CHAP 5

Finite Element Analysis of Contact Problem

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Introduction

- Contact is boundary nonlinearity
  - The graph of contact force versus displacement becomes vertical
  - Both displacement and contact force are unknown in the interface
- Objective of contact analysis
  1. Whether two or more bodies are in contact
  2. Where the location or region of contact is
  3. How much contact force or pressure occurs in the interface
  4. If there is a relative motion after contact in the interface
- Finite element analysis procedure for contact problem
  1. Find whether a material point in the boundary of a body is in contact with the other body
  2. If it is in contact, the corresponding contact force must be calculated

Introduction

- Equilibrium of elastic system:
  - finding a displacement field that minimizes the potential energy
- Contact condition (constrained minimization)
  - the potential is minimized while satisfying the contact constraint
- Convert to unconstrained optimization
  - Can be solved using either the penalty method or Lagrange multiplier method
- Slave-master concept for contact implementation
  - the nodes on the slave boundary cannot penetrate the surface elements on the master boundary

Goals

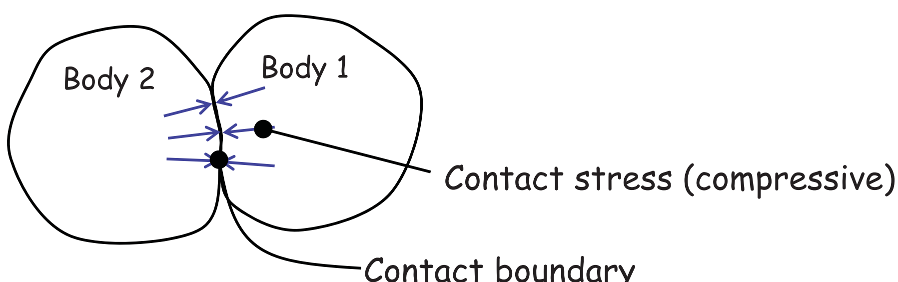
- Learn the computational difficulty in boundary nonlinearity
- Understand the concept of variational inequality and its relation with the constrained optimization
- Learn how to impose contact constraint and friction constraints using penalty method
- Understand difference between Lagrange multiplier method and penalty method
- Learn how to integrate contact constraint with the structural variational equation
- Learn how to implement the contact constraints in finite element analysis
- Understand collocational integration

5.2

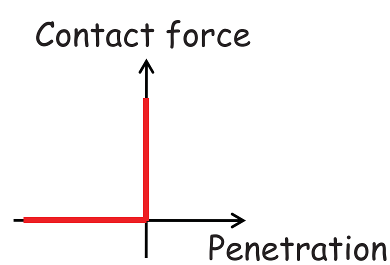
1D CONTACT EXAMPLES

Contact Problem – Boundary Nonlinearity

- Contact problem is categorized as boundary nonlinearity
- Body 1 cannot penetrate Body 2 (impenetrability)

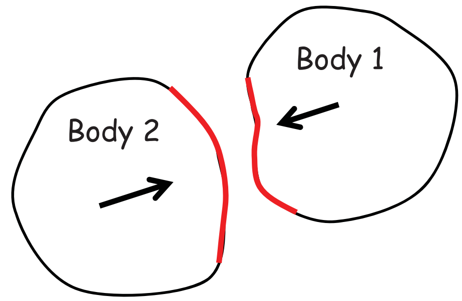


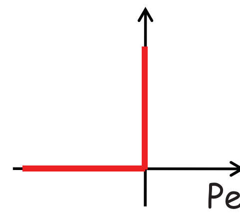
- Why nonlinear?
  - Both contact boundary and contact stress are unknown!!!
  - Abrupt change in contact force (difficult in NR iteration)



Why are contact problems difficult?

- Unknown boundary
  - Contact boundary is unknown a priori
  - It is a part of solution
  - Candidate boundary is often given
- Abrupt change in force
  - Extremely discontinuous force profile
  - When contact occurs, contact force cannot be determined from displacement
  - Similar to incompressibility (Lagrange multiplier or penalty method)
- Discrete boundary
  - In continuum, contact boundary varies smoothly
  - In numerical model, contact boundary varies node to node
  - Very sensitive to boundary discretization



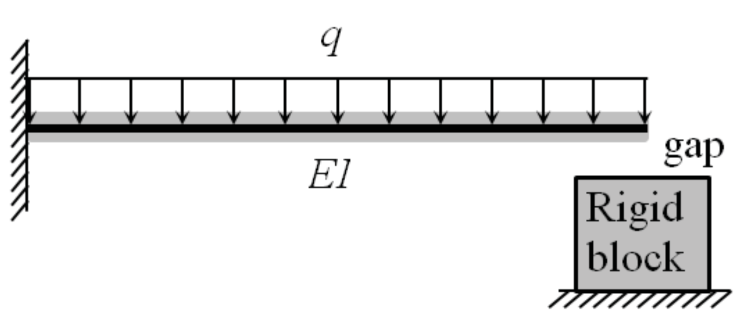


Contact of a Cantilever Beam with a Rigid Block

- q = 1 kN/m, L = 1 m, $EI = 10^5$ N·m^2, initial gap δ = 1 mm
- Trial-and-error solution
  - First assume that the deflection is smaller than the gap

$$v_N(x) = \frac{qx^2}{24EI}(x^2 + 6L^2 - 4Lx), \qquad v_N(L) = \frac{qL^4}{8EI} = 0.00125\text{m}$$

  - Since $v_N(L) > \delta$, the assumption is wrong, the beam will be in contact



Cantilever Beam Contact with a Rigid Block cont.

- Trial-and-error solution cont.
  - Now contact occurs. Contact in one-point (tip).
  - Contact force, λ, to prevent penetration

$$v_c(x) = \frac{-\lambda x^2}{6EI}(3L - x), \quad v_c(L) = \frac{-\lambda}{3 \times 10^5}$$

  - Determine the contact force from tip displacement = gap

$$v_{tip} = v_N(L) + v_c(L) = 0.00125 - \frac{\lambda}{3 \times 10^5} = 0.001 = \delta$$

$$\lambda = 75N$$

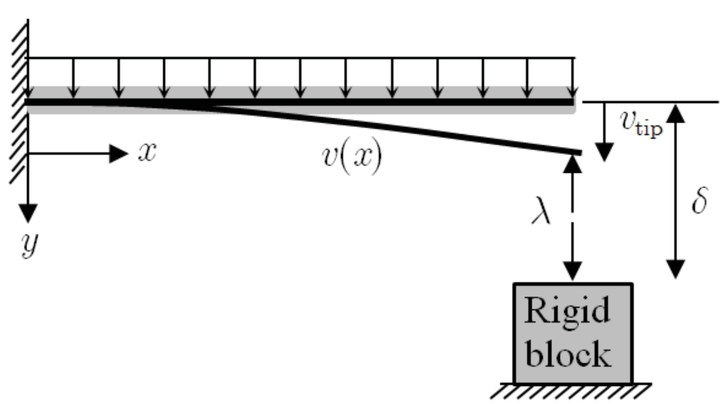
$$v(x) = v_N(x) + v_c(x)$$

Cantilever Beam Contact with a Rigid Block cont.

- Solution using contact constraint
  - Treat both contact force and gap as unknown and add constraint

$$v(x) = \frac{qx^2}{24EI}(x^2 + 6L^2 - 4Lx) - \frac{\lambda x^2}{6EI}(3L - x)$$

  - When λ = 0, no contact. Contact occurs when λ > 0. λ < 0 impossible
  - Gap condition: $\boxed{g = v_{tip} - \delta \le 0}$



Cantilever Beam Contact with a Rigid Block cont.

- Solution using contact constraint cont.
  - Contact condition

    No penetration: $g \le 0$

    Positive contact force: $\lambda \ge 0$

    Consistency condition: $\lambda g = 0$
  - Lagrange multiplier method

$$\lambda g = \lambda \left(0.00025 - \frac{\lambda}{3 \times 10^5} \right) = 0$$

  - When λ = 0N → g = 0.00025 > 0 → violate contact condition
  - When λ = 75N → g = 0 → satisfy contact condition

Lagrange multiplier, λ, is the contact force

Cantilever Beam Contact with a Rigid Block cont.

- Penalty method
  - Small penetration is allowed, and contact force is proportional to it
  - Penetration function

$$\phi_N = \frac{1}{2}\left(|g| + g\right)$$

ϕ_N = 0 when $g \le 0$
ϕ_N = g when $g > 0$

  - Contact force

$$\lambda = K_N \phi_N$$

K_N: penalty parameter

  - From $g = v_{tip} - \delta \le 0$

$$g = 0.00025 - \frac{K_N}{3 \times 10^5} \frac{1}{2}\left(|g| + g\right)$$

  - Gap depends on penalty parameter

Cantilever Beam Contact with a Rigid Block cont.

- Penalty method cont.
  - Large penalty allows small penetration

Penalty parameter	Penetration (m)	Contact force (N)
3×10^{5}	1.25×10^{-4}	37.50
3×10^{6}	2.27×10^{-5}	68.18
3×10^{7}	2.48×10^{-6}	74.26
3×10^{8}	2.50×10^{-7}	74.92
3×10^{9}	2.50×10^{-8}	75.00

Beam Contact with Friction

- Sequence: q is applied first, followed by the axial load P
- Assume no friction

P=100N, μ=0.5

$$u_{tip}^{\text{no-friction}} = \frac{PL}{EA} = 1.0\text{mm}$$

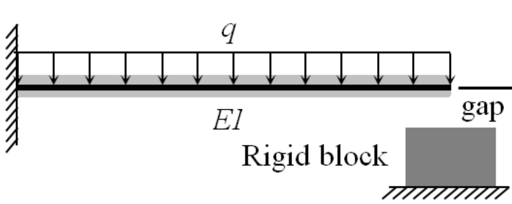
- Frictional constraint

Stick condition: $t - \mu\lambda < 0,\ u_{tip} = 0$

Slip condition: $t - \mu\lambda = 0,\ u_{tip} > 0$

Consistency condition: $u_{tip}(t - \mu\lambda) = 0$

t: tangential friction force



Beam Contact with Friction cont.

