CHAP 2
Nonlinear Finite Element Analysis Procedures

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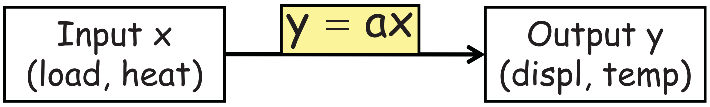
Goals

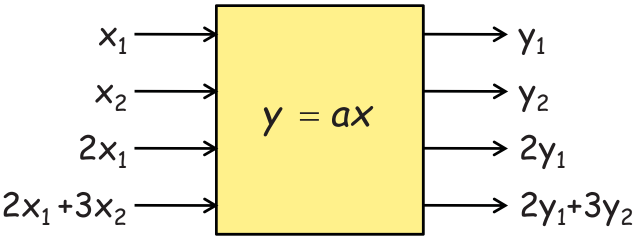
- What is a nonlinear problem?
- How is a nonlinear problem different from a linear one?
- What types of nonlinearity exist?
- How to understand stresses and strains
- How to formulate nonlinear problems
- How to solve nonlinear problems
- When does nonlinear analysis experience difficulty?

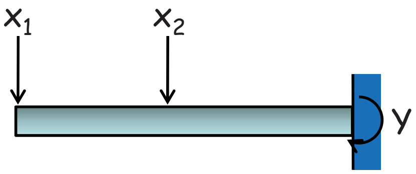
Nonlinear Structural Problems

- What is a nonlinear structural problem?
  - Everything except for linear structural problems
  - Need to understand linear problems first
- What is linearity?

$$A(\alpha u + \beta w) = \alpha A(u) + \beta A(w)$$

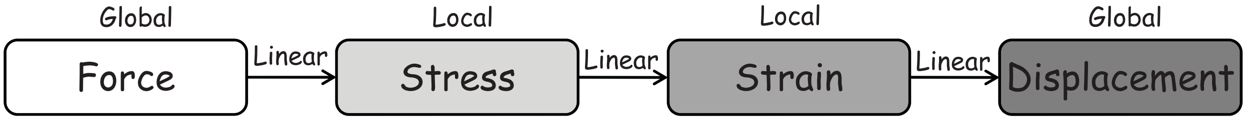


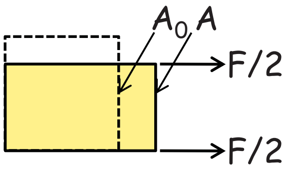




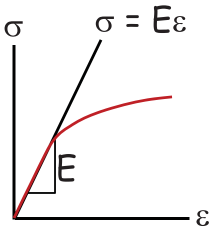
- Example: fatigue analysis

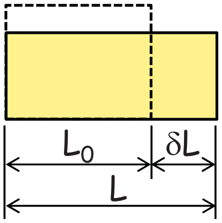
What is a linear structural problem?





$$\sigma = \frac{F}{A_0} \; ? \; \frac{F}{A(F)}$$





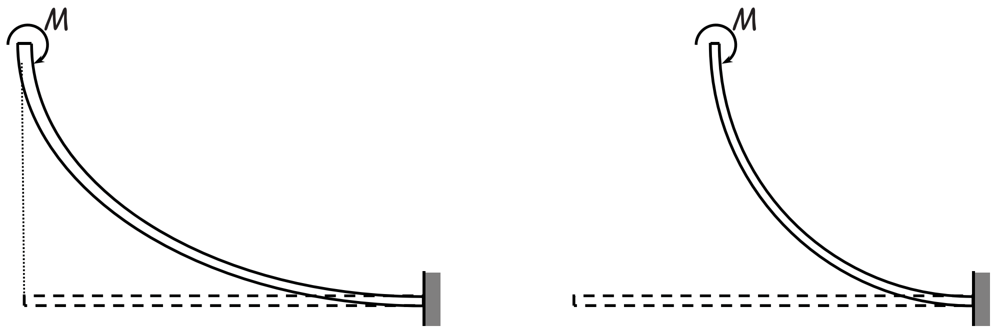
$$\varepsilon = \frac{\delta L}{L_0} \; ? \; \frac{\delta L}{L} \; ?$$

$$F = \sigma A_0 = A_0 E \varepsilon = \frac{A_0 E}{L_0} \delta L$$

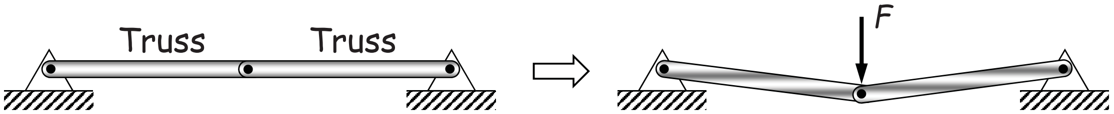
- Linearity is an approximation
- Assumptions:
  - Infinitesimal strain (<0.2%)
  - Infinitesimal displacement
  - Small rotation
  - Linear stress-strain relation

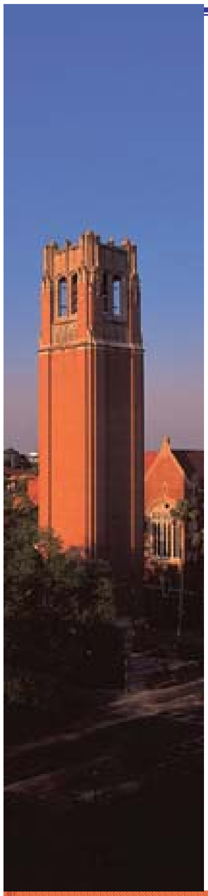
Observations in linear problems

- Which one will happen?



- Will this happen?





What types of nonlinearity exist?

It is at every stage of analysis

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Linear vs. Nonlinear Problems

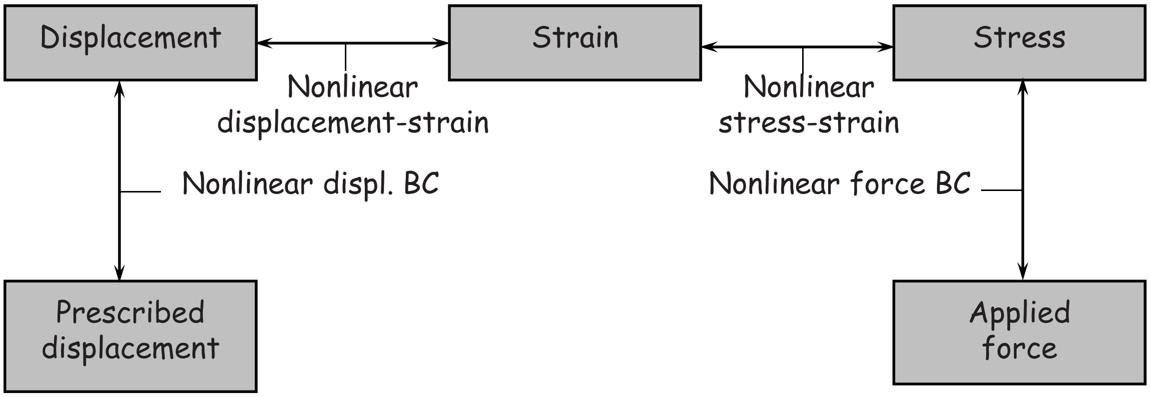
- Linear Problem:
  - Infinitesimal deformation: $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$
  - Linear stress-strain relation: $\sigma = D : \varepsilon$
  - Constant displacement BCs
  - Constant applied forces

Undeformed coord.

Constant

- Nonlinear Problem:
  - Everything except for linear problems!
  - Geometric nonlinearity: nonlinear strain-displacement relation
  - Material nonlinearity: nonlinear constitutive relation
  - Kinematic nonlinearity: Non-constant displacement BCs, contact
  - Force nonlinearity: follow-up loads

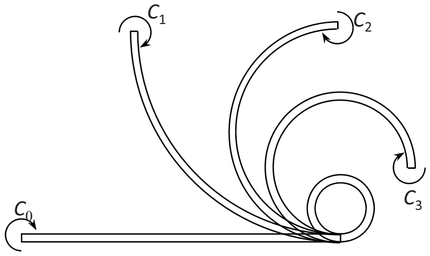
Nonlinearities in Structural Problems

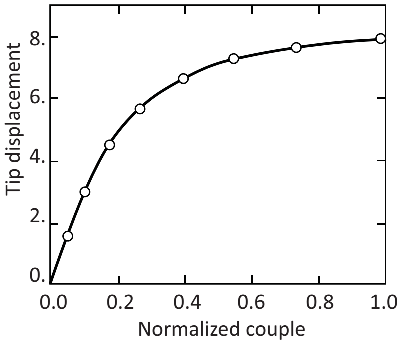


- **More than one nonlinearity can exist at the same time**

Geometric Nonlinearity

- Relations among kinematic quantities (i.e., displacement, rotation and strains) are nonlinear





- Displacement-strain relation
  - Linear: $\varepsilon(x) = \dfrac{du}{dx}$
  - Nonlinear: $E(x) = \dfrac{du}{dx} + \dfrac{1}{2}\left(\dfrac{du}{dx}\right)^2$

When du/dx is small

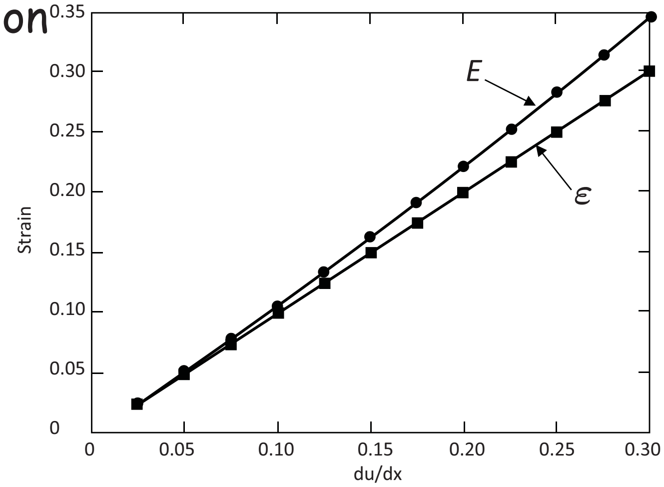
$$\left(\frac{du}{dx}\right)^2 \ll \frac{du}{dx}$$

H.O.T. can be ignored

$$\varepsilon(x) \approx E(x)$$

Geometric Nonlinearity cont.

