BAR & TRUSS FINITE ELEMENT

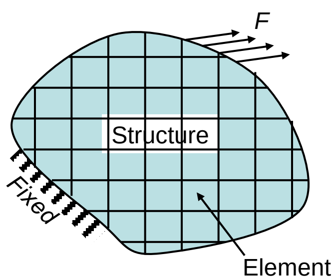
Direct Stiffness Method

FINITE ELEMENT ANALYSIS AND
APPLICATIONS

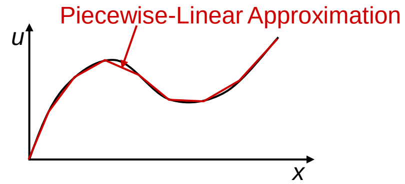
1

INTRODUCTION TO FINITE ELEMENT METHOD

- What is the finite element method (FEM)?
  - A technique for obtaining approximate solutions of differential equations.
  - Partition of the domain into a set of simple shapes (element)
  - Approximate the solution using piecewise polynomials within the element



$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + b_x = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + b_y = 0 \end{cases}$$



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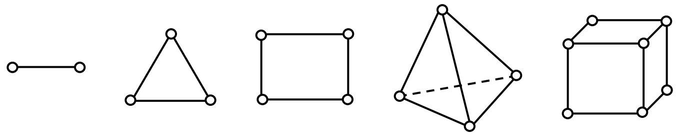
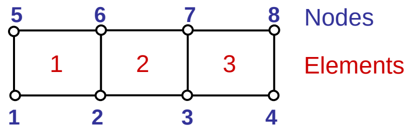
INTRODUCTION TO FEM *cont.*

- How to discretize the domain?
  - Using simple shapes (element)



  - All elements are connected using “nodes”.



  - Solution at Element 1 is described using the values at Nodes 1, 2, 6, and 5 (Interpolation).
  - Elements 1 and 2 share the solution at Nodes 2 and 6.

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INTRODUCTION TO FEM *cont.*

- Methods
  - Direct method: Easy to understand, limited to 1D problems
  - Variational method
  - Weighted residual method
- Objectives
  - Determine displacements, forces, and supporting reactions
  - Will consider only static problem

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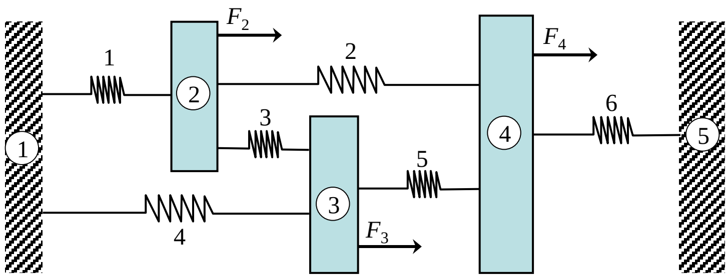
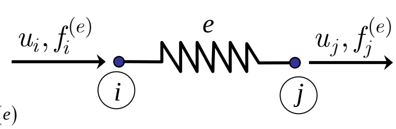
1-D SYSTEM OF SPRINGS



- Bodies move only in horizontal direction
- External forces, F_2, F_3, and F_4, are applied
- No need to discretize the system (it is already discretized!)
- Rigid body (including walls) ➡ **NODE**
- Spring ➡ **ELEMENT**

SPRING ELEMENT

- Element *e*
  - Consist of Nodes *i* and *j*
  - Spring constant $k^{(e)}$
  - Force applied to the nodes: $f_i^{(e)}, f_j^{(e)}$
  - Displacement u_i and u_j
  - Elongation: $\Delta^{(e)} = u_j - u_i$
  - Force in the spring: $P^{(e)} = k^{(e)}\Delta^{(e)} = k^{(e)}\left(u_j - u_i\right)$
  - Relation b/w spring force and nodal forces: $f_j^{(e)} = P^{(e)}$
  - Equilibrium: $f_i^{(e)} + f_j^{(e)} = 0 \quad or \quad f_i^{(e)} = -f_j^{(e)}$



SPRING ELEMENT *cont*.

- Spring Element e
  - Relation between nodal forces and displacements

$$f_i^{(e)} = k^{(e)}\left(u_i - u_j\right)$$
$$f_j^{(e)} = k^{(e)}\left(-u_i + u_j\right)$$

$$\begin{bmatrix} k^{(e)} & -k^{(e)} \\ -k^{(e)} & k^{(e)} \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} f_i^{(e)} \\ f_j^{(e)} \end{Bmatrix}$$

$$\left[\mathbf{k}^{(e)}\right] \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} f_i^{(e)} \\ f_j^{(e)} \end{Bmatrix}$$

  - Matrix notation:

$$[\mathbf{k}^{(e)}]\left\{\mathbf{q}^{(e)}\right\} = \left\{\mathbf{f}^{(e)}\right\}$$

$$\mathbf{k} \cdot \mathbf{q} = \mathbf{f}$$

  - **k**: stiffness matrix
  - **q**: vector of DOFs
  - **f**: vector of element forces

SPRING ELEMENT *cont*.

- Stiffness matrix
  - It is square as it relates to the same number of forces as the displacements.
  - It is symmetric.
  - It is singular, i.e., determinant is equal to zero and it cannot be inverted.
  - It is positive semi-definite

$$\begin{bmatrix} k^{(e)} & -k^{(e)} \\ -k^{(e)} & k^{(e)} \end{bmatrix}$$

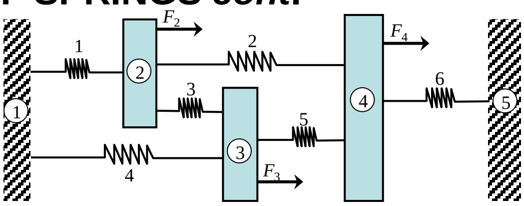
- Observation
  - For given nodal displacements, nodal forces can be calculated by

$$[\mathbf{k}^{(e)}]\left\{\mathbf{q}^{(e)}\right\} = \left\{\mathbf{f}^{(e)}\right\}$$