1. Trial-and-error solution
   - First assume stick condition

$$u_{tip} = \frac{PL}{EA} - \frac{tL}{EA} = 0, \quad \Rightarrow \quad t = P = 100\text{N}$$

Violate $t - \mu\lambda < 0$

   - Next, try slip condition $t = \mu\lambda = 37.5\text{N}$

$$u_{tip} = \frac{PL}{EA} - \frac{tL}{EA} = 0.625\text{mm}$$

Satisfies $u_{tip} > 0$, therefore, valid

Beam Contact with Friction cont.

2. Solution using frictional constraint (Lagrange multiplier)
   - Use consistency condition $u_{tip}(t - \mu\lambda) = 0$
   - Choose u_{tip} as a Lagrange multiplier and $t-\mu\lambda$ as a constraint

$$u_{tip}\left(P - \frac{EA}{L}u_{tip} - \mu\lambda\right) = 0$$

   - When $u_{tip} = 0$, $t = P$, and $t - \mu\lambda = 62.5 > 0$, violate the stick condition
   - When

$$u_{tip} = \frac{(P - \mu\lambda)L}{EA} = 0.625\text{mm} > 0$$

$t = \lambda\mu$ and the slip condition is satisfied, valid solution

Beam Contact with Friction cont.

3. Penalty method
   - Penalize when $t - \mu\lambda > 0$

$$\phi_T = \frac{1}{2}\left(\left|t - \mu\lambda\right| + t - \mu\lambda\right)$$

   - Slip displacement and frictional force

$$u_{tip} = K_T \phi_T$$

K_T: penalty parameter for tangential slip

   - When $t - \mu\lambda < 0$ (stick), $u_{tip} = 0$ (no penalization)
   - When $t - \mu\lambda > 0$ (slip), penalize to stay close $t = \mu\lambda$
   - Friction force

$$t = P - \frac{EA}{L} u_{tip}$$

Beam Contact with Friction cont.

- Penalty method cont.
  - Tip displacement $u_{tip} = \dfrac{K_T L(P - \mu\lambda)}{L + K_T EA}$
  - For large K_T, $u_{tip} \approx \dfrac{(P - \mu\lambda)L}{EA} \Longrightarrow t = P - \dfrac{EA}{L} u_{tip} \approx \mu\lambda$

Penalty parameter	Tip displacement (m)	Frictional force (N)
1×10^{-4}	5.68×10^{-4}	43.18
1×10^{-3}	6.19×10^{-4}	38.12
1×10^{-2}	6.24×10^{-4}	37.56
1×10^{-1}	6.25×10^{-4}	37.50
1×10^{0}	6.25×10^{-4}	37.50

Observations

- Due to unknown contact boundary, contact point should be found using either direct search (trial-and-error) or nonlinear constraint equation
  - Both methods requires iterative process to find contact boundary and contact force
  - Contact function replace the abrupt change in contact condition with a smooth but highly nonlinear function
- Friction force calculation depends on the sequence of load application (Path-dependent)
- Friction function regularizes the discontinuous friction behavior to a smooth one

5.3

GENERAL FORMULATION OF CONTACT PROBLEMS

General Contact Formulation

- Contact between a flexible body and a rigid body
- Point $\mathbf{x} \in \Omega$ contacts to $\mathbf{x}_c \in$ rigid surface (param ξ)

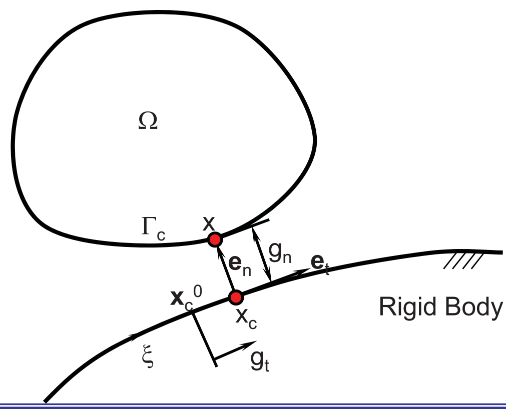
$$\boxed{\mathbf{x}_c = \mathbf{x}_c(\xi)}$$

- How to find $\mathbf{x}_c(\xi)$

$$\boxed{\varphi(\xi_c) = (\mathbf{x} - \mathbf{x}_c(\xi_c))^T \mathbf{e}_t(\xi_c) = 0} \qquad \mathbf{e}_t = \frac{\mathbf{t}}{\|\mathbf{t}\|} = \frac{\mathbf{x}_{c,\xi}}{\|\mathbf{x}_{c,\xi}\|}$$

Unit tangent vector

  - Closest projection onto the rigid surface



Contact Formulation cont.

- Gap function

$$\boxed{g_n = (\mathbf{x} - \mathbf{x}_c(\xi_c))^T \mathbf{e}_n(\xi_c)}$$

- Impenetrability condition

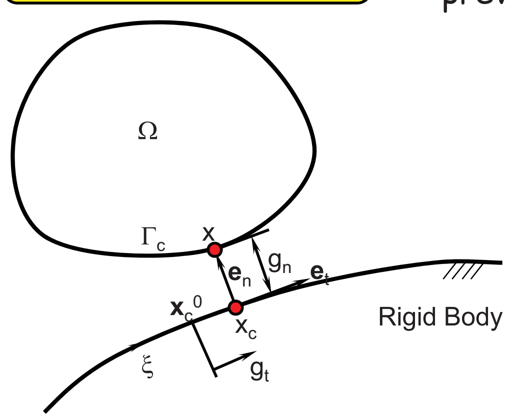
$$\boxed{g_n \geq 0, \quad \mathbf{x} \in \Gamma_c}$$

boundary that has a possibility of contact

- Tangential slip function

$$\boxed{g_t \equiv \|\mathbf{t}^0\|(\xi_c - \xi_c^0)}$$

Parameter at the previous contact point

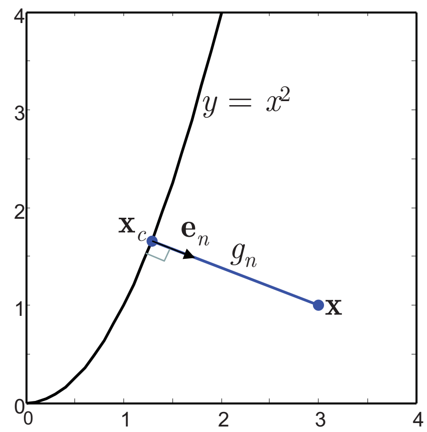


Ex) Project to a Parabola

- Projection of x={3, 1} to y = x^2
- Let $\mathbf{x}_c = \{\xi, \xi^2\}^T$

$$\mathbf{e}_t = \frac{\mathbf{x}_{c,\xi}}{\|\mathbf{x}_{c,\xi}\|} = \frac{1}{\sqrt{1+4\xi^2}}\begin{Bmatrix} 1 \\ 2\xi \end{Bmatrix}$$

$$\mathbf{e}_n = \mathbf{e}_t \times \mathbf{k} = \frac{1}{\sqrt{1+4\xi^2}}\begin{Bmatrix} 2\xi \\ -1 \end{Bmatrix}$$



- Projection point

$$\varphi(\xi) = (\mathbf{x} - \mathbf{x}_c(\xi))^T \mathbf{e}_t(\xi) = \frac{3 + \xi - 2\xi^3}{\sqrt{1+4\xi^2}} = 0 \quad \Rightarrow$$

ξ_c = 1.29

$\mathbf{x}_c$ = {1.29, 1.66}

- Gap

$$g_n = (\mathbf{x} - \mathbf{x}_c)^T \mathbf{e}_n = \frac{-\xi_c^2 + 6\xi_c - 1}{\sqrt{1+4\xi_c^2}} = 1.83$$

Variational Inequality

- Governing equation

$$\begin{cases} \nabla \cdot \sigma + \mathbf{f}^b = 0 & \mathbf{x} \in \Omega \\ \mathbf{u} = 0 & \mathbf{x} \in \Gamma^g \\ \sigma \mathbf{n} = \mathbf{f}^s & \mathbf{x} \in \Gamma^s \end{cases}$$

- Contact conditions (small deformation)

$$\left.\begin{aligned} &\mathbf{u}^T\mathbf{e}_n + g_n \ge 0 \\ &\sigma_n \ge 0 \\ &\sigma_n(\mathbf{u}^T\mathbf{e}_n + g_n) = 0 \end{aligned}\right\} \mathbf{x} \in \Gamma_c$$

- Contact set $\mathbb{K}$ (convex)

$$\boxed{\mathbb{K} = \left\{ \mathbf{w} \in [H^1(\Omega)]^N \,\middle|\, \mathbf{w}\big|_{\Gamma^g} = 0 \text{ and } \mathbf{w}^T\mathbf{e}_n + g_n \ge 0 \text{ on } \Gamma_c \right\}}$$

satisfies all kinematic constraints (displacement conditions)

Variational Inequality cont.

- Variational equation with $\bar{\mathbf{u}} = \mathbf{w} - \mathbf{u}$ (i.e., $\mathbf{w}$ is arbitrary)

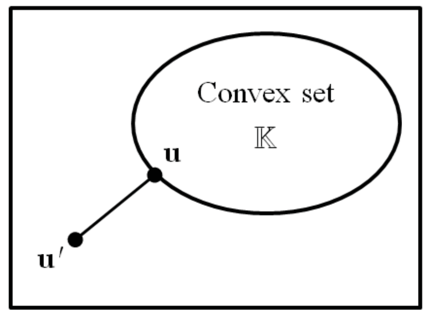
$$\iint_\Omega \sigma_{ij}\varepsilon_{ij}(\mathbf{w}-\mathbf{u})\,d\Omega = -\iint_\Omega \sigma_{ij,j}(w_i-u_i)\,d\Omega + \int_{\Gamma^s\cup\Gamma_c} \sigma_{ij}n_j(w_i-u_i)\,d\Gamma$$

$$\sigma_{ij,j} = -f_i^b\ (\mathbf{x}\in\Omega),\quad \sigma_{ij}n_j = f_i^s\ (\mathbf{x}\in\Gamma^s)$$

$$\iint_\Omega \sigma_{ij}\varepsilon_{ij}(\mathbf{w}-\mathbf{u})\,d\Omega = \ell(\mathbf{w}-\mathbf{u}) + \int_{\Gamma_c} \sigma_{ij}n_j(w_i-u_i)\,d\Gamma$$

$$\begin{aligned}\int_{\Gamma_c} \sigma_{ij}n_j(w_i-u_i)\,d\Gamma &= \int_{\Gamma_c} \sigma_n(w_n-u_n)\,d\Gamma\\ &= \int_{\Gamma_c} \sigma_n(w_n+g_n-u_n-g_n)\,d\Gamma\\ &= \int_{\Gamma_c} \sigma_n(w_n+g_n)\,d\Gamma \ge 0\end{aligned}$$