- Displacement-strain relation
  - E has a higher-order term
  - (du/dx) << 1 → $\varepsilon(x) \sim E(x)$.



- Domain of integration
  - Undeformed domain Ω_0
  - Deformed domain Ω_x

$$a(\mathbf{u}, \bar{\mathbf{u}}) = \iint_{\Omega} \varepsilon(\bar{\mathbf{u}}) : \sigma(\mathbf{u})\, d\Omega$$

Deformed domain is unknown

Material Nonlinearity

- Linear (elastic) material

$$\{\sigma\} = [\mathbf{D}]\{\varepsilon\}$$

  - Only for infinitesimal deformation
- Nonlinear (elastic) material

More generally, $\{\sigma\} = \{\mathbf{f}(\varepsilon)\}$

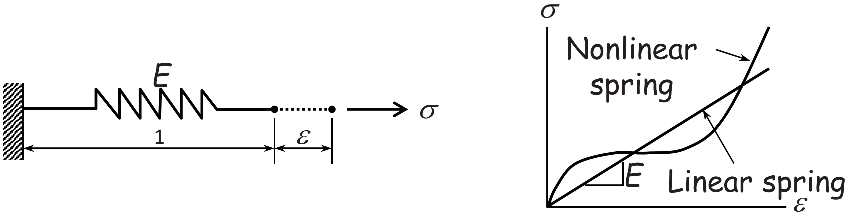
  - [D] is not a constant but depends on deformation
  - Stress by differentiating strain energy density U

$$\sigma = \frac{dU}{d\varepsilon}$$

  - Linear material:

$$U = \frac{1}{2}E\varepsilon^2 \qquad \sigma = \frac{dU}{d\varepsilon} = E\varepsilon$$

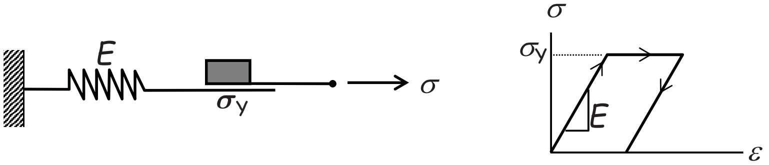
  - Stress is a function of strain (deformation): potential, path independent



Linear and nonlinear elastic spring models

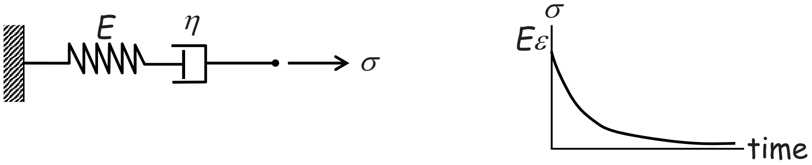
Material Nonlinearity cont.

- Elasto-plastic material (energy dissipation occurs)
  - Friction plate only support stress up to σ_y
  - Stress cannot be determined from stress alone
  - History of loading path is required: path-dependent



Elasto-plastic spring model

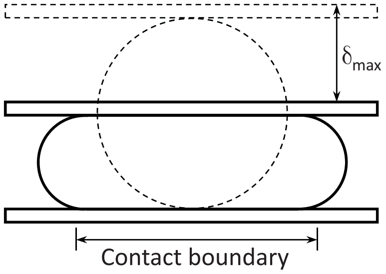
- Visco-elastic material
  - Time-dependent behavior
  - Creep, relaxation

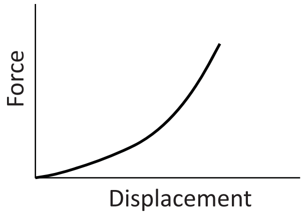


Visco-elastic spring model

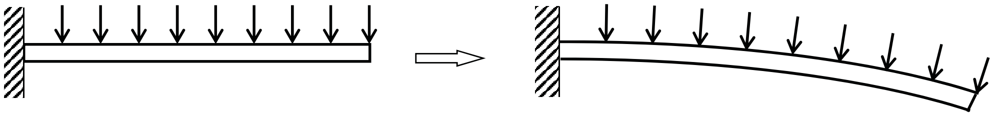
Boundary and Force Nonlinearities

- Nonlinear displacement BC (kinematic nonlinearity)
  - Contact problems, displacement dependent conditions





- Nonlinear force BC (Kinetic nonlinearity)

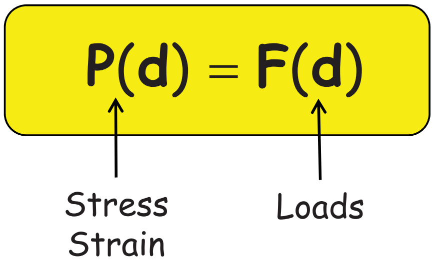


Mild vs. Rough Nonlinearity

- **Mild** Nonlinear Problems
  - Continuous, **history-independent** nonlinear relations between stress and strain
  - Nonlinear elasticity, Geometric nonlinearity, and deformation-dependent loads
- **Rough** Nonlinear Problems
  - Equality and/or inequality constraints in constitutive relations
  - **History-dependent** nonlinear relations between stress and strain
  - Elastoplasticity and contact problems

Nonlinear Finite Element Equations

- Equilibrium between internal and external forces



$$P(d) = F(d)$$

Stress
Strain

Loads

Linear problems
$[K]\{d\} = \{F\}$

- Kinetic and kinematic nonlinearities
  - Appears on the boundary
  - Handled by displacements and forces (global, explicit)
  - Relatively easy to understand (Not easy to implement though)
- Material & geometric nonlinearities
  - Appears in the domain
  - Depends on stresses and strains (local, implicit)

Solution Procedure

We can only solve for linear problems …

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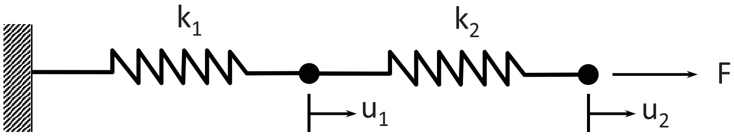
Example – Nonlinear Springs

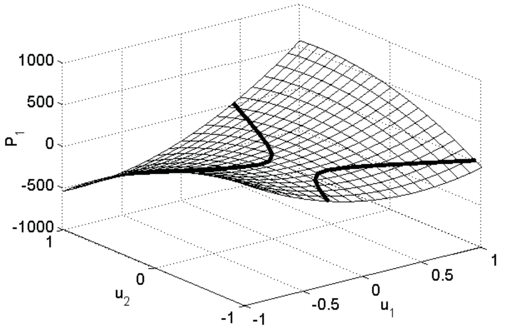
- Spring constants
  - $k_1 = 50 + 500u$
    $k_2 = 100 + 200u$

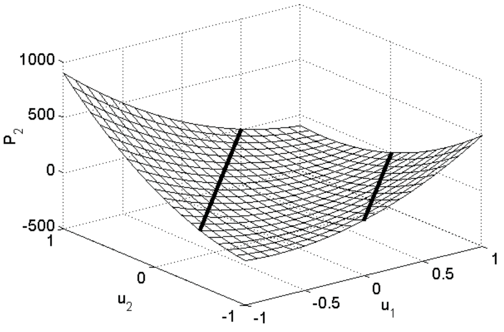


- Governing equation

$$\begin{cases} 300u_1^2 + 400u_1u_2 - 200u_2^2 + 150u_1 - 100u_2 = 0 & P_1 \\ 200u_1^2 - 400u_1u_2 + 200u_2^2 - 100u_1 + 100u_2 = 100 & P_2 \end{cases}$$

  - Solution is in the intersection between two zero contours
  - Multiple solutions may exist
  - No solution exists in a certain situation





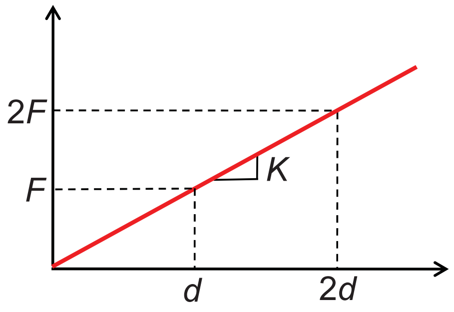
Solution Procedure

- Linear Problems

$$\mathbf{K} \cdot \mathbf{d} = \mathbf{F} \quad \text{or} \quad \mathbf{P}(\mathbf{d}) = \mathbf{F}$$

  - Stiffness matrix **K** is constant

$$\mathbf{P}(\mathbf{d}_1 + \mathbf{d}_2) = \mathbf{P}(\mathbf{d}_1) + \mathbf{P}(\mathbf{d}_2)$$
$$\mathbf{P}(\alpha\mathbf{d}) = \alpha\mathbf{P}(\mathbf{d}) = \alpha\mathbf{F}$$



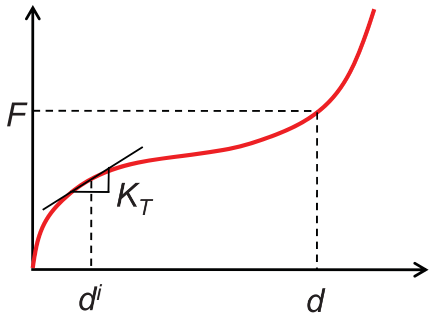
  - If the load is doubled, displacement is doubled, too
  - **Superposition** is possible

- Nonlinear Problems

$$\mathbf{P}(\mathbf{d}) = \mathbf{F}, \quad \mathbf{P}(2\mathbf{d}) \neq 2\mathbf{F}$$

  - How to find **d** for a given **F**?