  - For given nodal forces, nodal displacements cannot be determined uniquely

SYSTEM OF SPRINGS *cont.*

- Element equation and assembly



$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix} \Rightarrow \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} f_2^{(2)} \\ f_4^{(2)} \end{Bmatrix} \Rightarrow \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 + k_2 & 0 & -k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -k_2 & 0 & k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_2^{(2)} \\ 0 \\ f_4^{(2)} \\ 0 \end{Bmatrix}$$

SYSTEM OF SPRINGS *cont.*

$$\begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_2^{(3)} \\ f_3^{(3)} \end{Bmatrix} \Rightarrow \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 + k_2 + k_3 & -k_3 & -k_2 & 0 \\ 0 & -k_3 & k_3 & 0 & 0 \\ 0 & -k_2 & 0 & k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_2^{(2)} + f_2^{(3)} \\ f_3^{(3)} \\ f_4^{(2)} \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} k_4 & -k_4 \\ -k_4 & k_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1^{(4)} \\ f_3^{(4)} \end{Bmatrix} \Rightarrow \begin{bmatrix} k_1 + k_4 & -k_1 & -k_4 & 0 & 0 \\ -k_1 & k_1 + k_2 + k_3 & -k_3 & -k_2 & 0 \\ -k_4 & -k_3 & k_3 + k_4 & 0 & 0 \\ 0 & -k_2 & 0 & k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} + f_1^{(4)} \\ f_2^{(1)} + f_2^{(2)} + f_2^{(3)} \\ f_3^{(3)} + f_3^{(4)} \\ f_4^{(2)} \\ 0 \end{Bmatrix}$$

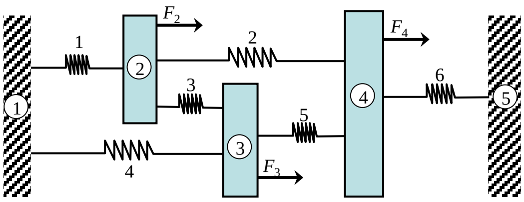
$$\begin{bmatrix} k_5 & -k_5 \\ -k_5 & k_5 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} f_3^{(5)} \\ f_4^{(5)} \end{Bmatrix}$$

$$\Rightarrow \begin{bmatrix} k_1 + k_4 & -k_1 & -k_4 & 0 & 0 \\ -k_1 & k_1 + k_2 + k_3 & -k_3 & -k_2 & 0 \\ -k_4 & -k_3 & k_3 + k_4 + k_5 & -k_5 & 0 \\ 0 & -k_2 & -k_5 & k_2 + k_5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} + f_1^{(4)} \\ f_2^{(1)} + f_2^{(2)} + f_2^{(3)} \\ f_3^{(3)} + f_3^{(4)} + f_3^{(5)} \\ f_4^{(2)} + f_4^{(5)} \\ 0 \end{Bmatrix}$$

SYSTEM OF SPRINGS *cont.*

$$\begin{bmatrix} k_6 & -k_6 \\ -k_6 & k_6 \end{bmatrix} \begin{Bmatrix} u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} f_4^{(6)} \\ f_5^{(6)} \end{Bmatrix} \Longrightarrow$$

$$\begin{bmatrix} k_1+k_4 & -k_1 & -k_4 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_3 & -k_2 & 0 \\ -k_4 & -k_3 & k_3+k_5+k_4 & -k_5 & 0 \\ 0 & -k_2 & -k_5 & k_2+k_5+k_6 & -k_6 \\ 0 & 0 & 0 & -k_6 & +k_6 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} + f_1^{(4)} \\ f_2^{(1)} + f_2^{(2)} + f_2^{(3)} \\ f_3^{(3)} + f_3^{(4)} + f_3^{(5)} \\ f_4^{(2)} + f_4^{(5)} + f_4^{(6)} \\ f_5^{(6)} \end{Bmatrix}$$



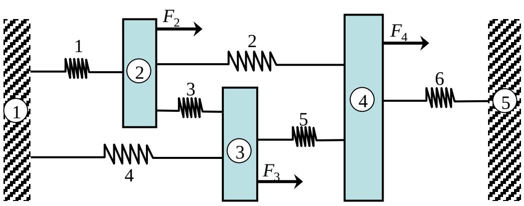
SYSTEM OF SPRINGS *cont.*

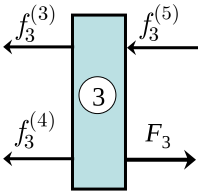
- Relation b/w element forces and external force
- Force equilibrium

$$F_i - \sum_{e=1}^{i_e} f_i^{(e)} = 0$$

$$F_i = \sum_{e=1}^{i_e} f_i^{(e)}, \quad i = 1,\ldots ND$$

- At each node, the summation of **element forces** is equal to the **applied, external force**





$$\begin{Bmatrix} f_1^{(1)} + f_1^{(4)} \\ f_2^{(1)} + f_2^{(2)} + f_2^{(3)} \\ f_3^{(3)} + f_3^{(4)} + f_3^{(5)} \\ f_4^{(2)} + f_4^{(5)} + f_4^{(6)} \\ f_5^{(6)} \end{Bmatrix} = \begin{Bmatrix} R_1 \\ F_2 \\ F_3 \\ F_4 \\ R_5 \end{Bmatrix}$$

SYSTEM OF SPRINGS *cont*.

- Assembled System of Matrix Equation:

$$\begin{bmatrix} k_1+k_4 & -k_1 & -k_4 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_3 & -k_2 & 0 \\ -k_4 & -k_3 & k_3+k_5+k_4 & -k_5 & 0 \\ 0 & -k_2 & -k_5 & k_2+k_5+k_6 & -k_6 \\ 0 & 0 & 0 & -k_6 & +k_6 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ F_2 \\ F_3 \\ F_4 \\ R_5 \end{Bmatrix}$$

$$[\mathbf{K}_s]\{\mathbf{Q}_s\} = \{\mathbf{F}_s\}$$

- $[\mathbf{K}_s]$ is square, symmetric, singular and positive semi-definite.
- When displacement is known, force is unknown

$u_1 = u_5 = 0 \implies$ R_1 and R_5 are unknown reaction forces

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SYSTEM OF SPRINGS *cont*.