Since it is arbitrary, we don't know the value, but it is non-negative

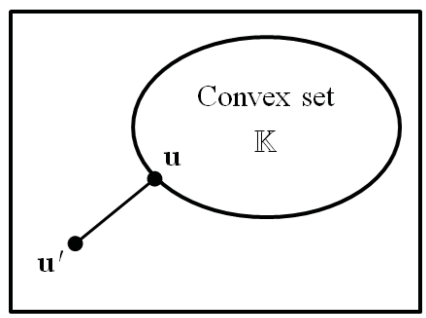


$$\boxed{\iint_\Omega \sigma_{ij}\varepsilon_{ij}(\mathbf{w}-\mathbf{u})\,d\Omega \ge \ell(\mathbf{w}-\mathbf{u}),\quad \forall \mathbf{w}\in\mathbb{K}}$$

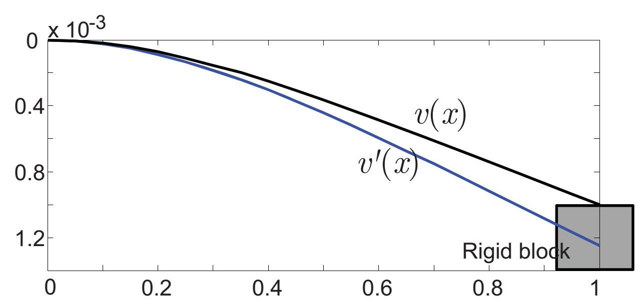
Variational inequality for contact problem

Illustration of Projection

$$\boxed{a(\mathbf{u},\mathbf{w}-\mathbf{u}) \ge \ell(\mathbf{w}-\mathbf{u}),\quad \forall \mathbf{w}\in\mathbb{K}}$$



- If the solution $\mathbf{u}'$ is out of set $\mathbb{K}$, it is projected to $\mathbf{u}$ on $\mathbb{K}$
- Beam deflection example (rigid block with initial gap of 1mm)
- $v'(x)$: beam deflection without rigid block $v' \in \mathbb{Z}$
  - Contact condition is violated (penetration to the block)
- $v(x)$: projection of $v'(x)$ onto convex set $\mathbb{K}$
  - by applying the contact force



Variational Inequality cont.

- For large deformation problem

$$\mathbb{K} = \left\{ \mathbf{w} \in [H^1(\Omega)]^N \;\middle|\; \mathbf{w}\big|_{\Gamma^g} = 0 \;\text{ and }\; (\mathbf{x} - \mathbf{x}_c(\xi_c))^T \mathbf{e}_n \geq 0 \;\text{ on }\; \Gamma_c \right\}.$$

- Variational inequality is not easy to solve directly
- We will show that V.I. is equivalent to constrained optimization of total potential energy
- The constraint will be imposed using either penalty method or Lagrange multiplier method

Potential Energy and Directional Derivative

- Potential energy

$$\Pi(\mathbf{u}) = \tfrac{1}{2} a(\mathbf{u},\mathbf{u}) - \ell(\mathbf{u})$$

- Directional derivative

$$\begin{aligned}
&\tfrac{d}{d\tau}\Big[\, \Pi(\mathbf{u} + \tau\mathbf{v}) \Big]\Big|_{\tau=0} \\
&\qquad = \tfrac{d}{d\tau}\Big[\, \tfrac{1}{2} a(\mathbf{u} + \tau\mathbf{v}, \mathbf{u} + \tau\mathbf{v}) - \ell(\mathbf{u} + \tau\mathbf{v}) \Big]\Big|_{\tau=0} \\
&\qquad = \tfrac{1}{2} a(\mathbf{u},\mathbf{v}) + \tfrac{1}{2} a(\mathbf{v},\mathbf{u}) - \ell(\mathbf{u}) \\
&\qquad = a(\mathbf{u},\mathbf{v}) - \ell(\mathbf{u})
\end{aligned}$$

- Directional derivative of potential energy

$$\left\langle D\Pi(\mathbf{u}), \mathbf{v} \right\rangle = a(\mathbf{u},\mathbf{v}) - \ell(\mathbf{v})$$

- For variational inequality

$$\boxed{a(\mathbf{u}, \mathbf{w} - \mathbf{u}) \geq \ell(\mathbf{w} - \mathbf{u}) \Leftrightarrow \left\langle D\Pi(\mathbf{u}), \mathbf{w} - \mathbf{u} \right\rangle \geq 0, \quad \forall \mathbf{w} \in \mathbb{K}}$$

Equivalence

- V.I. is equivalent to constrained optimization
  - For arbitrary $\mathbf{w} \in \mathbb{K}$

$$\begin{aligned}\Pi(\mathbf{w}) - \Pi(\mathbf{u}) &= \tfrac{1}{2}a(\mathbf{w},\mathbf{w}) - \ell(\mathbf{w}) - \tfrac{1}{2}a(\mathbf{u},\mathbf{u}) + \ell(\mathbf{u}) + a(\mathbf{u},\mathbf{w}-\mathbf{u}) - a(\mathbf{u},\mathbf{w}-\mathbf{u}) \\ &= a(\mathbf{u},\mathbf{w}-\mathbf{u}) - \ell(\mathbf{w}-\mathbf{u}) + \tfrac{1}{2}a(\mathbf{w},\mathbf{w}) - a(\mathbf{u},\mathbf{w}) + \tfrac{1}{2}a(\mathbf{u},\mathbf{u}) \\ &= \langle D\Pi(\mathbf{u}),\mathbf{w}-\mathbf{u}\rangle + \boxed{\tfrac{1}{2}a(\mathbf{w}-\mathbf{u},\mathbf{w}-\mathbf{u})} \longrightarrow \text{non-negative}\end{aligned}$$

$$\Rightarrow \quad \Pi(\mathbf{w}) \ge \Pi(\mathbf{u}) + \langle D\Pi(\mathbf{u}),\mathbf{w}-\mathbf{u}\rangle \qquad \forall \mathbf{w} \in \mathbb{K}$$

- Thus, $\Pi(\mathbf{u})$ is the smallest potential energy in $\mathbb{K}$

$$\boxed{\Pi(\mathbf{u}) = \min_{\mathbf{w}\in\mathbb{K}} \Pi(\mathbf{w}) = \min_{\mathbf{w}\in\mathbb{K}} \left[\tfrac{1}{2}a(\mathbf{w},\mathbf{w}) - \ell(\mathbf{w})\right]}$$

  - Unique solution if and only if $\Pi(\mathbf{w})$ is a convex function and set $\mathbb{K}$ is closed convex

Constrained Optimization

- PMPE minimizes the potential energy in the kinematically admissible space
- Contact problem minimizes the same potential energy in the contact constraint set $\mathbb{K}$

$$\mathbb{K} = \left\{ \mathbf{w} \in [H^1(\Omega)]^N \,\middle|\, \mathbf{w}|_{\Gamma^g} = 0 \text{ and } \underbrace{(\mathbf{x}-\mathbf{x}_c)^T \mathbf{e}_n}_{\longrightarrow\, g_n \ge 0 \text{ on } \Gamma_c} \ge 0 \text{ on } \Gamma_c \right\}$$

- The constrained optimization problem can be converted into unconstrained optimization problem using the penalty method or Lagrange multiplier method
  - If $g_n < 0$, penalize $\Pi(\mathbf{u})$ using

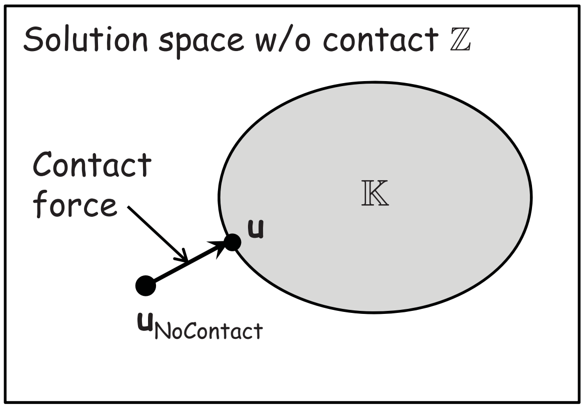
$$\boxed{P = \frac{1}{2}\underbrace{\omega_n}_{\text{penalty parameter}}\int_{\Gamma_C} g_n^2 \, d\Gamma + \frac{1}{2}\omega_t\int_{\Gamma_C} g_t^2 \, d\Gamma}$$

Constrained Optimization cont.

- Penalized unconstrained optimization problem

$$\Pi(\mathbf{u}) = \min_{\mathbf{w}\in\mathbb{K}} \Pi(\mathbf{w}) = \min_{\mathbf{w}\in\mathbb{Z}} \left[\Pi(\mathbf{w}) + P(\mathbf{w}) \right]$$

constrained unconstrained



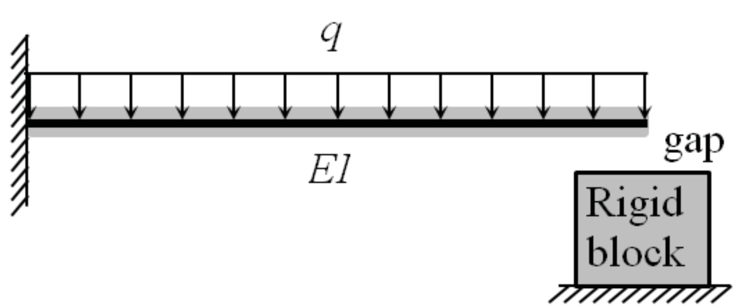
Ex) Beam Deflection with Rigid Block

- $q = 1$ kN/m, $L = 1$ m, $EI = 10^5$ N·m^2, initial gap $\delta = 1$ mm
- Assumed deflection: $v(x) = a_2x^2 + a_3x^3 + a_4x^4$
- Penalty function:

$$P = \tfrac{1}{2}\omega_n g_n^2 \qquad g_n = \delta - v_{tip} = \delta - a_2 - a_3 - a_4$$