Incremental Solution Procedure



Newton-Raphson Method

- Most popular method
- Assume $\mathbf{d}^i$ at i-th iteration is known
- Looking for $\mathbf{d}^{i+1}$ from first-order Taylor series expansion

$$\mathbf{P}(\mathbf{d}^{i+1}) \approx \mathbf{P}(\mathbf{d}^i) + \mathbf{K}_T^i(\mathbf{d}^i) \cdot \Delta\mathbf{d}^i = \mathbf{F}$$

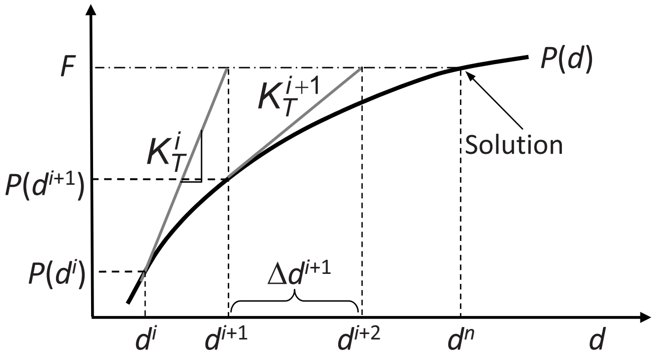
  - $\mathbf{K}_T^i(\mathbf{d}^i) \equiv \left(\dfrac{\partial \mathbf{P}}{\partial \mathbf{d}}\right)^i$: **Jacobian matrix or Tangent stiffness matrix**

- **Solve for incremental solution**

$$\mathbf{K}_T^i \Delta\mathbf{d}^i = \mathbf{F} - \mathbf{P}(\mathbf{d}^i)$$

- Update solution

$$\mathbf{d}^{i+1} = \mathbf{d}^i + \Delta\mathbf{d}^i$$



N-R Method cont.

- Observations:
  - **Second-order convergence** near the solution (Fastest method!)
  - Tangent stiffness $\mathbf{K}_T^i(\mathbf{d}^i)$ is not constant $\lim\limits_{n\to\infty} \dfrac{|u_{exact} - u_{n+1}|}{|u_{exact} - u_n|^2} = c$
  - The matrix equation solves for incremental displacement $\Delta\mathbf{d}^i$
  - RHS is not a force but a **residual force** $\mathbf{R}^i \equiv \mathbf{F} - \mathbf{P}(\mathbf{d}^i)$
  - Iteration stops when conv < tolerance

$$\text{conv} = \frac{\sum_{j=1}^{n}(R_j^{i+1})^2}{1 + \sum_{j=1}^{n}(F_j)^2} \qquad \text{Or,} \qquad \text{conv} = \frac{\sum_{j=1}^{n}(\Delta u_j^{i+1})^2}{1 + \sum_{j=1}^{n}(\Delta u_j^0)^2}$$

N-R Algorithm

1. Set tolerance = 0.001, k = 0, max_iter = 20, and initial estimate u = u_0
2. Calculate residual R = f – P(u)
3. Calculate conv. If conv < tolerance, stop
4. If k > max_iter, stop with error message
5. Calculate Jacobian matrix K_T
6. If the determinant of K_T is zero, stop with error message
7. Calculate solution increment Δu
8. Update solution by u = u + Δu
9. Set k = k + 1
10. Go to Step 2

Example – N-R Method

$$\mathbf{P}(\mathbf{d}) \equiv \begin{Bmatrix} d_1 + d_2 \\ d_1^2 + d_2^2 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 9 \end{Bmatrix} \equiv \mathbf{F} \qquad \mathbf{d}^0 = \begin{Bmatrix} 1 \\ 5 \end{Bmatrix} \qquad \mathbf{P}(\mathbf{d}^0) = \begin{Bmatrix} 6 \\ 26 \end{Bmatrix}$$

$$\mathbf{K}_T = \frac{\partial \mathbf{P}}{\partial \mathbf{d}} = \begin{bmatrix} 1 & 1 \\ 2d_1 & 2d_2 \end{bmatrix} \qquad \mathbf{R}^0 = \mathbf{F} - \mathbf{P}(\mathbf{d}^0) = \begin{Bmatrix} -3 \\ -17 \end{Bmatrix}$$

- Iteration 1

$$\begin{bmatrix} 1 & 1 \\ 2 & 10 \end{bmatrix} \begin{Bmatrix} \Delta d_1^0 \\ \Delta d_2^0 \end{Bmatrix} = \begin{Bmatrix} -3 \\ -17 \end{Bmatrix} \quad \Longrightarrow \quad \begin{Bmatrix} \Delta d_1^0 \\ \Delta d_2^0 \end{Bmatrix} = \begin{Bmatrix} -1.625 \\ -1.375 \end{Bmatrix}$$

$$\mathbf{d}^1 = \mathbf{d}^0 + \Delta\mathbf{d}^0 = \begin{Bmatrix} -0.625 \\ 3.625 \end{Bmatrix}$$

$$\mathbf{R}^1 = \mathbf{F} - \mathbf{P}(\mathbf{d}^1) = \begin{Bmatrix} 0 \\ -4.531 \end{Bmatrix}$$

Example – N-R Method cont.

- Iteration 2

$$\begin{bmatrix} 1 & 1 \\ -1.25 & 7.25 \end{bmatrix} \begin{Bmatrix} \Delta d_1^1 \\ \Delta d_2^1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -4.531 \end{Bmatrix} \quad \Longrightarrow \quad \begin{Bmatrix} \Delta d_1^1 \\ \Delta d_2^1 \end{Bmatrix} = \begin{Bmatrix} 0.533 \\ -0.533 \end{Bmatrix}$$

$$\mathbf{d}^2 = \mathbf{d}^1 + \Delta\mathbf{d}^1 = \begin{Bmatrix} -0.092 \\ 3.092 \end{Bmatrix} \qquad \mathbf{R}^2 = \mathbf{F} - \mathbf{P}(\mathbf{d}^2) = \begin{Bmatrix} 0 \\ -0.568 \end{Bmatrix}$$

- Iteration 3

$$\begin{bmatrix} 1 & 1 \\ -0.184 & 6.184 \end{bmatrix} \begin{Bmatrix} \Delta d_1^2 \\ \Delta d_2^2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -0.568 \end{Bmatrix} \quad \Longrightarrow \quad \begin{Bmatrix} \Delta d_1^2 \\ \Delta d_2^2 \end{Bmatrix} = \begin{Bmatrix} 0.089 \\ -0.089 \end{Bmatrix}$$

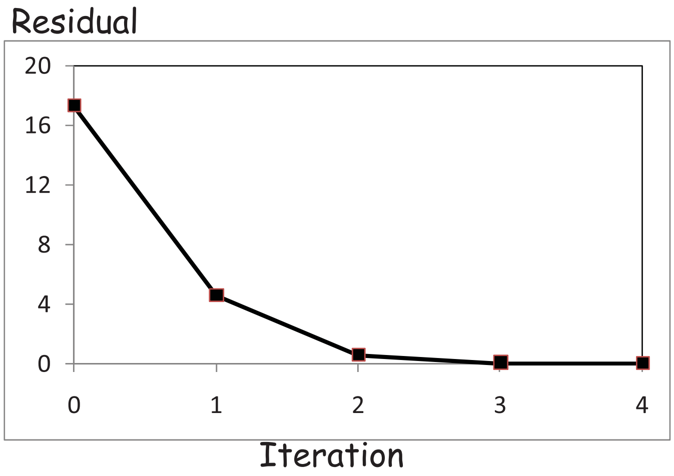
$$\mathbf{d}^3 = \mathbf{d}^2 + \Delta\mathbf{d}^2 = \begin{Bmatrix} -0.003 \\ 3.003 \end{Bmatrix} \qquad \mathbf{R}^3 = \mathbf{F} - \mathbf{P}(\mathbf{d}^3) = \begin{Bmatrix} 0 \\ -0.016 \end{Bmatrix}$$

Example – N-R Method cont.