- Imposing Boundary Conditions
  - Ignore the equations for which the RHS forces are unknown and strike out the corresponding rows in $[\mathbf{K}_s]$.
  - Eliminate the columns in $[\mathbf{K}_s]$ that multiply into zero values of displacements of the boundary nodes.

$$\begin{bmatrix} \cancel{k_1+k_4} & \cancel{-k_1} & \cancel{-k_4} & \cancel{0} & \cancel{0} \\ \cancel{-k_1} & k_1+k_2+k_3 & -k_3 & -k_2 & \cancel{0} \\ \cancel{-k_4} & -k_3 & k_3+k_5+k_4 & -k_5 & \cancel{0} \\ \cancel{0} & -k_2 & -k_5 & k_2+k_5+k_6 & \cancel{-k_6} \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{-k_6} & \cancel{+k_6} \end{bmatrix} \begin{Bmatrix} \cancel{u_1} \\ u_2 \\ u_3 \\ u_4 \\ \cancel{u_5} \end{Bmatrix} = \begin{Bmatrix} \cancel{R_1} \\ F_2 \\ F_3 \\ F_4 \\ \cancel{R_5} \end{Bmatrix}$$

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SYSTEM OF SPRINGS *cont*.

- Global Matrix Equation

$$\begin{bmatrix} k_1+k_2+k_3 & -k_3 & -k_2 \\ -k_3 & k_3+k_4+k_5 & -k_5 \\ -k_2 & -k_5 & k_2+k_5+k_6 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

$$[\mathbf{K}]\{\mathbf{Q}\} = \{\mathbf{F}\}$$

- Global Stiffness Matrix [**K**]
  - square, symmetric and positive definite and hence non-singular
- Solution

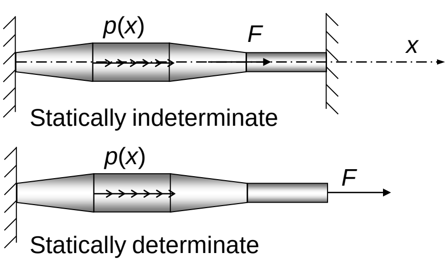
$$\{\mathbf{Q}\} = [\mathbf{K}]^{-1}\{\mathbf{F}\}$$

- Once nodal displacements are obtained, spring forces can be calculated from

$$P^{(e)} = k^{(e)}\Delta^{(e)} = k^{(e)}\left(u_j - u_i\right)$$

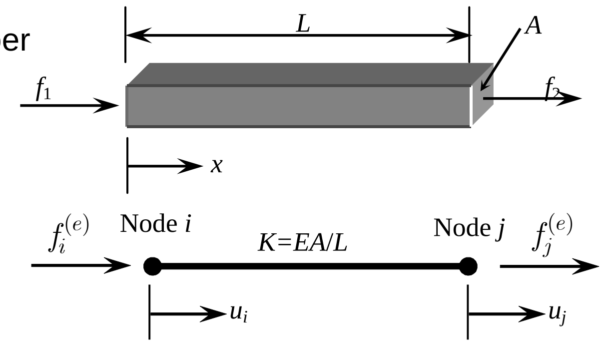
UNIAXIAL BAR

- For general uniaxial bar, we need to divide the bar into a set of elements and nodes
- Elements are connected by sharing a node
- Forces are applied at the nodes (distributed load must be converted to the equivalent nodal forces)
- Assemble all elements in the same way with the system of springs
- Solve the matrix equation for nodal displacements
- Calculate stress and strain using nodal displacements



1D BAR ELEMENT

- Two-force member
- Only constant cross-section
- Element force is proportional to relative displ
- First node: *i* second code: *j*
- Force-displacement relation



$$f_i^{(e)} = \left(\frac{AE}{L}\right)^{(e)} (u_i - u_j)$$

$$f_j^{(e)} = -f_i^{(e)} = \left(\frac{AE}{L}\right)^{(e)} (u_j - u_i)$$

Similar to the spring element

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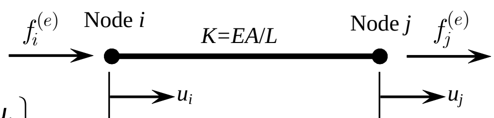
1D BAR ELEMENT *cont.*

– Matrix notation



$$\begin{Bmatrix} f_i^{(e)} \\ f_j^{(e)} \end{Bmatrix} = \left(\frac{AE}{L}\right)^{(e)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

$$\{\mathbf{f}^{(e)}\} = [\mathbf{k}^{(e)}]\{\mathbf{q}^{(e)}\}$$

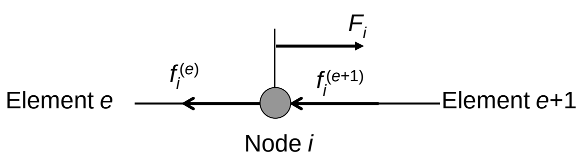
– Either force or displacement (not both) must be given at each node.
– Example: $u_i = 0$ and $f_j = 100$ *N*.
– What happens when f_i and f_j are given?

- Nodal equilibrium
  – Equilibrium of forces acting on Node I

$$F_i - f_i^{(e)} - f_i^{(e+1)} = 0 \quad \Longrightarrow \quad f_i^{(e)} + f_i^{(e+1)} = F_i$$

  – In general

$$F_i = \sum_{e=1}^{i_e} f_i^{(e)}$$



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1D BAR ELEMENT *cont*.

- Assembly
  - Similar process as spring elements
  - Replace all internal nodal forces with **External Applied Nodal Force**
  - Obtain system of equations

$$\boxed{[\mathbf{K}_s]\{\mathbf{Q}_s\} = \{\mathbf{F}_s\}}$$

[**K**$_S$]: Structural stiffness matrix
{**Q**$_S$} Vector of nodal DOFs
{**F**$_S$}: Vector of applied forces

- Property of [**K**$_S$]
  - Square, symmetric, positive semi-definite, singular, non-negative diagonal terms
- Applying boundary conditions
  - Remove rigid-body motion be fixing DOFs
  - Striking-the-nodes and striking-the-columns (Refer to sprint elements)

$$\boxed{[\mathbf{K}]\{\mathbf{Q}\} = \{\mathbf{F}\}}$$

[**K**]: Global stiffness matrix
{**Q**} Vector of unknown nodal DOFs
{**F**}: Vector of known applied forces

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1D BAR ELEMENT *cont*.