- Penalized potential energy

$$\Pi_a = \Pi + P = \frac{1}{2}\int_0^L EI(v_{,xx})^2\,dx - \int_0^L qv\,dx + \frac{1}{2}\omega_n g_n^2$$



Ex) Beam Deflection with Rigid Block

- Stationary condition

$$\frac{\partial \Pi_a}{\partial a_i} = 0 \Longrightarrow \begin{bmatrix} 4EI + \omega_n & 6EI + \omega_n & 8EI + \omega_n \\ 6EI + \omega_n & 12EI + \omega_n & 18EI + \omega_n \\ 8EI + \omega_n & 18EI + \omega_n & \frac{144}{5}EI + \omega_n \end{bmatrix} \begin{Bmatrix} a_2 \\ a_3 \\ a_4 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{3}q + \omega_n\delta \\ \frac{1}{4}q + \omega_n\delta \\ \frac{1}{5}q + \omega_n\delta \end{Bmatrix}$$

- Penetration: $a_2 + a_3 + a_4 - \delta$
- Contact force: $-\omega_n g_n$

Penalty parameter	a_1	a_2	a_3	Penetration (m)	Contact force (N)
3×10^5	2.31×10^{-3}	-1.60×10^{-3}	4.17×10^{-4}	1.25×10^{-4}	37.50
3×10^6	2.16×10^{-3}	-1.55×10^{-3}	4.17×10^{-4}	2.27×10^{-5}	68.18
3×10^7	2.13×10^{-3}	-1.54×10^{-3}	4.17×10^{-4}	2.48×10^{-6}	74.26
3×10^8	2.13×10^{-3}	-1.54×10^{-3}	4.17×10^{-4}	2.50×10^{-7}	74.92
3×10^9	2.13×10^{-3}	-1.54×10^{-3}	4.17×10^{-4}	2.50×10^{-8}	75.00
True value	2.13×10^{-3}	-1.54×10^{-3}	4.17×10^{-4}	0.0	75.00

Variational Equation

- Structural Equilibrium

$$\delta\Pi(\mathbf{u};\bar{\mathbf{u}}) + \delta P(\mathbf{u};\bar{\mathbf{u}}) = 0 \quad \Longrightarrow \quad a(\mathbf{u},\bar{\mathbf{u}}) - \ell(\bar{\mathbf{u}}) + \delta P(\mathbf{u};\bar{\mathbf{u}}) = 0$$

$$\delta P(\mathbf{u};\bar{\mathbf{u}}) = \underbrace{\omega_n \int_{\Gamma_C} g_n \bar{g}_n \, d\Gamma}_{b_N(\mathbf{u},\bar{\mathbf{u}})} + \underbrace{\omega_t \int_{\Gamma_C} g_t \bar{g}_t \, d\Gamma}_{b_T(\mathbf{u},\bar{\mathbf{u}}),} \equiv b(\mathbf{u},\bar{\mathbf{u}})$$

need to express in terms of u and ū

- Variational equation

$$\boxed{a(\mathbf{u},\bar{\mathbf{u}}) + b(\mathbf{u},\bar{\mathbf{u}}) = \ell(\bar{\mathbf{u}}), \quad \forall \bar{\mathbf{u}} \in \mathbb{Z}}$$

Frictionless Contact Formulation

- Variation of the normal gap

$$g_n = (\mathbf{x} - \mathbf{x}_c(\xi))^T \mathbf{e}_n \quad \Longrightarrow \quad \bar{g}_n = \bar{\mathbf{u}}^T \mathbf{e}_n$$

- Normal contact form

$$\boxed{b_N(\mathbf{u}, \bar{\mathbf{u}}) = \omega \int_{\Gamma_c} g_n \bar{\mathbf{u}}^T \mathbf{e}_n \, d\Gamma}$$

Linearization

- It is clear that b_N is independent of energy form
  - The same b_N can be used for elastic or plastic problem
- It is nonlinear with respect to u (need linearization)
- Increment of gap function

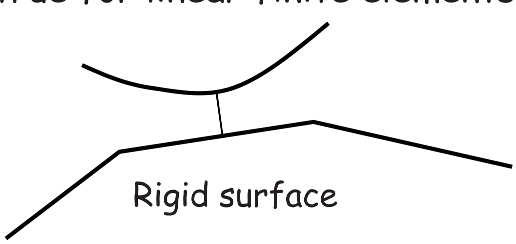
$$\Delta g_n = \Delta\left[(\mathbf{x} - \mathbf{x}_c(\xi))^T \mathbf{e}_n \right] = \Delta \mathbf{u}^T \mathbf{e}_n + (\mathbf{x} - \mathbf{x}_c(\xi))^T \Delta \mathbf{e}_n$$

$$\Delta g_n(\mathbf{u}; \Delta \mathbf{u}) = \Delta \mathbf{u}^T \mathbf{e}_n$$

$\Delta \mathbf{e}_n \parallel \mathbf{e}_t$

$\mathbf{x} - \mathbf{x}_c \parallel \mathbf{e}_n$

  - We assume that the contact boundary is straight line $\Delta \mathbf{e}_n = 0$
  - This is true for linear finite elements



Linearization cont.

- Linearization of contact form

$$b_N(\mathbf{u}, \bar{\mathbf{u}}) = \omega \int_{\Gamma_c} \langle g \rangle_{-} \bar{\mathbf{u}}^T \mathbf{e}_n \, d\Gamma$$

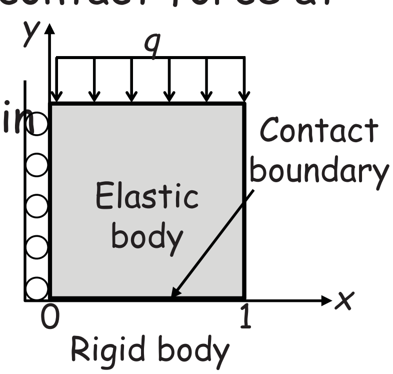
$$\boxed{b_N^*(\mathbf{u}; \Delta\mathbf{u}, \bar{\mathbf{u}}) = \omega \int_{\Gamma_c} \bar{\mathbf{u}}^T \mathbf{e}_n \mathbf{e}_n^T \Delta\mathbf{u} \, d\Gamma}$$

- N-R iteration

$$a^*(\mathbf{u}; \Delta\mathbf{u}, \bar{\mathbf{u}}) + b_N^*(\mathbf{u}; \Delta\mathbf{u}, \bar{\mathbf{u}}) = \ell(\bar{\mathbf{u}}) - a(\mathbf{u}, \bar{\mathbf{u}}) - b_N(\mathbf{u}, \bar{\mathbf{u}}), \quad \forall \bar{\mathbf{u}} \in \mathbb{Z}$$

Ex) Frictionless Contact of a Block

- Calculate displacement, penetration and contact force at the contact interface.



- $EA = 10^5$N, $\nu = 0$, $q = 1.0$kN/m, plane strain
- Plane strain: $\mathbf{u} = \{u_x, u_y\}^T \quad \bar{\mathbf{u}} = \{\bar{u}_x, \bar{u}_y\}^T$
- Contact boundary: $\mathbf{e}_n = \{0, 1\}^T$

$$\xi = x, \quad \mathbf{x}_c = \{\xi_c, 0\}^T, \quad \mathbf{x} = \{\xi_c, u_y\}^T$$

- Gap function: $g_n = (\mathbf{x} - \mathbf{x}_c)^T \mathbf{e}_n = u_y$
- Contact form: $b_N(\mathbf{u}, \bar{\mathbf{u}}) = \omega_n \int_0^1 u_y \bar{u}_y \big|_{y=0} \, dx$
- Penalized potential energy

$$\Pi + P = \frac{1}{2} \iint_A \varepsilon^T \mathbf{D} \varepsilon \, dA - \int_0^1 (-q) u_y \big|_{y=1} \, dx + \frac{1}{2} \omega_n \int_0^1 g_n^2 \big|_{y=1} \, dx$$

Ex) Frictionless Contact of a Block

- From $\nu = 0$, **D** becomes a diagonal matrix, decoupled x & y
- Since load is only y-direction, $\varepsilon_{xx} = \gamma_{xy} = 0$
- Linear displacement in y-direction

$$u_y = a_0 + a_1 y \quad \bar{u}_y = \bar{a}_0 + \bar{a}_1 y$$

- Penalized potential

$$\bar{\Pi} + \bar{P} = \iint_A E a_1 \bar{a}_1 \, dA - \int_0^1 (-q)(\bar{a}_0 + \bar{a}_1)\, dx + \omega_n \int_0^1 a_0 \bar{a}_0 \, dx = 0$$

- Need to satisfy for arbitrary $\bar{a}_0, \bar{a}_1$

$$a_0 = -\frac{q}{\omega_n}, \quad a_1 = -\frac{q}{EA}$$

$$u_y = -\frac{q}{\omega_n} - \frac{q}{EA} y, \quad 0 \le y \le 1$$

$$-\omega_n g_n = -\omega_n u_y \big|_{y=0} = q$$

As ω_n increases, penetration decreases but contact force remains constant

Frictional Contact Formulation

- frictional contact depends on load history
- Frictional interface law – regularization of Coulomb law
- Friction form

$$b_T(\mathbf{u}, \bar{\mathbf{u}}) = \omega_t \int_{\Gamma_c} g_t \bar{g}_t \, d\Gamma$$

$$\bar{g}_t = \left\| \mathbf{t}^0 \right\| \bar{\xi}_c = \nu \bar{\mathbf{u}}^T \mathbf{e}_t$$

$$\Longrightarrow \boxed{b_T(\mathbf{u}, \bar{\mathbf{u}}) = \omega_t \int_{\Gamma_c} \nu g_t \bar{\mathbf{u}}^T \mathbf{e}_t \, d\Gamma}$$

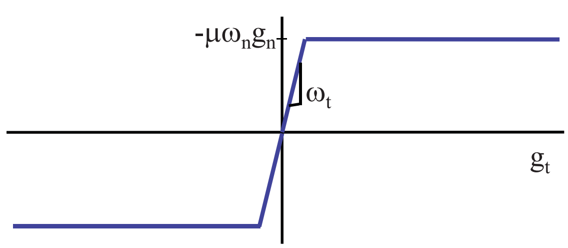
$$\alpha = \mathbf{e}_n^T \mathbf{x}_{c,\xi\xi}$$

$$\beta = \mathbf{e}_t^T \mathbf{x}_{c,\xi\xi}$$

$$\gamma = \mathbf{e}_n^T \mathbf{x}_{c,\xi\xi\xi}$$

$$c = \left\| \mathbf{t} \right\|^2 - g_n \alpha$$

$$\nu = \left\| \mathbf{t} \right\| \left\| \mathbf{t}^0 \right\| / c$$



Friction Force

- During stick condition, $f_T = \omega_t g_t$ (ω_t: regularization param.)