- Iteration 4

$$\begin{bmatrix} 1 & 1 \\ -0.005 & 6.005 \end{bmatrix} \begin{Bmatrix} \Delta d_1^3 \\ \Delta d_2^3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -0.016 \end{Bmatrix} \quad \Longrightarrow \quad \begin{Bmatrix} \Delta d_1^3 \\ \Delta d_2^3 \end{Bmatrix} = \begin{Bmatrix} 0.003 \\ -0.003 \end{Bmatrix}$$

$$\mathbf{d}^4 = \mathbf{d}^3 + \Delta\mathbf{d}^3 = \begin{Bmatrix} -0.000 \\ 3.000 \end{Bmatrix} \qquad \mathbf{R}^4 = \mathbf{F} - \mathbf{P}(\mathbf{d}^4) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

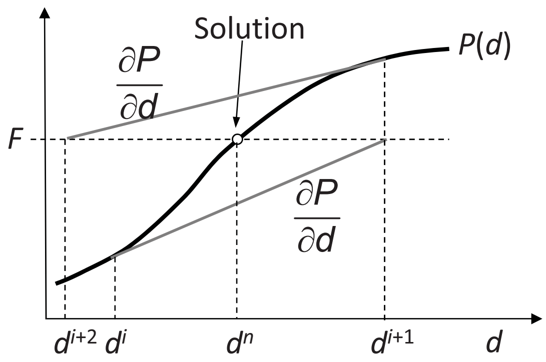


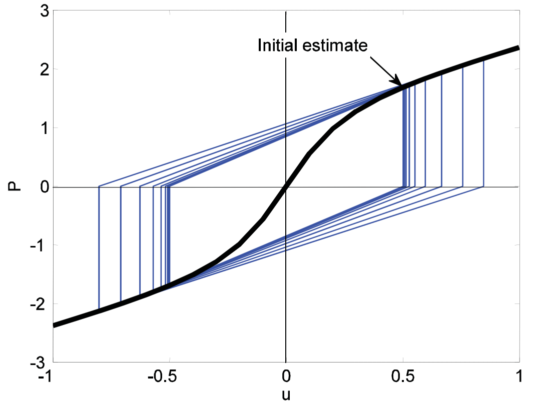
Iter	||R||
0	17.263
1	4.531
2	0.016
3	0.0

Quadratic convergence

When N-R Method Does Not Converge

- Difficulties
  - **Convergence is not always guaranteed**
  - Automatic load step control and/or line search techniques are often used
  - Difficult/expensive to calculate $\mathbf{K}_T^i(\mathbf{d}^i)$

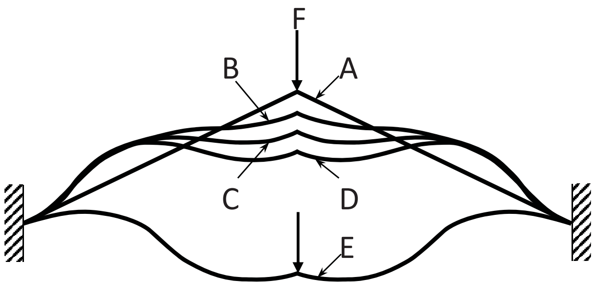


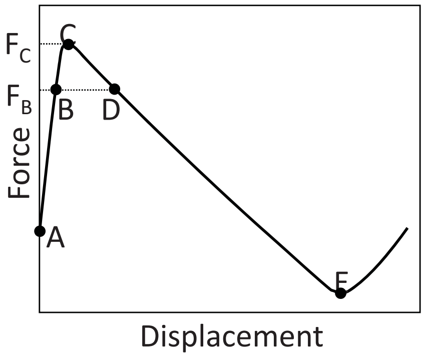


When N-R Method Does Not Converge cont.

- Convergence difficulty occurs when
  - Jacobian matrix is not positive-definite

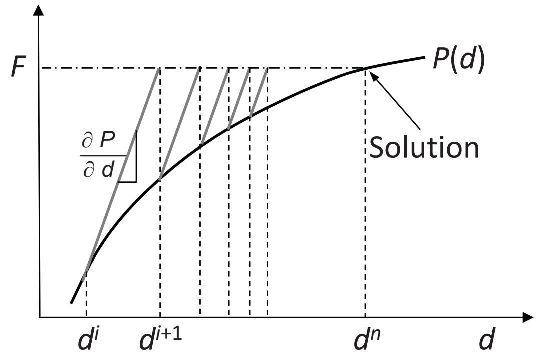
    P.D. Jacobian: in order to increase displ., force must be increased
  - Bifurcation & snap-through require a special algorithm





Modified N-R Method

- Constructing $\mathbf{K}_T^i(\mathbf{d}^i)$ and solving $\mathbf{K}_T^i \Delta\mathbf{d}^i = \mathbf{R}^i$ is expensive
- Computational Costs (Let the matrix size be N x N)
  - L-U factorization ~ N^3
  - Forward/backward substitution ~ N
- **Use L-U factorized $\mathbf{K}_T^i(\mathbf{d}^i)$ repeatedly**
- More iteration is required, but each iteration is fast
- **More stable than N-R method**
- Hybrid N-R method



Example – Modified N-R Method

- Solve the same problem using modified N-R method

$$\mathbf{P}(\mathbf{d}) \equiv \begin{Bmatrix} d_1 + d_2 \\ d_1^2 + d_2^2 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 9 \end{Bmatrix} \equiv \mathbf{F} \qquad \mathbf{d}^0 = \begin{Bmatrix} 1 \\ 5 \end{Bmatrix} \qquad \mathbf{P}(\mathbf{d}^0) = \begin{Bmatrix} 6 \\ 26 \end{Bmatrix}$$

$$\mathbf{K}_T = \frac{\partial \mathbf{P}}{\partial \mathbf{d}} = \begin{bmatrix} 1 & 1 \\ 2d_1 & 2d_2 \end{bmatrix} \qquad \mathbf{R}^0 = \mathbf{F} - \mathbf{P}(\mathbf{d}^0) = \begin{Bmatrix} -3 \\ -17 \end{Bmatrix}$$

- Iteration 1

$$\begin{bmatrix} 1 & 1 \\ 2 & 10 \end{bmatrix} \begin{Bmatrix} \Delta d_1^0 \\ \Delta d_2^0 \end{Bmatrix} = \begin{Bmatrix} -3 \\ -17 \end{Bmatrix} \quad \Longrightarrow \quad \begin{Bmatrix} \Delta d_1^0 \\ \Delta d_2^0 \end{Bmatrix} = \begin{Bmatrix} -1.625 \\ -1.375 \end{Bmatrix}$$

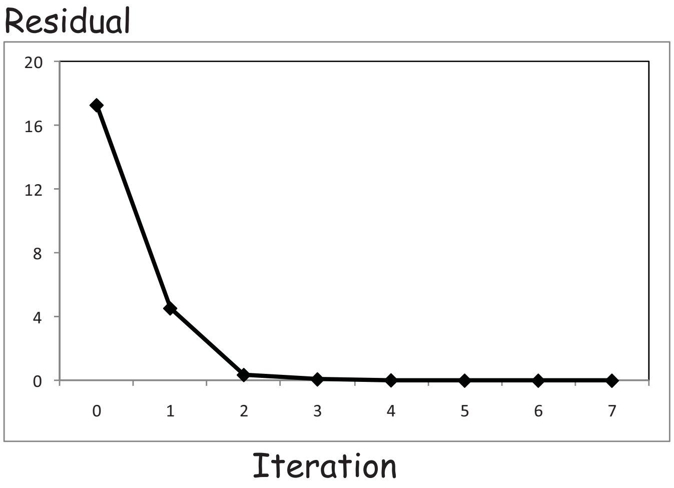
$$\mathbf{d}^1 = \mathbf{d}^0 + \Delta\mathbf{d}^0 = \begin{Bmatrix} -0.625 \\ 3.625 \end{Bmatrix} \qquad \mathbf{R}^1 = \mathbf{F} - \mathbf{P}(\mathbf{d}^1) = \begin{Bmatrix} 0 \\ -4.531 \end{Bmatrix}$$

Example – Modified N-R Method cont.

- Iteration 2

$$\begin{bmatrix}1 & 1\\ 2 & 10\end{bmatrix}\begin{Bmatrix}\Delta d_1^1\\ \Delta d_2^1\end{Bmatrix} = \begin{Bmatrix}0\\ -4.531\end{Bmatrix} \Longrightarrow \begin{Bmatrix}\Delta d_1^1\\ \Delta d_2^1\end{Bmatrix} = \begin{Bmatrix}0.566\\ -0.566\end{Bmatrix}$$

$$\mathbf{d}^2 = \mathbf{d}^1 + \Delta\mathbf{d}^1 = \begin{Bmatrix}-0.059\\ 3.059\end{Bmatrix} \qquad \mathbf{R}^2 = \mathbf{F} - \mathbf{P}(\mathbf{d}^2) = \begin{Bmatrix}0\\ -0.358\end{Bmatrix}$$



Iter	||R||
0	17.263
1	4.5310
2	0.3584
3	0.0831
4	0.0204
5	0.0051
6	0.0013
7	0.0003

Incremental Secant Method

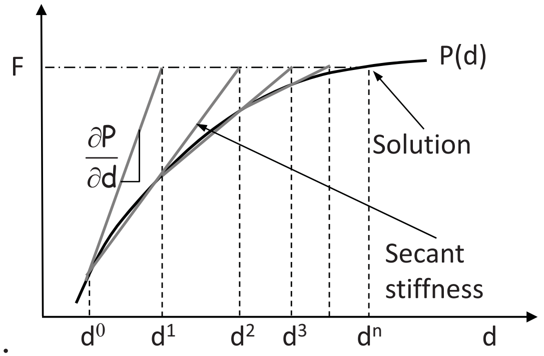
- Secant matrix
  - Instead of using tangent stiffness, **approximate it** using the solution from the previous iteration
  - At i-th iteration

$$\mathbf{K}_s^i \Delta\mathbf{d}^i = \mathbf{F} - \mathbf{P}(\mathbf{d}^i)$$

  - The secant matrix satisfies

$$\mathbf{K}_s^i \cdot (\mathbf{d}^i - \mathbf{d}^{i-1}) = \mathbf{P}(\mathbf{d}^i) - \mathbf{P}(\mathbf{d}^{i-1})$$

  - Not a unique process in high dimension



- **Start from initial K_T matrix, iteratively update it**
  - Rank-1 or rank-2 update
  - The textbook has Broyden's algorithm (Rank-1 update)
  - Here we will discuss BFGS method (Rank-2 update)

Incremental Secant Method cont.