- Applying boundary conditions cont.
  - [**K**] is square, symmetric, positive definite, non-singular, invertible, and positive diagonal terms
  - Can obtain unique {**Q**}
- Element forces
  - After solving nodal displacements, the element force can be calculated

$$P^{(e)} = \left(\frac{AE}{L}\right)^{(e)} \left(u_j - u_i\right) = f_j^{(e)}$$

$$\begin{Bmatrix} -P_i^{(e)} \\ +P_j^{(e)} \end{Bmatrix} = \left(\frac{AE}{L}\right)^{(e)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

  - Element stress $\sigma = \dfrac{P^{(e)}}{A^{(e)}}$

Note $P_i = P_j$

- Reaction Forces
  - Use [**K**$_S$]{**Q**$_S$} = {**F**$_S$}: the rows that have been deleted (strike-the-rows)
  - Or, use

$$F_i = \sum_{e=1}^{i_e} f_i^{(e)}$$

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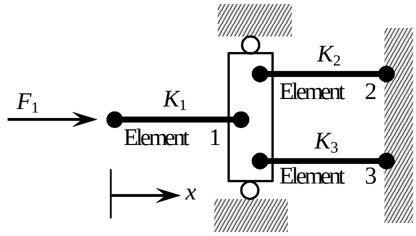
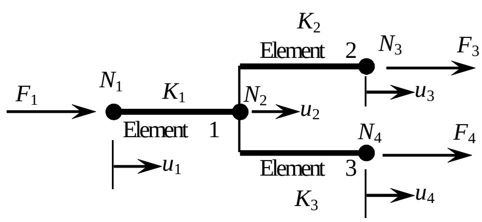
EXAMPLE

- 3 elements and 4 nodes
- At node 2:

$$F_2 = f_2^{(1)} + f_2^{(2)} + f_2^{(3)}$$





- Equation for each element:

$$\begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix} = \begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix} = \begin{bmatrix} K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} f_2^{(3)} \\ f_4^{(3)} \end{Bmatrix} = \begin{bmatrix} K_3 & -K_3 \\ -K_3 & K_3 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_4 \end{Bmatrix}$$

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EXAMPLE *cont.*

- How can we combine different element equations? (Assembly)
  - First, prepare global matrix equation:

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Displacement vector

Stiffness matrix

Applied force vector

  - Write the equation of element 1 in the corresponding location

$$\begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} K_1 & -K_1 & 0 & 0 \\ -K_1 & K_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

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EXAMPLE *cont.*

– Write the equation of element 2:

$$\begin{Bmatrix} 0 \\ f_2^{(2)} \\ f_3^{(2)} \\ 0 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & K_2 & -K_2 & 0 \\ 0 & -K_2 & K_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

– Combine two equations of elements 1 and 2

$$\begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_2^{(2)} \\ f_3^{(2)} \\ 0 \end{Bmatrix} = \begin{bmatrix} K_1 & -K_1 & 0 & 0 \\ -K_1 & K_1 + K_2 & -K_2 & 0 \\ 0 & -K_2 & K_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

EXAMPLE *cont.*

– Write the equation of element 3

$$\begin{Bmatrix} 0 \\ f_2^{(3)} \\ 0 \\ f_4^{(3)} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & K_3 & 0 & -K_3 \\ 0 & 0 & 0 & 0 \\ 0 & -K_3 & 0 & K_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

– Combine with other two elements

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_2^{(2)} + f_2^{(3)} \\ f_3^{(2)} \\ f_4^{(3)} \end{Bmatrix} = \underbrace{\begin{bmatrix} K_1 & -K_1 & 0 & 0 \\ -K_1 & (K_1 + K_2 + K_3) & -K_2 & -K_3 \\ 0 & -K_2 & K_2 & 0 \\ 0 & -K_3 & 0 & K_3 \end{bmatrix}}_{\text{Structural Stiffness Matrix}} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

EXAMPLE *cont.*

- Substitute boundary conditions and solve for the unknown displacements.
  - Let K_1 = 50 N/cm, K_2 = 30 N/cm, K_3 = 70 N/cm and f_1 = 40 N.

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{bmatrix} 50 & -50 & 0 & 0 \\ -50 & (50+30+70) & -30 & -70 \\ 0 & -30 & 30 & 0 \\ 0 & -70 & 0 & 70 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

  - Knowns: F_1, F_2, u_3, and u_4
  - Unknowns: F_3, F_4, u_1, and u_2

$$\begin{Bmatrix} 40 \\ 0 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{bmatrix} 50 & -50 & 0 & 0 \\ -50 & (50+30+70) & -30 & -70 \\ 0 & -30 & 30 & 0 \\ 0 & -70 & 0 & 70 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ 0 \\ 0 \end{Bmatrix}$$

EXAMPLE *cont.*

  - Remove zero-displacement columns: u_3 and u_4.

$$\begin{Bmatrix} 40 \\ 0 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{bmatrix} 50 & -50 \\ -50 & 150 \\ 0 & -30 \\ 0 & -70 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

  - Remove unknown force rows: F_3 and F_4.

$$\begin{Bmatrix} 40 \\ 0 \end{Bmatrix} = \begin{bmatrix} 50 & -50 \\ -50 & 150 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

  - Now, the matrix should not be singular. Solve for u_1 and u_2.

$$u_1 = 1.2\ \text{cm}$$
$$u_2 = 0.4\ \text{cm}$$

  - Using u_1 and u_2, Solve for F_3 and F_4.

$$F_3 = 0u_1 - 30u_2 = -12\ \text{N}$$
$$F_4 = 0u_1 - 70u_2 = -28\ \text{N}$$

EXAMPLE *cont.*

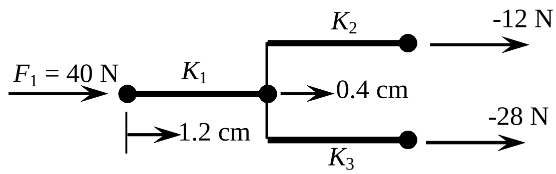
- Recover element data

Element force

$$\begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix} = \begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 50 & -50 \\ -50 & 50 \end{bmatrix} \begin{Bmatrix} 1.2 \\ 0.4 \end{Bmatrix} = \begin{Bmatrix} 40 \\ -40 \end{Bmatrix}$$