$$\omega_t g_t \le \left|\mu\omega_n g_n\right|$$

- When slip occurs, $\omega_t g_t = -\mu\omega_n g_n$
- Modified friction form

$$b_T(\mathbf{u},\bar{\mathbf{u}}) = \begin{cases} \omega_t \int_{\Gamma_c} \nu g_t \bar{\mathbf{u}}^T \mathbf{e}_t \, d\Gamma, & \text{if } \left|\omega_t g_t\right| \le \left|\mu\omega_n g_n\right| \\ -\mu\omega_n \,\mathrm{sgn}(g_t) \int_{\Gamma_c} \nu g_n \bar{\mathbf{u}}^T \mathbf{e}_t \, d\Gamma, & \text{otherwise.} \end{cases}$$

Linearization of Stick Form

- Increment of slip

$$\Delta g_t(\mathbf{u};\Delta\mathbf{u}) = \left\|\mathbf{t}^0\right\|\Delta\xi_c = \nu\,\mathbf{e}_t^T\Delta\mathbf{u}$$

- Increment of tangential vector

$$\Delta\mathbf{e}_t = -\mathbf{e}_3 \times \Delta\mathbf{e}_n = \frac{\alpha}{c}\mathbf{e}_n(\mathbf{e}_t^T\Delta\mathbf{u})$$

- Incremental slip form for the stick condition

$$\begin{aligned} b_T^*(\mathbf{u};\Delta\mathbf{u},\bar{\mathbf{u}}) &= \omega_t \int_{\Gamma_c} \nu^2 \bar{\mathbf{u}}^T \mathbf{e}_t \mathbf{e}_t^T \Delta\mathbf{u}\, d\Gamma \\ &+\omega_t \int_{\Gamma_c} \frac{\alpha\nu g_t}{c} \bar{\mathbf{u}}^T (\mathbf{e}_n\mathbf{e}_t^T + \mathbf{e}_t\mathbf{e}_n^T)\Delta\mathbf{u}\, d\Gamma \\ &+\omega_t \int_{\Gamma_c} \frac{\nu g_t}{c^2}\left((\gamma\left\|\mathbf{t}\right\| - 2\alpha\beta)g_n - \beta\left\|\mathbf{t}\right\|^2\right) \bar{\mathbf{u}}^T \mathbf{e}_t \mathbf{e}_t^T \Delta\mathbf{u}\, d\Gamma \end{aligned}$$

Linearization of Slip Form

- Parameter for slip condition: $\omega_t = -\mu\omega_n \,\mathrm{sgn}(g_t)$.
- Friction form: $b_T(\mathbf{u},\bar{\mathbf{u}}) = \omega_t \int_{\Gamma_c} \nu\, g_n \bar{\mathbf{u}}^T \mathbf{e}_t \, d\Gamma$
- Linearized slip form (not symmetric!!)

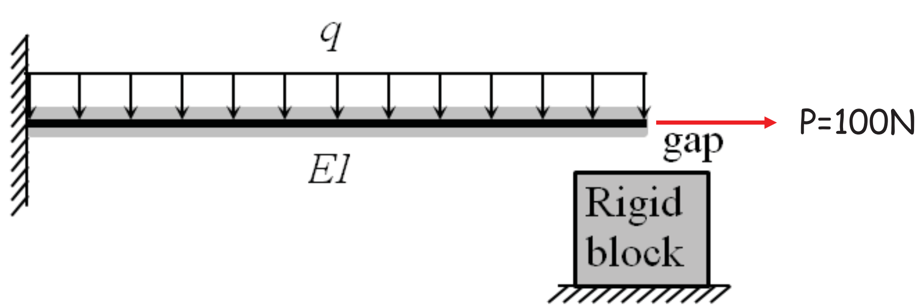
$$
\begin{aligned}
b_T^*(\mathbf{u};\Delta\mathbf{u},\bar{\mathbf{u}}) = {} & \omega_t \int_{\Gamma_c} \nu \bar{\mathbf{u}}^T \mathbf{e}_t \mathbf{e}_n^T \Delta\mathbf{u} \, d\Gamma \\
& + \omega_t \int_{\Gamma_c} \frac{\alpha\nu g_n}{c} \bar{\mathbf{u}}^T (\mathbf{e}_n \mathbf{e}_t^T + \mathbf{e}_t \mathbf{e}_n^T) \Delta\mathbf{u} \, d\Gamma \\
& + \omega_t \int_{\Gamma_c} \frac{\nu g_n}{c^2} \left((\gamma \|\mathbf{t}\| - 2\alpha\beta) g_n - \beta \|\mathbf{t}\|^2 \right) \bar{\mathbf{u}}^T \mathbf{e}_t \mathbf{e}_t^T \Delta\mathbf{u} \, d\Gamma
\end{aligned}
$$

Ex) Frictional Slip of a Cantilever Beam

- Distributed load q → axial load P, ω_t = 106, μ = 0.5
- Contact force: $F_c = -\omega_n g_n$ = 75N
- Penalized potential energy (axial alone)

$$\Pi_a = \int_0^L EA(u_{,x})^2 \, dx - Pu(L) + \frac{1}{2}\omega_t g_t^2 \Big|_{x=L}$$

$$\bar{\Pi}_a = \int_0^L EAu_{,x}\bar{u}_{,x} \, dx - P\bar{u}(L) + \omega_t g_t \bar{g}_t \Big|_{x=L} = 0, \quad \forall \bar{u} \in \mathbb{Z}$$



Ex) Frictional Slip of a Cantilever Beam

- Linear axial displacement field: $u(x) = a_0 + a_1 x$

$$u(0) = 0 \quad \Rightarrow \quad u(x) = a_1 x \quad u_{,x} = a_1 \quad \bar{u}_{,x} = \bar{a}_1$$

- Tangential slip in terms of displacement
  - Parametric coordinate ξ has an origin at $x = L$, and it has the same length as the x-coordinate

$$\left\| \mathbf{x}_{c,\xi} \right\| = \left\| \mathbf{t} \right\| = \left\| \mathbf{t}^0 \right\| = 1 \qquad \xi_c^0 = 0 \qquad g_t = \xi_c = u(L) = a_1$$

- Assume the stick condition: $\omega_t g_t \le \left| \mu \omega_n g_n \right|$

$$\bar{a}_1 (EAa_1 + \omega_t a_1 - P) = 0, \quad \forall \bar{a}_1 \in \mathbb{R}$$

$$\Rightarrow u(x) = \frac{Px}{EA + \omega_t} = 9.09 \times 10^{-5} x$$

$$\omega_t g_t = 90.9 > 37.5 = \left| \mu \omega_n g_n \right|$$

The assumption is violated!!

Ex) Frictional Slip of a Cantilever Beam

- Assume the slip condition:

$$\bar{\Pi}_a = \int_0^L EAu_{,x}\bar{u}_{,x}\, dx - P\bar{u}(L) - \mu\omega_n \operatorname{sgn}(g_t) g_n \bar{g}_t \Big|_{x=L} = 0, \quad \forall \bar{u} \in \mathbb{Z}$$

  - With contact force = 70N, $-\mu\omega_n g_n = 37.5\text{N}$

$$\Rightarrow \bar{a}_1 (EAa_1 - P + 37.5) = 0, \quad \forall \bar{a}_1 \in \mathbb{R}$$

$$\Rightarrow a_1 = 62.5 \times 10^{-5}$$

$$\Rightarrow u_{tip} = 0.625\text{mm}$$

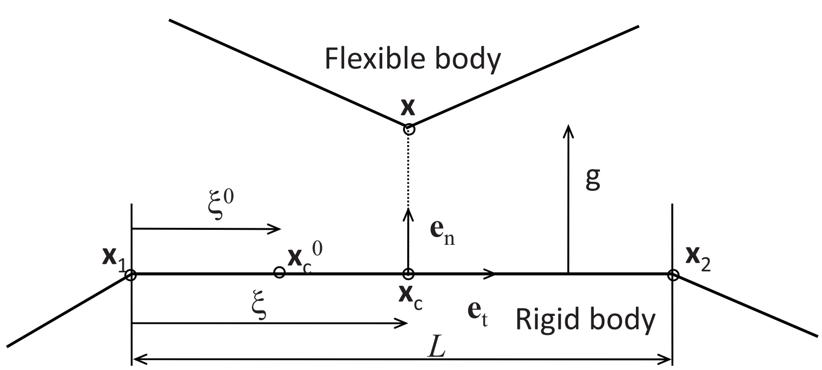
5.4

FINITE ELEMENT FORMULATION OF CONTACT PROBLEMS

Finite Element Formulation

- Slave-Master contact
  - The rigid body has fixed or prescribed displacement
  - Point **x** is projected onto the piecewise linear segments of the rigid body with $\mathbf{x}_c$ ($\xi = \xi_c$) as the projected point
  - Unit normal and tangent vectors

$$\mathbf{e}_t = \frac{\mathbf{x}_2 - \mathbf{x}_1}{L}, \qquad \mathbf{e}_n = \mathbf{e}_3 \times \mathbf{e}_t$$



Finite Element Formulation cont.

- Parameter at contact point

$$\xi_c = \frac{1}{L}(\mathbf{x} - \mathbf{x}_1)^T \mathbf{e}_t$$

- Gap function

$$g_n = (\mathbf{x} - \mathbf{x}_1)^T \mathbf{e}_n \geq 0 \qquad \text{Impenetrability condition}$$

- Penalty function

$$P = \frac{1}{2}\sum_{I=1}^{NC}\left(\omega\langle g\rangle_-^2\right)_I$$

  - Note: we don't actually integrate the penalty function. We simply added the integrand at all contact node
  - This is called **collocation** (a kind of integration)
  - In collocation, the integrand is function value × weight

Finite Element Formulation cont.