- BFGS (Broyden, Fletcher, Goldfarb and Shanno) method
  - Stiffness matrix must be symmetric and positive-definite

$$\Delta \mathbf{d}^i = [\mathbf{K}_s^i]^{-1}\{\mathbf{F} - \mathbf{P}(\mathbf{d}^i)\} \equiv [\mathbf{H}_s^i]\{\mathbf{F} - \mathbf{P}(\mathbf{d}^i)\}$$

  - Instead of updating **K**, update **H** (saving computational time)

$$\mathbf{H}_s^i = (\mathbf{I} + \mathbf{w}^i\mathbf{v}^{iT})\mathbf{H}_s^{i-1}(\mathbf{I} + \mathbf{w}^i\mathbf{v}^{iT})$$

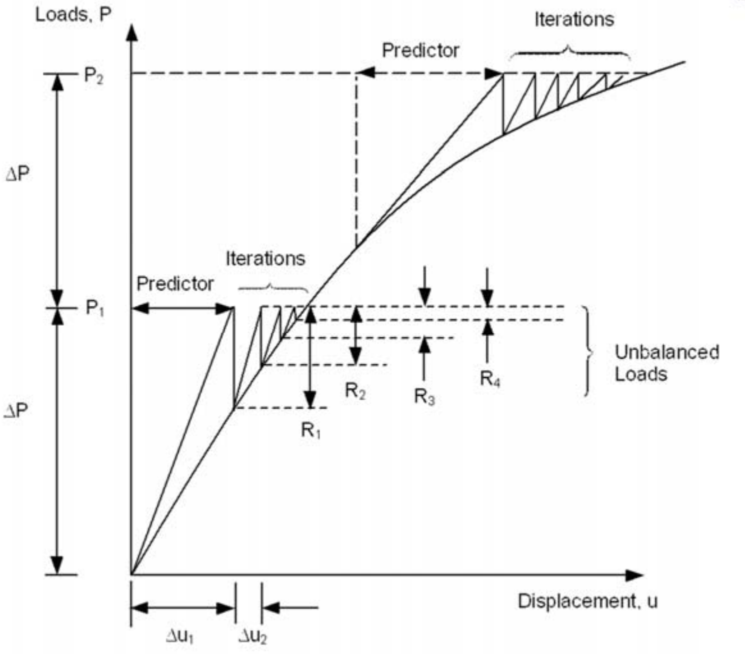
$$\mathbf{v}^i = \mathbf{R}^{i-1}\left(1 - \frac{(\Delta\mathbf{d}^{i-1})^T(\mathbf{R}^{i-1} - \mathbf{R}^i)}{(\Delta\mathbf{d}^i)^T\mathbf{R}^{i-1}}\right) - \mathbf{R}^i$$

$$\mathbf{w}^i = \frac{\Delta\mathbf{d}^{i-1}}{(\Delta\mathbf{d}^{i-1})^T(\mathbf{R}^{i-1} - \mathbf{R}^i)}$$

- Become unstable when the No. of iterations is increased

Incremental Force Method

- N-R method converges fast if the initial estimate is close to the solution
- Solid mechanics: initial estimate = undeformed shape
- Convergence difficulty occurs when the applied load is large (deformation is large)
- IFM: apply loads in increments. Use the solution from the previous increment as an initial estimate
- Commercial programs call it "**Load Increment**" or "**Time Increment**"



Incremental Force Method cont.

- Load increment does not have to be uniform
  - Critical part has smaller increment size
- Solutions in the intermediate load increments
  - History of the response can provide insight into the problem
  - Estimating the bifurcation point or the critical load
  - Load increments greatly affect the accuracy in path-dependent problems

---

Load Increment in Commercial Software

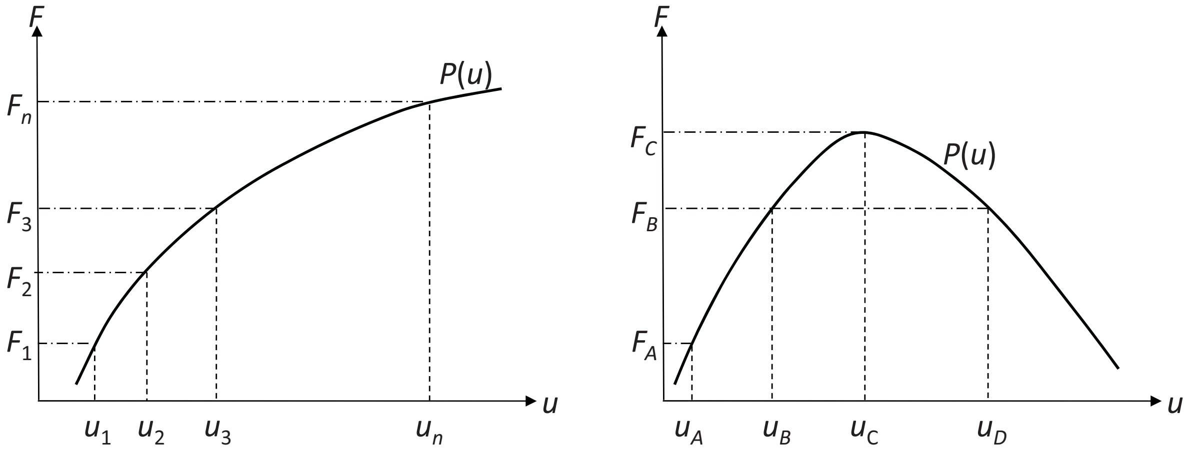
- Use "Time" to represent load level
  - In a static problem, "Time" means a pseudo-time
  - Required Starting time, (T_{start}), Ending time (T_{end}) and increment
  - Load is gradually increased from zero at T_{start} and full load at T_{end}
  - Load magnitude at load increment T^n:

$$F^n = \frac{T^n - T_{start}}{T_{end} - T_{start}} F \qquad T^n = n \times \Delta T \leq T_{end}$$

- Automatic time stepping
  - Increase/decrease next load increment based on the number of convergence iteration at the current load
  - User provide initial load increment, minimum increment, and maximum increment
  - Bisection of load increment when not converged

Force Control vs. Displacement Control

- Force control: gradually increase applied forces and find equilibrium configuration
- Displ. control: gradually increase prescribed displacements
  - Applied load can be calculated as a reaction
  - More stable than force control.
  - Useful for softening, contact, snap-through, etc.



Nonlinear Solution Steps

1. Initialization: $\mathbf{d}^0 = \mathbf{0};\ i = 0$
2. Residual Calculation $\mathbf{R}^i = \mathbf{F} - \mathbf{P}(\mathbf{d}^i)$
3. Convergence Check (If converged, stop)
4. Linearization
   - Calculate tangent stiffness $\mathbf{K}_T^i(\mathbf{d}^i)$
5. Incremental Solution:
   - Solve $\mathbf{K}_T^i(\mathbf{d}^i)\Delta\mathbf{d}^i = \mathbf{R}^i$
6. State Determination
   - Update displacement and stress $\mathbf{d}^{i+1} = \mathbf{d}^i + \Delta\mathbf{d}^i$ $\sigma^{i+1} = \sigma^i + \Delta\sigma^i$
7. Go To Step 2

Nonlinear Solution Steps cont.

- State determination
  - For a given displ $\mathbf{d}^k$, determine current state (strain, stress, etc)

$$\mathbf{u}^k(\mathbf{x}) = \mathbf{N}(\mathbf{x}) \cdot \mathbf{d}^k \qquad \varepsilon^k = \mathbf{B} \cdot \mathbf{d}^k \qquad \sigma^k = f(\varepsilon^k)$$

  - Sometimes, stress cannot be determined using strain alone

- Residual calculation
  - Applied nodal force – Nodal forces due to internal stresses

Weak form: $\iiint_\Omega \varepsilon(\bar{\mathbf{u}})^T \sigma \, d\Omega = \iint_{\Gamma_s} \bar{\mathbf{u}}^T \mathbf{t} \, d\Gamma + \iiint_\Omega \bar{\mathbf{u}}^T \mathbf{f}^b \, d\Omega, \quad \forall \bar{\mathbf{u}} \in \mathbb{Z}$

Discretization: $\bar{\mathbf{d}}^T \left(\iiint_\Omega \mathbf{B}^T \sigma \, d\Omega = \iint_{\Gamma_s} \mathbf{N}^T \mathbf{t} \, d\Gamma + \iiint_\Omega \mathbf{N}^T \mathbf{f}^b \, d\Omega \right), \quad \forall \bar{\mathbf{d}} \in \mathbb{Z}_h$