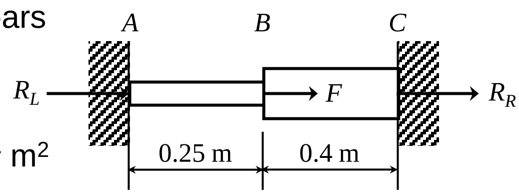
$$\begin{Bmatrix} f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix} = \begin{bmatrix} K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 30 & -30 \\ -30 & 30 \end{bmatrix} \begin{Bmatrix} 0.4 \\ 0.0 \end{Bmatrix} = \begin{Bmatrix} 12 \\ -12 \end{Bmatrix}$$

$$\begin{Bmatrix} f_2^{(3)} \\ f_4^{(3)} \end{Bmatrix} = \begin{bmatrix} K_3 & -K_3 \\ -K_3 & K_3 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_4 \end{Bmatrix} = \begin{bmatrix} 70 & -70 \\ -70 & 70 \end{bmatrix} \begin{Bmatrix} 0.4 \\ 0.0 \end{Bmatrix} = \begin{Bmatrix} 28 \\ -28 \end{Bmatrix}$$



EXAMPLE

- Statically indeterminate bars
- E = 100 GPa
- F = 10,000 N
- $A_1 = 10^{-4}$ m^2, $A_2 = 2\times10^{-4}$ m^2
- Element stiffness matrices:



$$[\mathbf{k}^{(1)}] = \frac{10^{11}\times10^{-4}}{0.25}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^7 \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \end{matrix}$$

$$[\mathbf{k}^{(2)}] = \frac{10^{11}\times2\times10^{-4}}{0.4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^7 \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} \begin{matrix} u_2 \\ u_3 \end{matrix}$$

- Assembly

$$10^7 \begin{bmatrix} 4 & -4 & 0 \\ -4 & 9 & -5 \\ 0 & -5 & 5 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 10{,}000 \\ F_3 \end{Bmatrix}$$

EXAMPLE *cont.*

- Applying BC

$$10^7[9]\{u_2\}=\{10,000\}\Rightarrow u_2=1.11\times10^{-4}m$$

- Element forces or Element stresses

$$P=\frac{AE}{L}\left(u_j-u_i\right)$$

$$P^{(1)}=4\times10^7\left(u_2-u_1\right)=4,444N$$

$$P^{(2)}=5\times10^7\left(u_3-u_2\right)=-5,556N$$

- Reaction forces

$$R_L=-P^{(1)}=-4,444N,$$

$$R_R=+P^{(2)}=-5,556N$$

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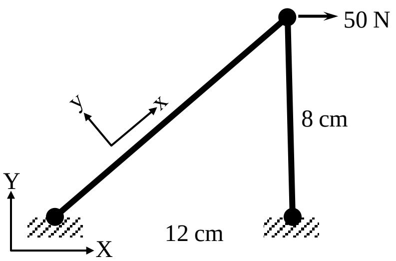
PLANE TRUSS ELEMENT

- What is the difference between 1D and 2D finite elements?
  - 2D element can move x- and y-direction (2 DOFs per node).
  - However, the stiffness can be applied only axial direction.
- Local Coordinate System
  - 1D FE formulation can be used if a body-fixed local coordinate system is constructed along the length of the element
  - The global coordinate system (*X* and *Y* axes) is chosen to represent the entire structure
  - The local coordinate system (*x* and *y* axes) is selected to align the *x*-axis along the length of the element



$$\begin{Bmatrix} f_{1\bar{x}} \\ f_{2\bar{x}} \end{Bmatrix}=\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix}$$

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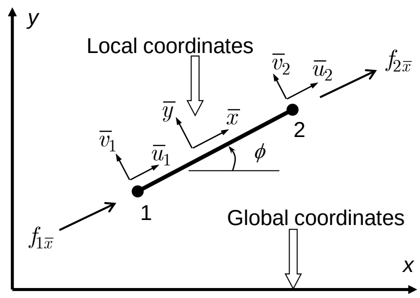
PLANE TRUSS ELEMENT *cont.*

- Element Equation (Local Coordinate System)
  - Axial direction is the local x-axis.
  - 2D element equation

$$\begin{Bmatrix} f_{1\bar{x}} \\ f_{1\bar{y}} \\ f_{2\bar{x}} \\ f_{2\bar{y}} \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \end{Bmatrix}$$

$$\{\bar{\mathbf{f}}\} = [\bar{\mathbf{k}}]\{\bar{\mathbf{q}}\}$$



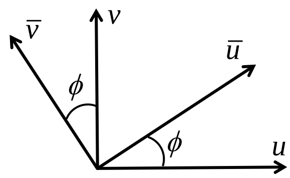
  - $[\bar{\mathbf{k}}]$ is square, symmetric, positive semi-definite, and non-negative diagonal components.
- How to connect to the neighboring elements?
  - Cannot connect to other elements because LCS is different
  - Use coordinate transformation

COORDINATE TRANSFORMATION

- Transform to the global coord. and assemble

$$\boxed{\begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}}$$

$$\boxed{\begin{Bmatrix} \bar{u}_2 \\ \bar{v}_2 \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix}}$$