- Contact form (normal)

$$b_N(\mathbf{u}, \bar{\mathbf{u}}) = \sum_{I=1}^{NC}\left(\omega\langle g_n\rangle_- \bar{\mathbf{d}}^T \mathbf{e}_n\right)_I = \{\bar{\mathbf{d}}\}^T\{\mathbf{f}^c\}$$

  - (ωg_n): **contact force**, proportional to the violation
  - Contact form is a virtual work done by contact force through normal virtual displacement

- Linearization

Heaviside step function
H(x) = 1 if x > 0
= 0 otherwise

$$\Delta\langle g_n\rangle_- = H(-g_n)\mathbf{e}_n^T \Delta\mathbf{u} = H(-g_n)\mathbf{e}_n^T \Delta\mathbf{d}$$

$$b_N^*(\mathbf{u}; \Delta\mathbf{u}, \bar{\mathbf{u}}) = \sum_{I=1}^{NC}\left(\omega H(-g_n)\bar{\mathbf{d}}^T \mathbf{e}_n \mathbf{e}_n^T \Delta\mathbf{d}\right)_I = \{\bar{\mathbf{d}}\}^T \sum_{I=1}^{NC}\left(\omega H(-g_n)\mathbf{e}_n \mathbf{e}_n^T\right)_I \{\Delta\mathbf{d}\}$$

$$= \{\bar{\mathbf{d}}\}^T [\mathbf{K}_c]\{\Delta\mathbf{d}\}$$

Contact stiffness

Finite Element Formulation cont.

- Global finite element matrix equation for increment

$$\boxed{\{\bar{\mathbf{d}}\}^T[\mathbf{K}_T + \mathbf{K}_c]\{\Delta\mathbf{d}\} = \{\bar{\mathbf{d}}\}^T\{\mathbf{f}^{ext} - \mathbf{f}^{int} - \mathbf{f}^c\}}$$

  - Since the contact forms are independent of constitutive relation, the above equation can be applied for different materials
- A similar approach can be used for flexible-flexible body contact
  - One body being selected as a slave and the other as a master
- Computational challenge in finding contact points (for example, out of 10,000 possible master segments, how can we find the one that will actually in contact?)

Finite Element Formulation cont.

- Frictional slip

$$g_t = l^0(\xi_c - \xi_c^0)$$

- Friction force and tangent stiffness (stick condition)

$$\mathbf{f}_t^c = -\omega_t g_t \mathbf{e}_t$$

$$\mathbf{k}_t^c = \omega_t \mathbf{e}_t \mathbf{e}_t^T$$

- Friction force and tangent stiffness (slip condition)

$$\mathbf{f}_t^c = \mu\omega_n \,\mathrm{sgn}(g_t) g_n \mathbf{e}_t, \qquad \text{if } |\omega_t g_t| \geq |\mu\omega_n g_n|$$

$$\mathbf{k}_t^c = \mu\omega_n \,\mathrm{sgn}(g_t) \mathbf{e}_t \mathbf{e}_n^T$$

5.6

CONTACT ANALYSIS PROCEDURE AND MODELING ISSUES

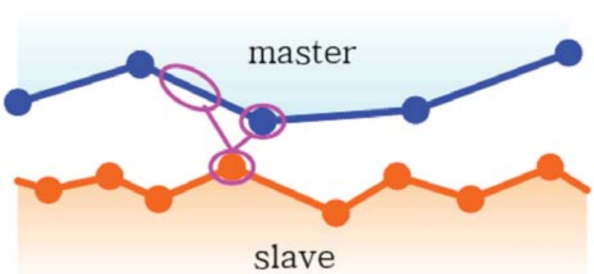
Types of Contact Interface

- Weld contact
  - A slave node is bonded to the master segment (no relative motion)
  - Conceptually same with rigid-link or MPC
  - For contact purpose, it allows a slight elastic deformation
  - Decompose forces in normal and tangential directions
- Rough contact
  - Similar to weld, but the contact can be separated
- Stick contact
  - The relative motion is within an elastic deformation
  - Tangent stiffness is symmetric,
- Slip contact
  - The relative motion is governed by Coulomb friction model
  - Tangent stiffness become unsymmetric

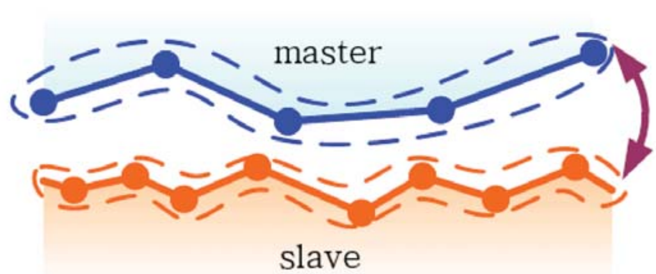
Contact Search

- Easiest case
  - User can specify which slave node will contact with which master segment
  - This is only possible when deformation is small and no relative motion exists in the contact surface
  - Slave and master nodes are often located at the same position and connected by a compression-only spring (node-to-node contact)
  - Works for very limited cases
- General case
  - User does not know which slave node will contact with which master segment
  - But, user can specify candidates
  - Then, the contact algorithm searches for contacting master segment for each slave node
  - Time consuming process, because this needs to be done at every iteration

Contact Search cont.



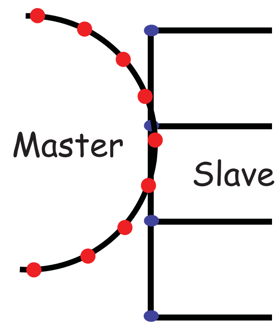
Node-to-surface contact search



Surface-to-surface contact search

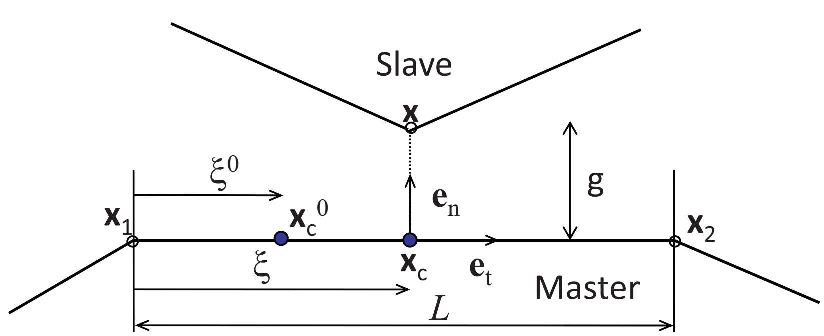
Slave-Master Contact

- Theoretically, there is no need to distinguish Body 1 from Body 2
- However, the distinction is often made for numerical convenience
- One body is called a slave body, while the other body is called a master body
- Contact condition: the slave body cannot penetrate into the master body
- The master body can penetrate into the slave body (physically not possible, but numerically it's not checked)



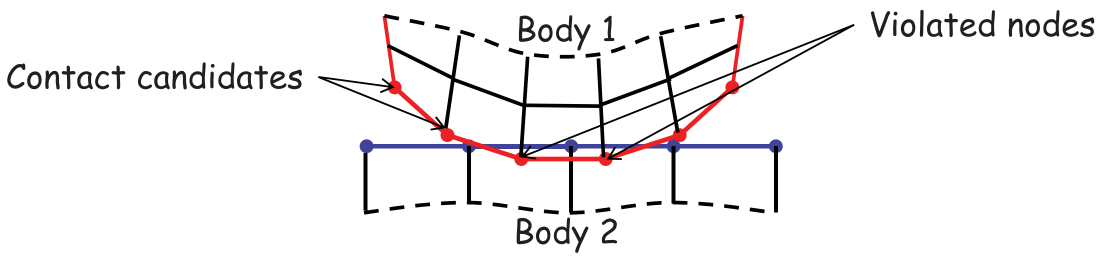
Slave-Master Contact cont.

- Contact condition between a slave node and a master segment
- In 2D, contact pair is often given in terms of $\{\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2\}$
- Slave node $\mathbf{x}$ is projected onto the piecewise linear segments of the master segment with $\mathbf{x}_c$ ($\xi = \xi_c$) as the projected point
- Gap: $\boxed{g = (\mathbf{x} - \mathbf{x}_1) \cdot \mathbf{e}_n \geq 0}$
- $g > 0$: no contact
- $g < 0$: contact

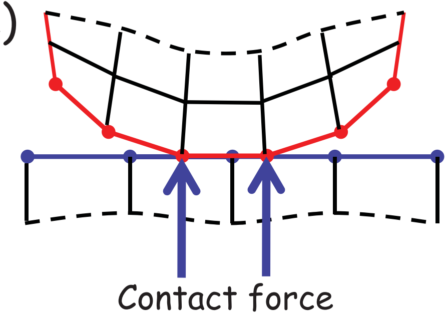


Contact Formulation (Two-Step Procedure)

1. Search nodes/segments that violate contact constraint

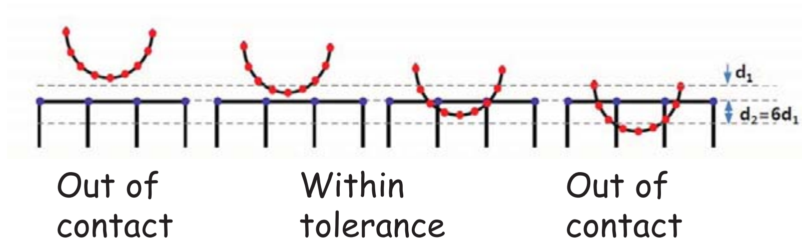


2. Apply contact force for the violated nodes/segments (contact force)

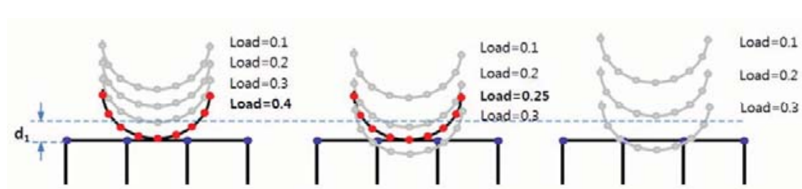


Contact Tolerance and Load Increment

- Contact tolerance
  - Minimum distance to search for contact (1% of element length)



- Load increment and contact detection
  - Too large load increment may miss contact detection