Residual: $\mathbf{R}^k = \iint_{\Gamma_s} \mathbf{N}^T \mathbf{t} \, d\Gamma + \iiint_\Omega \mathbf{N}^T \mathbf{f}^b \, d\Omega - \iiint_\Omega \mathbf{B}^T \sigma^k \, d\Omega$

Example – Linear Elastic Material

- Governing equation (Scalar equation)

$$\iiint_\Omega \varepsilon(\bar{\mathbf{u}})^T \sigma \, d\Omega = \iint_{\Gamma_s} \bar{\mathbf{u}}^T \mathbf{t} \, d\Gamma + \iiint_\Omega \bar{\mathbf{u}}^T \mathbf{f}^b \, d\Omega$$

$$\bar{\mathbf{u}} = \mathbf{N} \cdot \bar{\mathbf{d}}$$
$$\varepsilon(\bar{\mathbf{u}}) = \mathbf{B} \cdot \bar{\mathbf{d}}$$

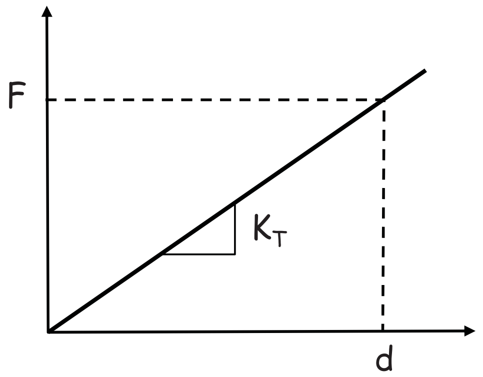
- Collect $\bar{\mathbf{d}}$

$$\bar{\mathbf{d}}^T \left(\underbrace{\iiint_\Omega \mathbf{B}^T \sigma \, d\Omega}_{\mathbf{P}(\mathbf{d})} = \underbrace{\iint_{\Gamma_s} \mathbf{N}^T \mathbf{t} \, d\Gamma + \iiint_\Omega \mathbf{N}^T \mathbf{f}^b \, d\Omega}_{\mathbf{F}} \right)$$

- Residual $\mathbf{R} = \mathbf{F} - \mathbf{P}(\mathbf{d})$
- Linear elastic material

$$\sigma = \mathbf{D} \cdot \varepsilon = \mathbf{D} \cdot \mathbf{B} \cdot \mathbf{d}$$

$$\mathbf{K}_T = \frac{\partial \mathbf{P}(\mathbf{d})}{\partial \mathbf{d}} = \iiint_\Omega \mathbf{B}^T \mathbf{D} \mathbf{B} \, d\Omega$$



Example – Nonlinear Bar

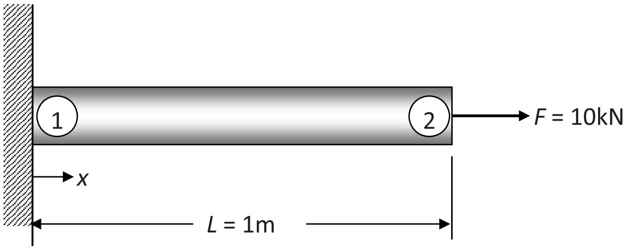
- Rubber bar $\sigma = E\tan^{-1}(m\varepsilon)$
- Discrete weak form $\bar{\mathbf{d}}^T \int_0^L \mathbf{B}^T \sigma A\, dx = \bar{\mathbf{d}}^T \mathbf{F}$
- Scalar equation $R = F - \int_0^L \frac{\sigma A}{L} dx$

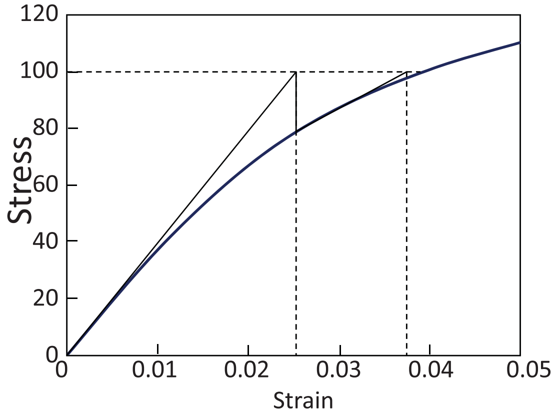
$$\Rightarrow R = F - \sigma(d)A$$

$$\bar{\mathbf{d}} = \begin{Bmatrix} \bar{d}_1 \\ \bar{d}_2 \end{Bmatrix}$$

$$\mathbf{F} = \begin{Bmatrix} R \\ F \end{Bmatrix}$$

$$\mathbf{B} = \frac{1}{L}\lfloor -1 \quad 1 \rfloor$$





Example – Nonlinear Bar cont.

- Jacobian

$$\frac{dP}{dd} = \frac{d\sigma(d)}{dd}A = \frac{d\sigma}{d\varepsilon}\frac{d\varepsilon}{dd}A = \frac{1}{L}mAE\cos^2\left(\frac{\sigma}{E}\right)$$

- N-R equation

$$\left[\frac{1}{L}mAE\cos^2\left(\frac{\sigma^k}{E}\right)\right]\Delta d^k = F - \sigma^k A$$

- Iteration 1

$$\frac{mAE}{L}\Delta d^0 = F$$

$$d^1 = d^0 + \Delta d^0 = 0.025\text{m}$$
$$\varepsilon^1 = d^1 / L = 0.025$$
$$\sigma^1 = E\tan^{-1}(m\varepsilon^1) = 78.5\text{MPa}$$

- Iteration 2

$$\left[\frac{mAE}{L}\cos^2\left(\frac{\sigma^1}{E}\right)\right]\Delta d^1 = F - \sigma^1 A$$

$$d^2 = d^1 + \Delta d^1 = 0.0357\text{m}$$
$$\varepsilon^2 = d^2 / L = 0.0357$$
$$\sigma^2 = E\tan^{-1}(m\varepsilon^2) = 96\text{MPa}$$

N-R or Modified N-R?

- It is always recommended to use the Incremental Force Method
  - Mild nonlinear: ~10 increments
  - Rough nonlinear: 20 ~ 100 increments
  - For rough nonlinear problems, analysis results depends on increment size
- Within an increment, N-R or modified N-R can be used
  - N-R method calculates K_T at every iteration
  - Modified N-R method calculates K_T once at every increment
  - N-R is better when: mild nonlinear problem, tight convergence criterion
  - Modified N-R is better when: computation is expensive, small increment size, and when N-R does not converge well
- Many FE programs provide automatic stiffness update option
  - Depending on convergence criteria used, material status change, etc

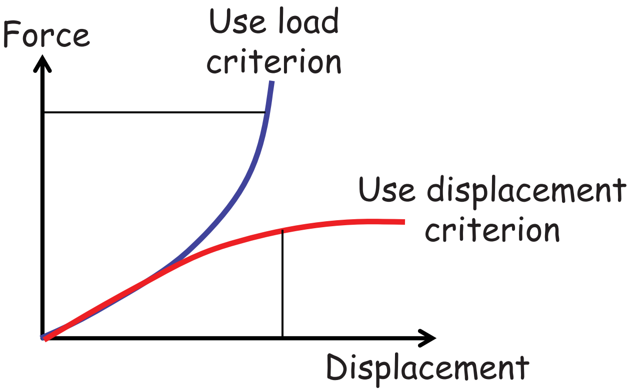
---

Accuracy vs. Convergence

- Nonlinear solution procedure requires
  - Internal force **P**(**d**)
  - Tangent stiffness $\mathbf{K}_T(\mathbf{d}) = \dfrac{\partial \mathbf{P}}{\partial \mathbf{d}}$
  - They are often implemented in the same routine
- Internal force **P**(**d**) needs to be accurate
  - We solve equilibrium of **P**(**d**) = **F**
- Tangent stiffness $\mathbf{K}_T(\mathbf{d})$ contributes to convergence
  - Accurate $\mathbf{K}_T(\mathbf{d})$ provides quadratic convergence near the solution
  - Approximate $\mathbf{K}_T(\mathbf{d})$ requires more iteration to converge
  - Wrong $\mathbf{K}_T(\mathbf{d})$ causes lack of convergence

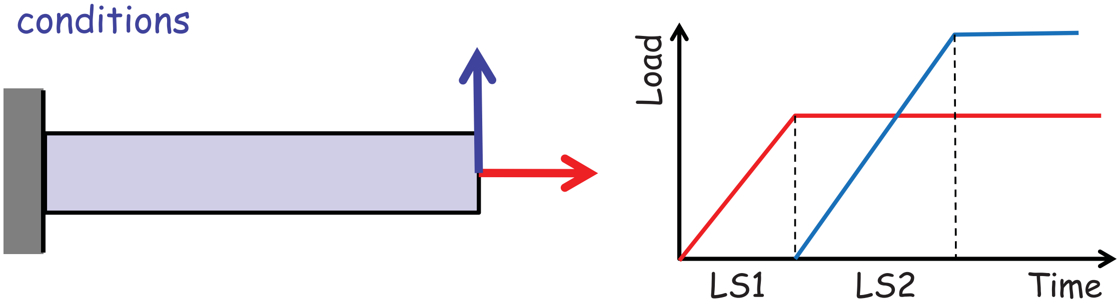
Convergence Criteria

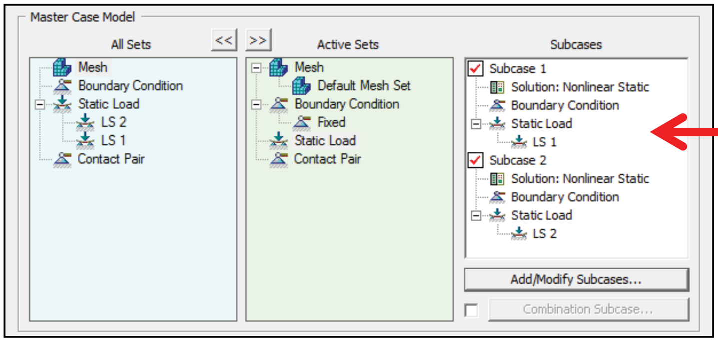
- Most analysis programs provide three convergence criteria
  - Work, displacement, load (residual)
  - Work = displacement * load
  - At least two criteria needs to be converged
- Traditional convergence criterion is load (residual)
  - Equilibrium between internal and external forces **P**(**d**) = **F**(**d**)
- Use displacement criterion for load insensitive system