- Transformation matrix

$$\underset{local}{\begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \end{Bmatrix}} = \begin{bmatrix} \cos\phi & \sin\phi & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi \\ 0 & 0 & -\sin\phi & \cos\phi \end{bmatrix} \underset{global}{\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}}$$

$$\boxed{\{\bar{\mathbf{q}}\} = [\mathbf{T}]\{\mathbf{q}\}}$$

Transformation matrix

COORDINATE TRANSFORMATION *cont.*

- The same transformation for force vector

$$\underset{local}{\begin{Bmatrix} f_{1\bar{x}} \\ f_{1\bar{y}} \\ f_{2\bar{x}} \\ f_{2\bar{y}} \end{Bmatrix}} = \begin{bmatrix} \cos\phi & \sin\phi & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi \\ 0 & 0 & -\sin\phi & \cos\phi \end{bmatrix} \underset{global}{\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix}}$$

$$\boxed{\{\bar{\mathbf{f}}\} = [\mathbf{T}]\{\mathbf{f}\}}$$

- Property of transformation matrix

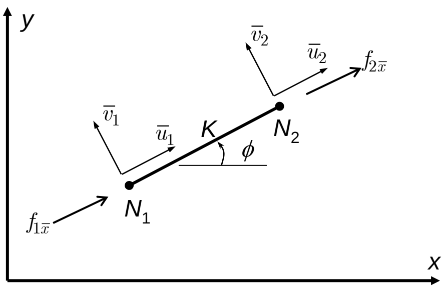
$$[\mathbf{T}]^{-1} = [\mathbf{T}]^T \qquad \{\bar{\mathbf{f}}\} = [\mathbf{T}]\{\mathbf{f}\} \implies \{\mathbf{f}\} = [\mathbf{T}]^T\{\bar{\mathbf{f}}\}$$

ELEMENT STIFFNESS IN GLOBAL COORD.

- Element 1

$$\begin{Bmatrix} f_{1\bar{x}} \\ f_{1\bar{y}} \\ f_{2\bar{x}} \\ f_{2\bar{y}} \end{Bmatrix} = \underbrace{\boxed{\frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}}_{\text{element stiffness matrix}} \begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \end{Bmatrix}$$

$$\{\bar{\mathbf{f}}\} = [\bar{\mathbf{k}}]\{\bar{\mathbf{q}}\}$$



- Transform to the global coordinates

$$[\mathbf{T}]\{\mathbf{f}\} = [\bar{\mathbf{k}}][\mathbf{T}]\{\mathbf{q}\} \implies \underset{\text{global}}{\{\mathbf{f}\}} = [\mathbf{T}]^{-1}[\bar{\mathbf{k}}][\mathbf{T}]\underset{\text{global}}{\{\mathbf{q}\}}$$

$$\Downarrow$$

$$\boxed{[\mathbf{k}] = [\mathbf{T}]^{-1}[\bar{\mathbf{k}}][\mathbf{T}]} \Longleftarrow \{\mathbf{f}\} = [\mathbf{k}]\{\mathbf{q}\}$$

ELEMENT STIFFNESS IN GLOBAL COORD. *cont.*

- Element stiffness matrix in global coordinates

$$[\mathbf{k}] = [\mathbf{T}]^T [\bar{\mathbf{k}}][\mathbf{T}]$$

$$[\mathbf{k}] = \frac{EA}{L}\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi & -\cos^2\phi & -\cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi & -\cos\phi\sin\phi & -\sin^2\phi \\ -\cos^2\phi & -\cos\phi\sin\phi & \cos^2\phi & \cos\phi\sin\phi \\ -\cos\phi\sin\phi & -\sin^2\phi & \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

  - Depends on Young's modulus (E), cross-sectional area (A), length (L), and angle of rotation (ϕ)
  - Axial rigidity = EA
  - Square, symmetric, positive semi-definite, singular, and non-negative diagonal terms

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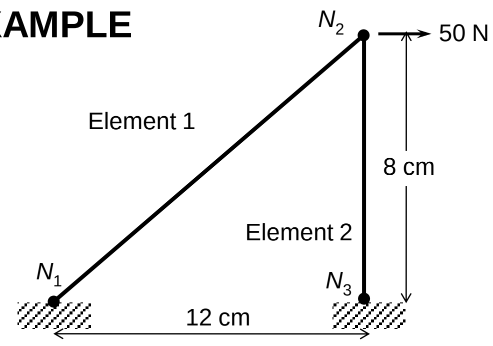
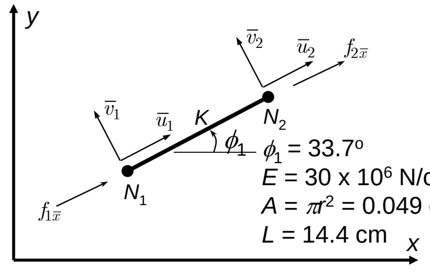
EXAMPLE

- Two-bar truss
  - Diameter = 0.25 cm
  - $E = 30\times10^6$ N/cm^2
- Element 1
  - In local coordinate



$$\{\bar{\mathbf{f}}^{(1)}\} = [\bar{\mathbf{k}}^{(1)}]\{\bar{\mathbf{q}}^{(1)}\}$$

$$\begin{Bmatrix} f_{1\bar{x}} \\ f_{1\bar{y}} \\ f_{2\bar{x}} \\ f_{2\bar{y}} \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \end{Bmatrix}$$



$\phi_1 = 33.7^o$
$E = 30 \times 10^6$ N/cm^2
$A = \pi r^2 = 0.049$ cm^2
$L = 14.4$ cm

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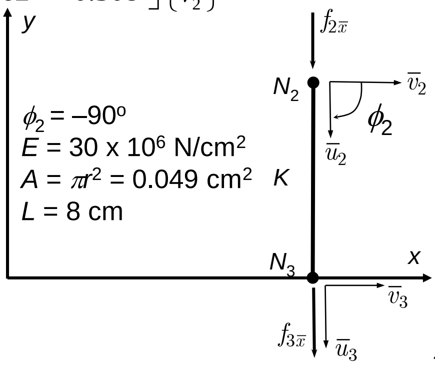
EXAMPLE *cont.*

- Element 1 cont.
  - Element equation in the global coordinates

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{1y}^{(1)} \\ f_{2x}^{(1)} \\ f_{2y}^{(1)} \end{Bmatrix} = 102150 \begin{bmatrix} 0.692 & 0.462 & -0.692 & -0.462 \\ 0.462 & 0.308 & -0.462 & -0.308 \\ -0.692 & -0.462 & 0.692 & 0.462 \\ -0.462 & -0.308 & 0.462 & 0.308 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \qquad \{\mathbf{f}^{(1)}\} = [\mathbf{k}^{(1)}]\{\mathbf{q}^{(1)}\}$$

- Element 2

$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{2y}^{(2)} \\ f_{3x}^{(2)} \\ f_{3y}^{(2)} \end{Bmatrix} = 184125 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$