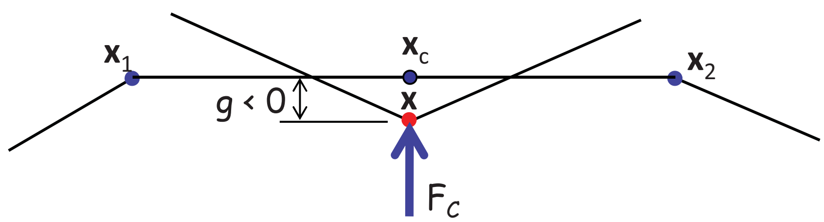


Contact Force

- For those contacting pairs, penetration needs to be corrected by applying a force (contact force)
- More penetration needs more force
- Penalty-based contact force (compression-only spring)

$$\boxed{F_C = K_n \langle -g \rangle}$$

- Penalty parameter (Kn): Contact stiffness
  - It allows a small penetration ($g < 0$)
  - It depends on material stiffness
  - The bigger Kn, the less allowed penetration



Contact Stiffness

- Contact stiffness depends on the material stiffness of contacting two bodies
- Large contact stiffness reduces penetration, but can cause problem in convergence
- Proper contact stiffness can be determined from allowed penetration (need experience)
- Normally expressed as a scalar multiple of material's elastic modulus

$$K_n = SF \cdot E \qquad SF \approx 1.0$$

- Start with small initial SF and increase it gradually until reasonable penetration

Lagrange Multiplier Method

- In penalty method, the contact force is calculated from penetration
  - Contact force is a function of deformation
- Lagrange multiplier method can impose contact condition exactly
  - Contact force is a Lagrange multiplier to impose impenetrability condition
  - Contact force is an independent variable
- Complimentary condition

$$\boxed{F_C\langle -g\rangle = 0} \qquad \begin{array}{lll} \langle -g\rangle = 0 & F_C > 0 & \text{contact} \\ \langle -g\rangle > 0 & F_C = 0 & \text{no contact} \end{array}$$

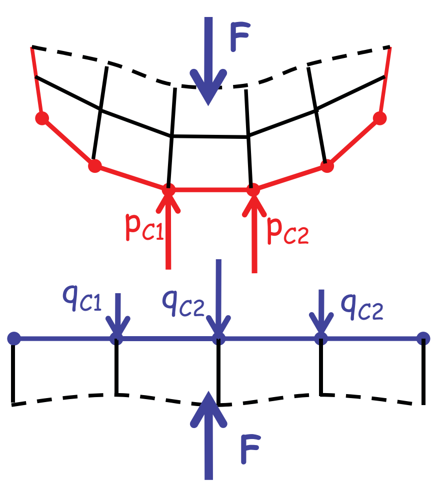
- Stiffness matrix is positive semi-definite

$$\begin{bmatrix} K & A \\ A^T & 0 \end{bmatrix} \begin{Bmatrix} d \\ F_C \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix}$$

- Contact force is applied in the normal direction to the master segment

Observations

- Contact force is an internal force at the interface
  - Newton's 3rd law: equal and opposite forces act on interface

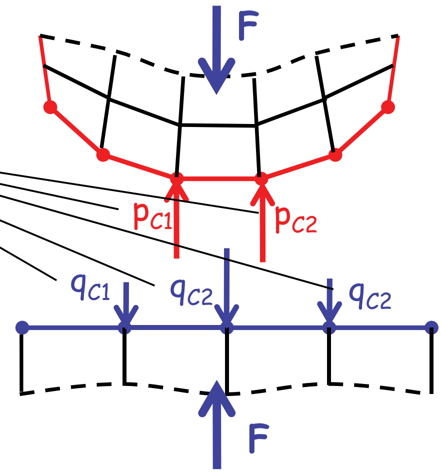


$$\mathbf{F} = \sum_{i=1}^{N_p} p_{ci} = \sum_{i=1}^{N_q} q_{ci}$$

- Due to discretization, force distribution can be different, but the resultants should be the same

Contact Formulation

- Add contact force as an external force
- Ex) Linear elastic materials



$$[K]\{d\} = \{F\} + \boxed{\{F_C(d)\}}$$

Internal force	External force	Contact force

- Contact force depends on displacement (nonlinear, unknown)

$$[K + K_C]\{\Delta d\} = \{F\} - \{F_C(d)\} - [K]\{d\}$$

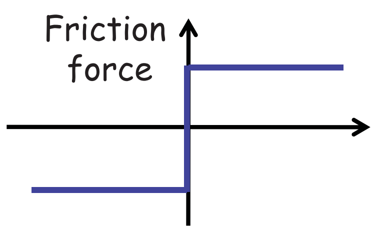
Tangent stiffness — Residual

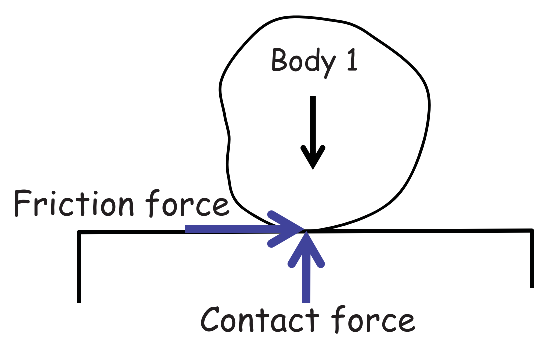
$$[K_C] = \left[\frac{\partial F_C}{\partial d}\right] \quad \text{Contact stiffness}$$

Friction Force

- So far, contact force is applied to the normal direction
  - It is independent of load history (potential problem)
- Friction force is produced by a relative motion in the interface
  - Friction force is applied to the parallel direction
  - It depends on load history (path dependent)
- Coulomb friction model

$$\begin{aligned} F_f &< \mu F_C \quad \text{Stick} \\ &= \mu F_C \quad \text{Slip} \end{aligned}$$



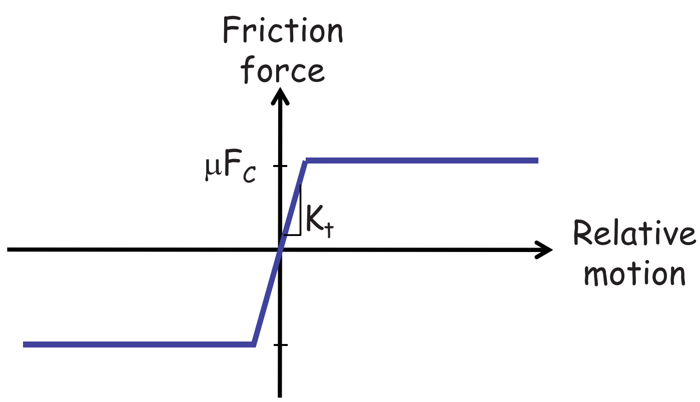


Friction Force cont.

- Coulomb friction force is indeterminate when two bodies are stick (no unique determination of friction force)
- In reality, there is a small elastic deformation before slip
- Regularized friction model
  - Similar to elasto-perfectly-plastic model

$$F_f = K_t s \quad \text{Stick}$$
$$= \mu F_C \quad \text{Slip}$$

K_t: tangential stiffness



Tangential Stiffness

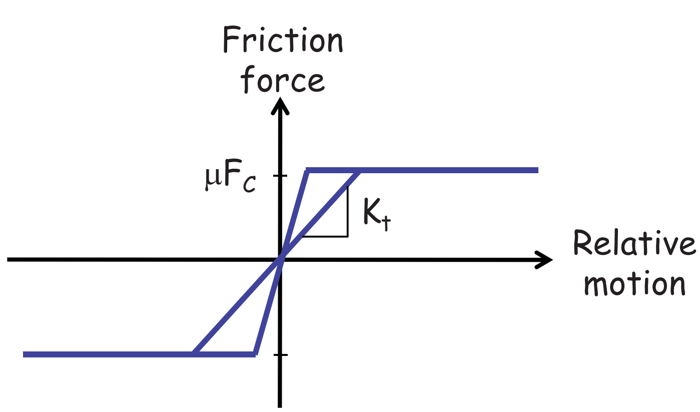
- Tangential stiffness determines the stick case

$$F_f = K_t s \quad \text{Stick}$$
$$= \mu F_C \quad \text{Slip}$$

- It is related to shear strength of the material

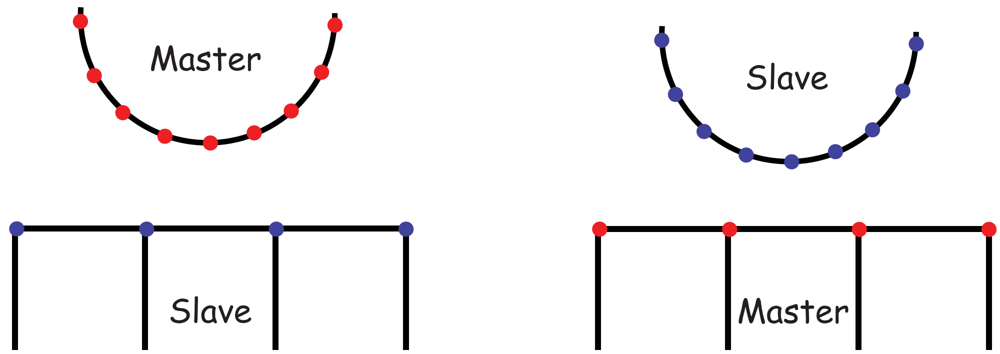
$$K_t = SF \cdot E \qquad SF \approx 0.5$$

- Contact surface with a large Kt behaves like a rigid body
- Small Kt elongate elastic stick condition too much (inaccurate)



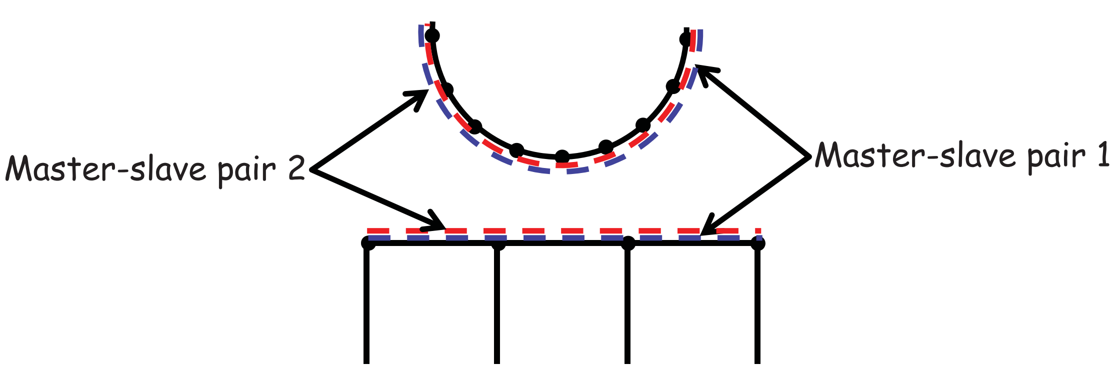
Selection of Master and Slave

- Contact constraint
  - A slave node CANNOT penetrate the master segment
  - A master node CAN penetrate the slave segment
  - There is not much difference in a fine mesh, but the results can be quite different in a coarse mesh
- How to choose master and slave
  - Rigid surface to a master
  - Convex surface to a slave
  - Fine mesh to a slave