Solution Strategies

- Load Step (subcase or step)
  - Load step is a set of loading and boundary conditions to define an analysis problem
  - Multiple load steps can be used to define a sequence of loading conditions

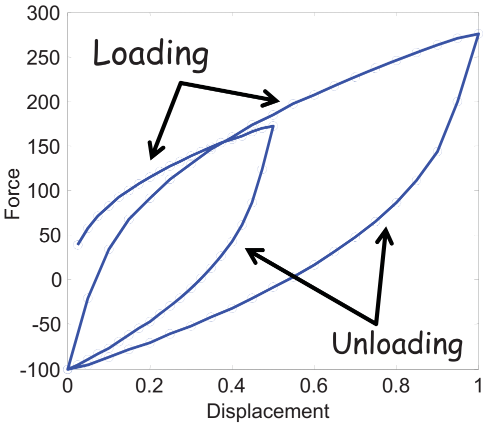




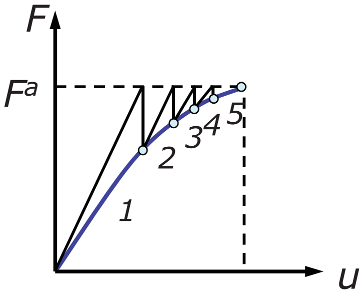
NASTRAN
SPC = 1
SUBCASE 1
  LOAD = 1
SUBCASE 2
  LOAD = 2

Solution Strategies

- Load Increment (substeps)
  - Linear analysis concerns max load
  - Nonlinear analysis depends on load path (history)
  - Applied load is gradually increased within a load step
  - Follow load path, improve accuracy, and easy to converge



- Convergence Iteration
  - Within a load increment, an iterative method (e.g., NR method) is used to find nonlinear solution
  - Bisection, linear search, stabilization, etc

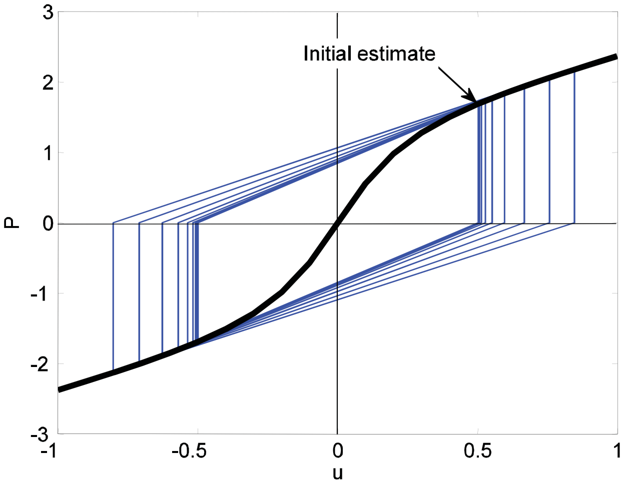


Solution Strategies cont.

- Automatic (Variable) Load Increment
  - Also called Automatic Time Stepping
  - Load increment may not be uniform
  - When convergence iteration diverges, the load increment is halved
  - If a solution converges in less than 4 iterations, increase time increment by 25%
  - If a solution converges in more than 8 iterations, decrease time increment by 25%
- Subincrement (or bisection)
  - When iterations do not converge at a given increment, analysis goes back to previously converged increment and the load increment is reduced by half
  - This process is repeated until max number of subincrements

When nonlinear analysis does not converge

- NR method assumes a constant curvature locally
- When a sign of curvature changes around the solution, NR method oscillates or diverges
- Often the residual changes sign between iterations
- Line search can help to converge

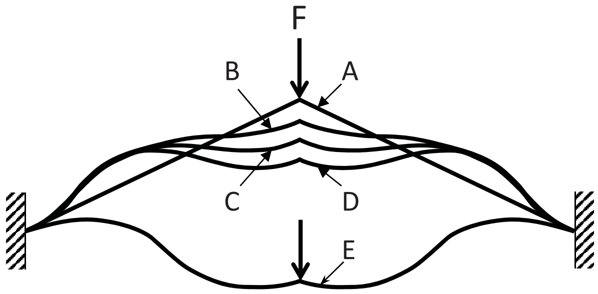


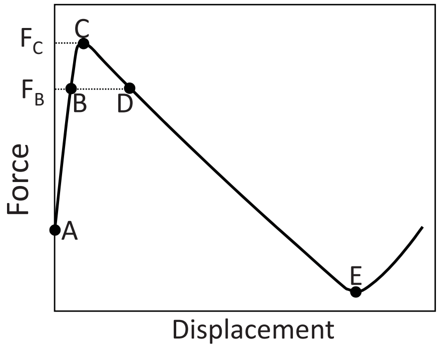
$$P(u) = u + \tan^{-1}(5u)$$

$$\frac{dP}{du} = 1 + 5\cos^2(\tan^{-1}(5u))$$

When nonlinear analysis does not converge

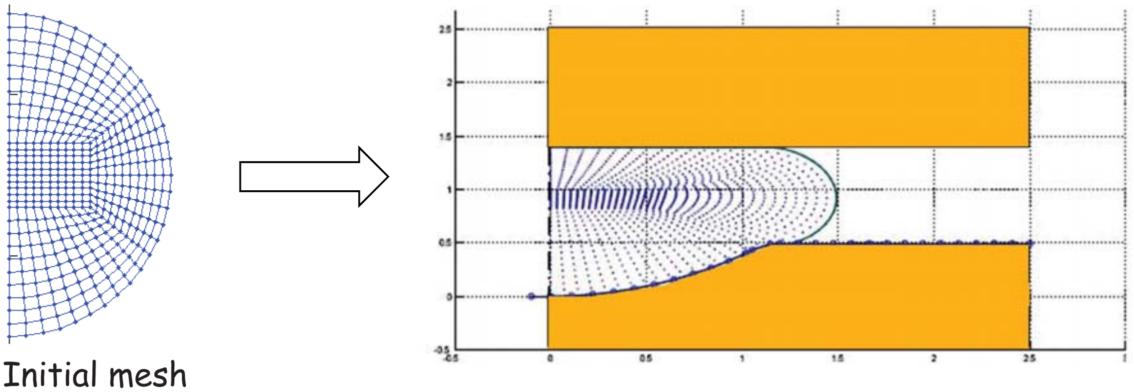
- Displacement-controlled vs. force-controlled procedure
  - Almost all linear problems are force-controlled
  - Displacement-controlled procedure is more stable for nonlinear analysis
  - Use reaction forces to calculate applied forces





When nonlinear analysis does not converge

- Mesh distortion
  - Most FE programs stop analysis when mesh is distorted too much
  - Initial good mesh may be distorted during a large deformation
  - Many FE programs provide remeshing capability, but it is still inaccurate or inconvenient
  - It is best to make mesh in such a way that the mesh quality can be maintained after deformation (need experience)



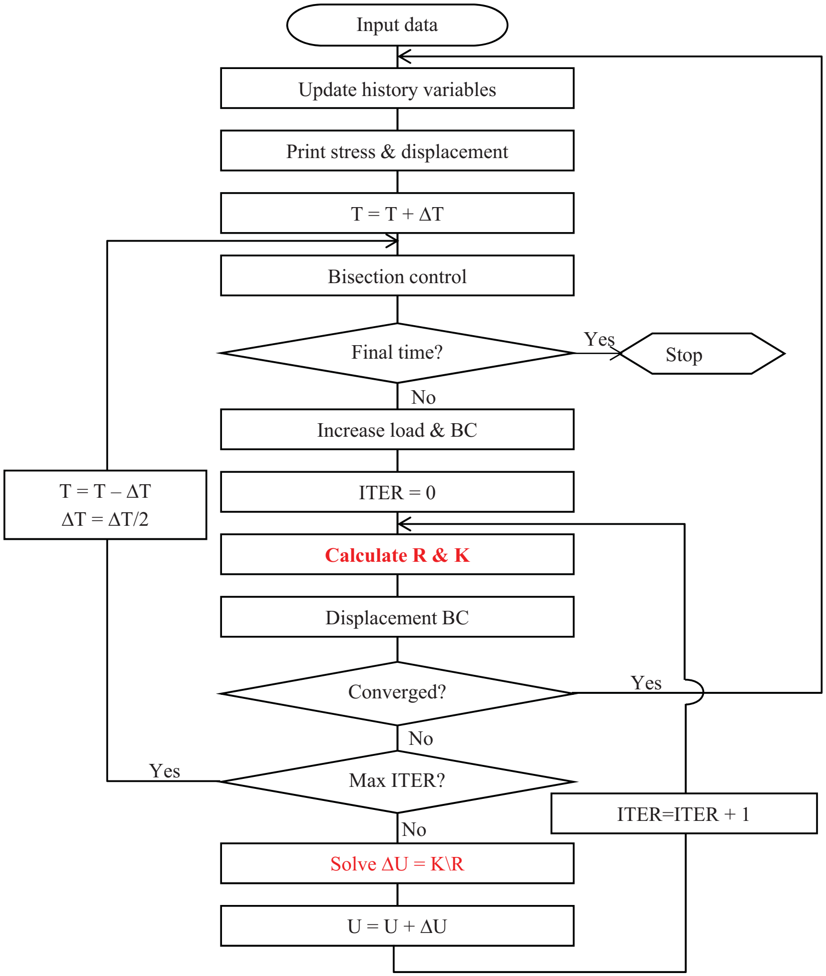
Initial mesh

MATLAB Code for Nonlinear FEA

NLFEA.m

- Nonlinear finite element analysis program
  - Incremental force method with N-R method
  - Bisection method when N-R is failed to converge
  - Can solve for linear elastic, hyperelastic and elasto-plastic material nonlinearities with large deformation
- Global arrays