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EXAMPLE *cont.*

- Assembly
  - After transforming to the global coordinates

Element 1

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix} = \begin{bmatrix} 70687 & 47193 & -70687 & -47193 & 0 & 0 \\ 47193 & 31462 & -47193 & -31462 & 0 & 0 \\ -70687 & -47193 & 70687 & 47193 & 0 & 0 \\ -47193 & -31462 & 47193 & 215587 & 0 & -184125 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -184125 & 0 & 184125 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

Element 2

- Boundary Conditions
  - Nodes 1 and 3 are fixed.
  - Node 2 has known applied forces: F_{2x} = 50 N, F_{2y} = 0 N

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EXAMPLE *cont.*

- Boundary conditions (striking-the-columns)

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ 50 \\ 0 \\ F_{3x} \\ F_{3y} \end{Bmatrix} = \begin{bmatrix} 70687 & 47193 & -70687 & -47193 & 0 & 0 \\ 47193 & 31462 & -47193 & -31462 & 0 & 0 \\ -70687 & -47193 & 70687 & 47193 & 0 & 0 \\ -47193 & -31462 & 47193 & 215587 & 0 & -184125 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -184125 & 0 & 184125 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ 0 \\ 0 \end{Bmatrix}$$

  - Striking-the-rows

$$\begin{Bmatrix} 50 \\ 0 \end{Bmatrix} = \begin{bmatrix} 70687 & 47193 \\ 47193 & 215587 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix}$$

- Solve the global matrix equation

$$u_2 = 8.28 \times 10^{-4} \text{ cm}$$

$$v_2 = -1.81 \times 10^{-4} \text{ cm}$$

EXAMPLE *cont.*

- Support reactions

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{3x} \\ F_{3y} \end{Bmatrix} = \begin{bmatrix} -70687 & -47193 \\ -47193 & -31462 \\ 0 & 0 \\ 0 & -184125 \end{bmatrix} \begin{Bmatrix} 8.28 \times 10^{-4} \\ -1.81 \times 10^{-4} \end{Bmatrix} = \begin{Bmatrix} -50 \\ -33.39 \\ 0 \\ 33.39 \end{Bmatrix} N$$

  - The reaction force is parallel to the element length (two-force member)

- Element force and stress (Element 1)
  - Need to transform to the element local coordinates

$$\begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \end{Bmatrix} = \begin{bmatrix} .832 & .555 & 0 & 0 \\ -.555 & .832 & 0 & 0 \\ 0 & 0 & .832 & .555 \\ 0 & 0 & -.555 & .832 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 5.89 \times 10^{-4} \\ -6.11 \times 10^{-4} \end{Bmatrix}$$

EXAMPLE *cont.*

- Element force and stress (Element 1) *cont.*
  - Element force can only be calculated using local element equation

$$\begin{Bmatrix} f_{1\bar{x}} \\ f_{1\bar{y}} \\ f_{2\bar{x}} \\ f_{2\bar{y}} \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 5.89\times10^{-4} \\ -6.11\times10^{-4} \end{Bmatrix} = \begin{Bmatrix} -60.2 \\ 0 \\ 60.2 \\ 0 \end{Bmatrix} N$$

  - There is no force components in the local y-direction
  - In x-direction, two forces are equal and opposite
  - The force in the second node is equal to the element force
  - Normal stress = 60.2 / 0.049 = 1228 N/cm^2.

OTHER WAY OF ELEMENT FORCE CALCULATION

- Element force for plane truss

$$P^{(e)} = \left(\frac{AE}{L}\right)^{(e)} \Delta^{(e)} = \left(\frac{AE}{L}\right)^{(e)} \left(\bar{u}_j - \bar{u}_i\right)$$

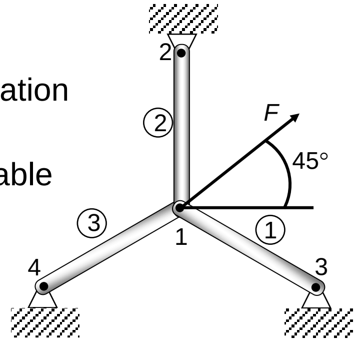
  - Write in terms of global displacements

$$\begin{aligned} P^{(e)} &= \left(\frac{AE}{L}\right)^{(e)} \left(\left(lu_j + mv_j\right) - \left(lu_i + mv_i\right)\right) \\ &= \left(\frac{AE}{L}\right)^{(e)} \left(l\left(u_j - u_i\right) + m\left(v_j - v_i\right)\right) \end{aligned}$$

$$l = \cos\phi \qquad m = \sin\phi$$

EXAMPLE

- Directly assembling global matrix equation (applying BC in the element level)
- Element property & direction cosine table



Elem	AE/L	i -> j	ϕ	$l = \cos\phi$	$m = \sin\phi$
1	206×10^5	1 -> 3	-30	0.866	–0.5
2	206×10^5	1 -> 2	90	0	1
3	206×10^5	1 -> 4	210	–0.866	–0.5

- Since u_3 and v_3 will be deleted after assembly, it is not necessary to keep them

$$\left[\mathbf{k}^{(1)}\right] = \left(\frac{EA}{L}\right)^{(1)} \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\ \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_3 \\ v_3 \end{array} \Rightarrow \left[\mathbf{k}^{(1)}\right] = \left(\frac{EA}{L}\right)^{(1)} \begin{array}{c} \begin{array}{cc} u_1 & v_1 \end{array} \\ \begin{bmatrix} l^2 & lm \\ lm & m^2 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ v_1 \end{array}$$

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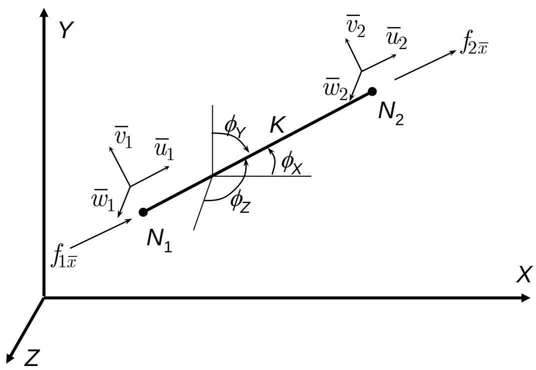
SPACE TRUSS ELEMENT

- A similar extension using coordinate transformation
- 3DOF per node
  - u, v, and w
  - f_x, f_y, and f_z
- Element stiffness matrix is 6x6



- FE equation in the local coord.

$$\begin{Bmatrix} f_{i\bar{x}} \\ f_{j\bar{x}} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \bar{u}_i \\ \bar{u}_j \end{Bmatrix} \qquad \{\bar{\mathbf{f}}\} = [\bar{\mathbf{k}}]\{\bar{\mathbf{q}}\}$$

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SPACE TRUSS ELEMENT cont.