Selection of Master and Slave

- How to prevent penetration?
  - Can define master-slave pair twice by changing the role
  - Some surface-to-surface contact algorithms use this
  - Careful in defining master-slave pairs



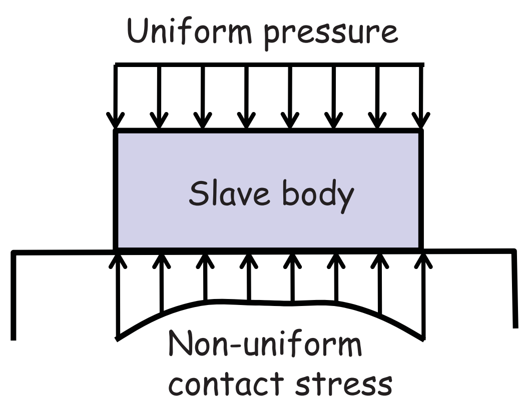
Flexible or Rigid Bodies?

- Flexible-Flexible Contact
  - Body1 and Body2 have a similar stiffness and both can deform
- Flexible-Rigid Contact
  - Stiffness of Body2 is significantly larger than that of Body1
  - Body2 can be assumed to be a rigid body (no deformation but can move)
  - Rubber contacting to steel
- Why not flexible-flexible?
  - When two bodies have a large difference in stiffness, the matrix becomes ill-conditioned
  - Enough to model contacting surface only for Body2
- Friction in the interface
  - Friction makes the contact analysis path-dependent
  - Careful in the order of load application

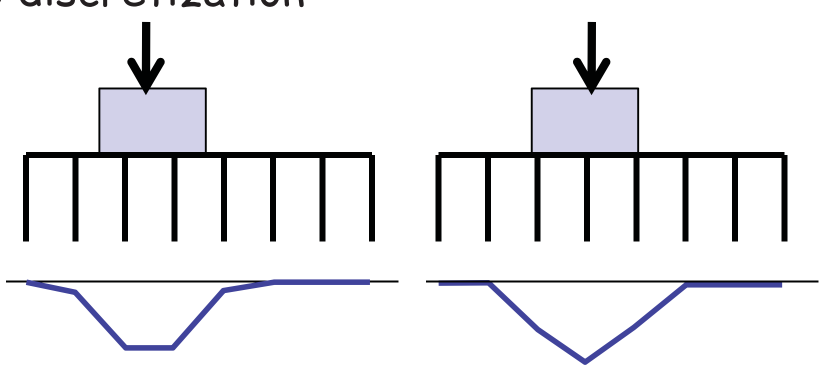
$$\begin{bmatrix} 10^7 & & & \\ & 10^7 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

Effect of discretization

- Contact stress (pressure) is high on the edge

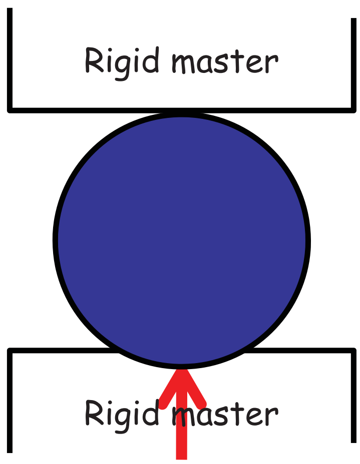


- Contact stress is sensitive to discretization



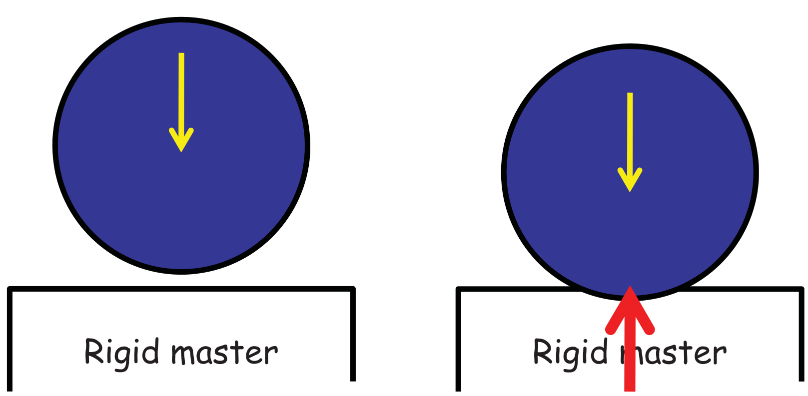
Rigid-Body Motion

- Contact constraint is different from displacement BC
  - Contact force is proportional to the penetration amount
  - A slave body between two rigid-bodies can either fly out or oscillate between two surfaces
  - Always better to remove rigid-body motion without contact



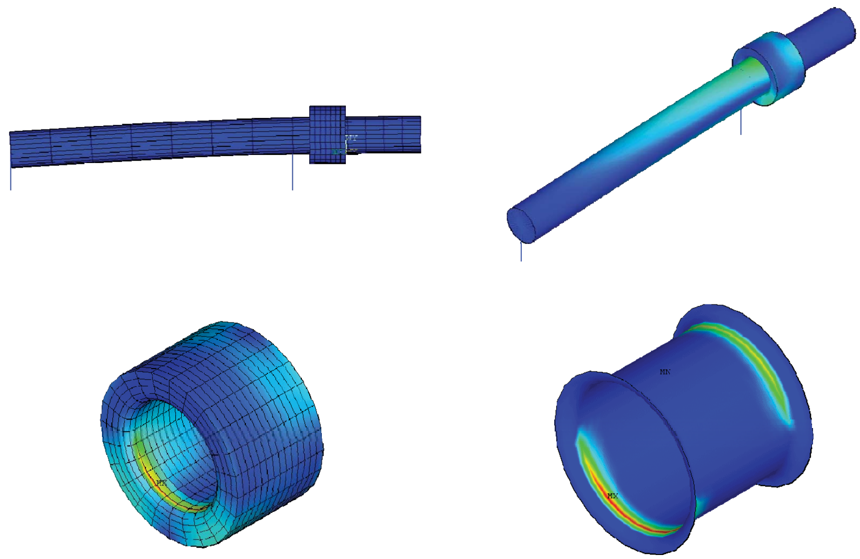
Rigid-Body Motion

- When a body has rigid-body motion, an initial gap can cause singular matrix (infinite/very large displacements)
- Same is true for initial overlap



Rigid-Body Motion

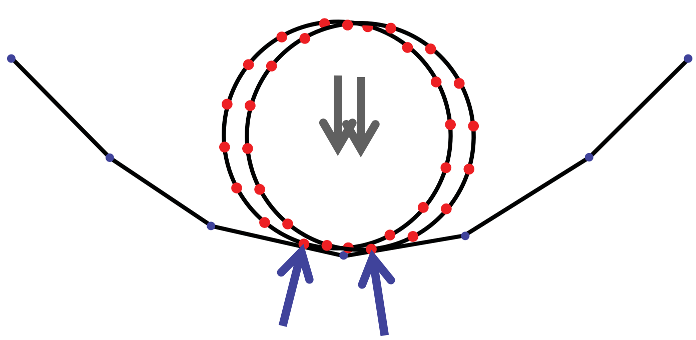
- Removing rigid-body motion
  - A small, artificial bar elements can be used to remove rigid-body motion without affecting analysis results much



Contact stress at bushing due to shaft bending

Convergence Difficulty at a Corner

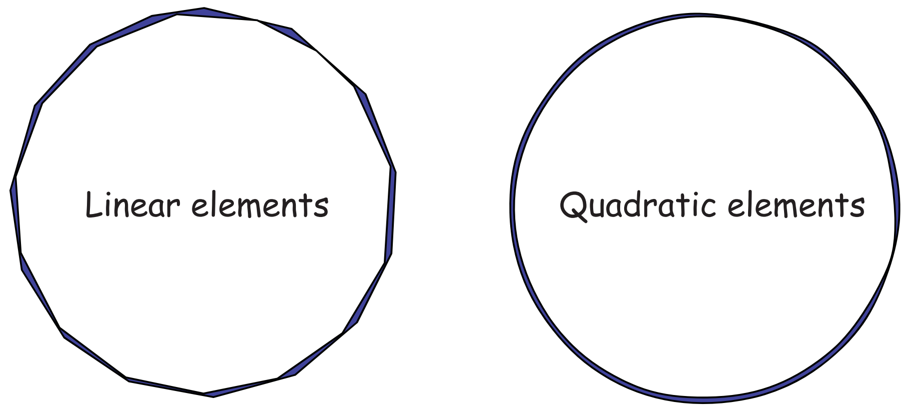
- Convergence iteration is stable when a variable (force) varies smoothly
- The slope of finite elements are discontinuous along the curved surface
- This can cause oscillation in residual force (not converging)



- Need to make the corner smooth using either higher-order elements or many linear elements
  - About 10 elements in 90 degrees, or use higher-order elements

Curved Contact Surface

- Curved surface
  - Contact pressure is VERY sensitive to the curvature
  - Linear elements often yield unsmooth contact pressure distribution
  - Less quadratic elements is better than many linear elements



Summary

- Contact condition is a rough boundary nonlinearity due to discontinuous contact force and unknown contact region
  - Both force and displacement on the contact boundary are unknown
  - Contact search is necessary at each iteration
- Penalty method or Lagrange multiplier method can be used to represent the contact constraint
  - Penalty method allows a small penetration, but easy to implement
  - Lagrange multiplier method can impose contact condition accurately, but requires additional variables and the matrix become positive semi-definite
- Numerically, slave-master concept is used along with collocation integration (at slave nodes)
- Friction makes the contact problem path-dependent
- Discrete boundary and rigid-body motion makes the contact problem difficult to solve