Name	Dimension	Contents
GKF	NEQ x NEQ	Tangent matrix
FORCE	NEQ x 1	Residual vector
DISPTD	NEQ x 1	Displacement vector
DISPDD	NEQ x 1	Displacement increment
SIGMA	6 x 8 x NE	Stress at each integration point
XQ	7 x 8 x NE	History variable at each integration point



NLFEA.m cont.

- Nodal coordinates and element connectivity
  - the node numbers are in sequence
  - nodal coordinates in XYZ(NNODE , 3)
  - eight-node hexahedral elements LE(NELEN, 8)
- Applied forces and prescribed displacements
  - EXTFORCE(NFORCE, 3): [node, DOF, value] format
  - SDISPT(NDISPT, 3)
- Load steps and increments
  - TIMS(NTIME,5): [T_{start}, T_{end}, T_{inc}, $LOAD_{init}$, $LOAD_{final}$] format
- Material properties
  - Mooney-Rivlin hyperelasticity (MID = -1), PROP = [A10, A01, K]
  - infinitesimal elastoplasticity (MID = 1), PROP = [LAMBDA, MU, BETA, H, Y0]

NLFEA.m cont.

- Control parameters
  - ITRA: maximum number of convergence iterations
  - if residual > ATOL, then solution diverges, bisection starts
  - The total number of bisections is limited by NTOL
  - The convergence iteration converges when residual < TOL
  - Program prints out results to NOUT after convergence

```
function NLFEA(ITRA,TOL,ATOL,NTOL,TIMS,NOUT,MID,PROP,EXTFORCE,SDISPT,XYZ,LE)
%***************************************************************************
% MAIN PROGRAM FOR HYPERELASTIC/ELASTOPLASTIC ANALYSIS
%***************************************************************************
```

Extension of a Single Element Example

```
%
% Nodal coordinates
XYZ=[0 0 0;1 0 0;1 1 0;0 1 0;0 0 1;1 0 1;1 1 1;0 1 1];
%
% Element connectivity
LE=[1 2 3 4 5 6 7 8];
%
% External forces [Node, DOF, Value]
EXTFORCE=[5 3 10.0; 6 3 10.0; 7 3 10.0; 8 3 10.0];
%
% Prescribed displacements [Node, DOF, Value]
SDISPT=[1 1 0;1 2 0;1 3 0;2 2 0;2 3 0;3 3 0;4 1 0;4 3 0];
%
% Load increments [Start End Increment InitialLoad FinalLoad]
TIMS=[0.0 0.5 0.1 0.0 0.5; 0.5 1.0 0.1 0.5 1.0]';
%
% Material properties
%PROP=[LAMBDA MU BETA H Y0]
MID=1;
PROP=[110.747, 80.1938, 0.0, 5., 35.0];
%
% Set program parameters
ITRA=20; ATOL=1.0E5; NTOL=5; TOL=1E-6;
%
% Calling main function
NOUT = fopen('output.txt','w');
NLFEA(ITRA, TOL, ATOL, NTOL, TIMS, NOUT, MID, PROP, EXTFORCE, SDISPT, XYZ, LE);
fclose(NOUT);
```

Tension of Elastoplastic Bar Example

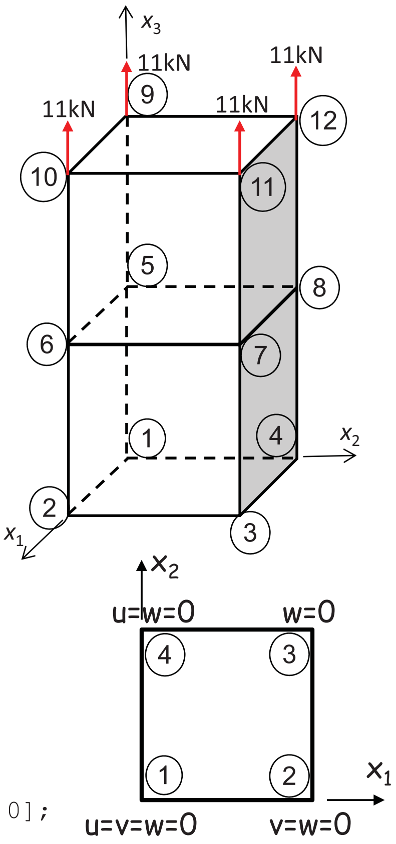
Material properties

λ	μ	σ_Y	H
110.7 GPa	80.2 GPa	400 MPa	100 MPa

Uniaxial stress condition $\sigma_{33} \neq 0$

When F_i = 10 kN, σ = 400 MPa (Elastic limit)

Elastoplastic when F_i = 10 ~ 11kN



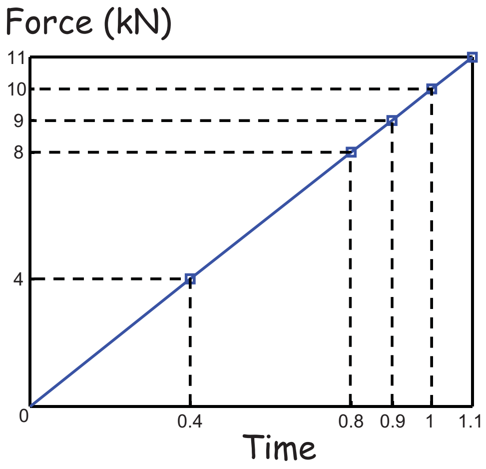
```
%
% Two-element example
%
% Nodal coordinates
XYZ=[0 0 0; 1 0 0; 1 1 0; 0 1 0;
     0 0 1; 1 0 1; 1 1 1; 0 1 1;
     0 0 2; 1 0 2; 1 1 2; 0 1 2]*0.01;
%
% Element connectivity
LE=[1 2 3 4 5  6  7  8;
    5 6 7 8 9 10 11 12];
%
% Prescribed displacements [Node, DOF, Value]
SDISPT=[1 1 0;1 2 0;1 3 0;2 2 0;2 3 0;3 3 0;4 1 0;4 3 0];
```

Tension of Elastoplastic Bar Example

```
%
% External forces [Node, DOF, Value]
EXTFORCE=[9 3 10.0E3; 10 3 10.0E3; 11 3 10.0E3; 12 3 10.0E3];
%
% Load increments [Start End Increment InitialFactor FinalFactor]
TIMS=[0.0 0.8 0.4 0.0 0.8; 0.8 1.1 0.1 0.8 1.1]';
```

10kN * 1.1 = 11kN

```
%
% Material properties PROP=[LAMDA MU BETA H Y0]
MID=1;
PROP=[110.747E9   80.1938E9   0.0    1.E8  4.0E8];
%
% Set program parameters
ITRA=70; ATOL=1.0E5; NTOL=6; TOL=1E-6;
%
% Calling main function
NOUT = fopen('output.txt','w');
NLFEA(ITRA, TOL, ATOL, NTOL, TIMS, NOUT, MID, PROP, EXTFORCE, SDISPT, XYZ, LE);
fclose(NOUT);
```



Tension of Elastoplastic Bar Example

Convergence iteration outputs (output.txt)

```
   Time   Time step   Iter       Residual
0.40000   4.000e-01      2    3.80851e-12

   Time   Time step   Iter       Residual
0.80000   4.000e-01      2    4.32010e-12

   Time   Time step   Iter       Residual
0.90000   1.000e-01      2    3.97904e-12

   Time   Time step   Iter       Residual
1.00000   1.000e-01      2    3.63798e-12

   Time   Time step   Iter       Residual
1.10000   1.000e-01      2    6.66390e+02
                         3    1.67060e-09
```

Linear elastic region

Elastoplastic region

Load factor	u_{5z}	u_{9z}	S_{33} Elem1	S_{33} Elem2	State
0.4	7.73×10^{-6}	1.55×10^{-5}	160 MPa	160 MPa	Elastic
0.8	1.55×10^{-5}	3.09×10^{-5}	320 MPa	320 MPa	Elastic
0.9	1.74×10^{-5}	3.48×10^{-5}	360 MPa	360 MPa	Elastic
1.0	1.93×10^{-5}	3.87×10^{-5}	400 MPa	400 MPa	Elastic
1.1	4.02×10^{-3}	8.04×10^{-3}	440 MPa	440 MPa	Plastic