielding occurs for tensile test :
  1) max. principal stress $\sigma_1 = Y$.
  2) max. principal strain $\varepsilon_1 = \varepsilon_Y \ (= Y/E)$
  3) strain energy density $U_0 = \frac{Y^2}{2E}$
  4) max. shear stress $\tau_{max} = \tau_Y \ (= Y/2)$
  5) distortion energy density $U_d = U_{dY} \ (= \frac{Y^2}{6G})$
  6) Octahedral stress $\tau_{oct} = \frac{\sqrt{2}}{3} Y$.

6. <u>Interpretation of Failure Criteria</u>

- Max. difference b/t Tresca & von Mises is shear.

HW4: Solve problems 4.4 4.17 4.26 4.33 55

4.4. The members AD and CF in Figure P4.4 are made of elastic–perfectly plastic structural steel, and member BE is made of 7075-T6 aluminum alloy (see Appendix A for properties). The members each have a cross-sectional area of 100 mm^2. Determine the load $P = P_Y$ that initiates yield of the structure and the fully plastic load P_P for which all the members yield.

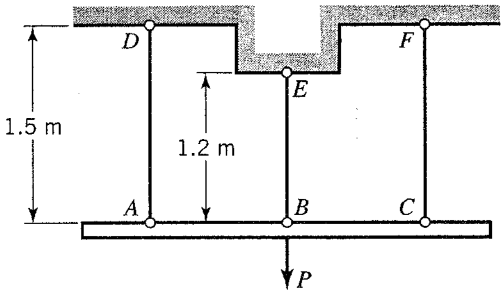


FIGURE P4.4

4.17. The structural steel in a T-beam has a yield strength Y = 260 MPa (Figure P4.17). The critical section of the beam is subjected to a moment that causes compression in the top fibers of the beam. Determine the fully plastic moment M_P.

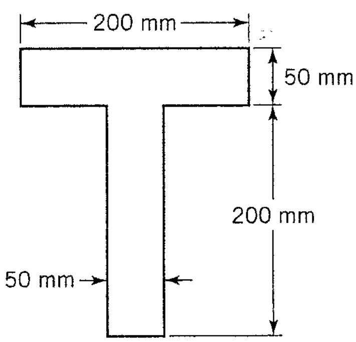


FIGURE P4.17

4.26. Consider a cantilever beam of rectangular cross section of width b and depth h. A lateral force P is applied at the free end of the beam (Figure P4.26). The beam material is elastic–perfectly plastic (see Figure 4.4a).

a. Determine the load $P = P_Y$ that causes initial yield in the beam.

b. Determine the load $P = P_P$ that produces complete yielding of the cross section in flexure.

c. Compute the ratio P_P/P_Y.

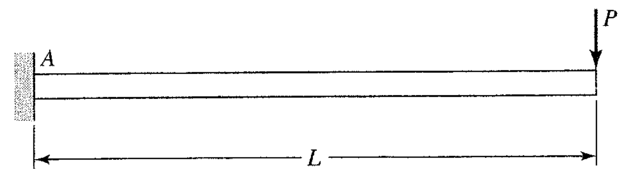


FIGURE P4.26

4.33. The shaft in Figure P4.33 is supported in flexible bearings at A and D, and two gears B and C are attached to the shaft at the locations shown. The gears are acted on by tangential forces as shown by the end view. The shaft is made of a ductile steel having a yield stress Y = 290 MPa. If the factor of safety for the design of the shaft is SF = 1.85, determine the diameter of the shaft using the maximum shear-stress criterion for the initiation of yielding failure.

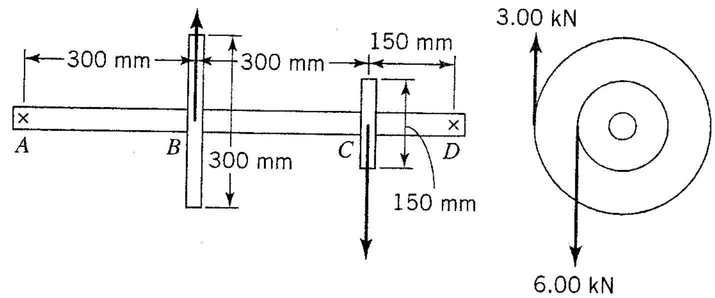


FIGURE P4.33