- Relation between local and global displacements
  - Each node has 3 DOFs (u_i, v_i, w_i)

$$\begin{Bmatrix} \bar{u}_i \\ \bar{u}_j \end{Bmatrix} = \begin{bmatrix} l & m & n & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & n \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ w_i \\ u_j \\ v_j \\ w_j \end{Bmatrix} \qquad \underset{(2\times 1)}{\{\bar{\mathbf{q}}\}} = \underset{(2\times 6)}{[\mathbf{T}]} \cdot \underset{(6\times 1)}{\{\mathbf{q}\}}$$

  - Direction cosines

$$l = \cos\phi_x = \frac{x_j - x_i}{L}, \quad m = \cos\phi_y = \frac{y_j - y_i}{L}, \quad n = \cos\phi_z = \frac{z_j - z_i}{L}$$

$$L = \sqrt{\left(x_j - x_i\right)^2 + \left(y_j - y_i\right)^2 + \left(z_j - z_i\right)^2}$$

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SPACE TRUSS ELEMENT cont.

- Relation between local and global force vectors

$$\begin{Bmatrix} f_{ix} \\ f_{iy} \\ f_{iz} \\ f_{jx} \\ f_{jy} \\ f_{jz} \end{Bmatrix} = \begin{bmatrix} l & 0 \\ m & 0 \\ n & 0 \\ 0 & l \\ 0 & m \\ 0 & n \end{bmatrix} \begin{Bmatrix} f_{i\bar{x}} \\ f_{j\bar{x}} \end{Bmatrix} \qquad \{\mathbf{f}\} = [\mathbf{T}]^T \{\bar{\mathbf{f}}\}$$

- Stiffness matrix

$$\{\bar{\mathbf{f}}\} = [\bar{\mathbf{k}}]\{\bar{\mathbf{q}}\} \implies [\mathbf{T}]^T\{\bar{\mathbf{f}}\} = [\mathbf{T}]^T[\bar{\mathbf{k}}][\mathbf{T}]\{\mathbf{q}\} \implies \{\mathbf{f}\} = [\mathbf{k}]\{\mathbf{q}\}$$

$$\Downarrow$$

$$[\mathbf{k}] = \frac{EA}{L} \begin{array}{c} \begin{bmatrix} l^2 & lm & ln & -l^2 & -lm & -ln \\ & m^2 & mn & -lm & -m^2 & -mn \\ & & n^2 & -ln & -mn & -n^2 \\ & & & l^2 & lm & ln \\ & \text{sym} & & & m^2 & mn \\ & & & & & n^2 \end{bmatrix} \end{array} \begin{array}{c} u_i \\ v_i \\ w_i \\ u_j \\ v_j \\ w_j \end{array} \Longleftarrow [\mathbf{k}] = [\mathbf{T}]^T[\bar{\mathbf{k}}][\mathbf{T}]$$

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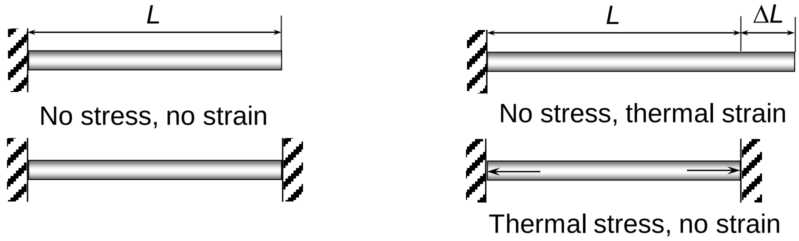
THERMAL STRESSES

- Temperature change causes thermal strain



(a) at $T = T_{\text{ref}}$ (b) at $T = T_{\text{ref}} + \Delta T$

- Constraints cause thermal stresses
- Thermo-elastic stress-strain relationship

$$\boxed{\sigma = E(\varepsilon - \alpha\Delta T)} \qquad \boxed{\varepsilon = \frac{\sigma}{E} + \alpha\Delta T}$$

Thermal expansion coefficient

THERMAL STRESSES cont.

- Force-displacement relation

$$P = AE\left(\frac{\Delta L}{L} - \alpha\Delta T\right) = AE\frac{\Delta L}{L} - AE\alpha\Delta T$$

- Finite element equation

$$\{\bar{\mathbf{f}}^{(e)}\} = [\bar{\mathbf{k}}^{(e)}]\{\bar{\mathbf{q}}^{(e)}\} - \{\bar{\mathbf{f}}_T^{(e)}\}$$

Thermal force vector

$$\{\bar{\mathbf{f}}_T^{(e)}\} = AE\alpha\Delta T\begin{Bmatrix} -1 \\ 0 \\ +1 \\ 0 \end{Bmatrix}\begin{matrix} \bar{u}_i \\ \bar{v}_i \\ \bar{u}_j \\ \bar{v}_j \end{matrix}$$

- For plane truss, transform to the global coord.

$$\{\mathbf{f}\} = [\mathbf{k}]\{\mathbf{q}\} - \{\mathbf{f}_T\}$$
$$\Downarrow$$
$$[\mathbf{k}]\{\mathbf{q}\} = \{\mathbf{f}\} + \{\mathbf{f}_T\}$$
$$\Downarrow$$
$$[\mathbf{K}_s]\{\mathbf{Q}_s\} = \{\mathbf{F}_s\} + \{\mathbf{F}_{Ts}\}$$

$$\boxed{\{\mathbf{f}_T\} = AE\alpha\Delta T\begin{Bmatrix} -l \\ -m \\ +l \\ +m \end{Bmatrix}\begin{matrix} u_i \\ v_i \\ u_j \\ v_j \end{matrix}}$$