Chapter 3: BASIC ELEMENTS
Section 3.1: Preliminaries (review of solid mechanics)

Outline

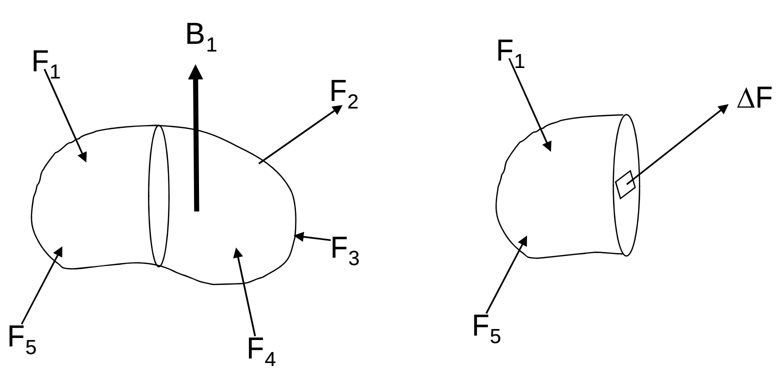
- Most structural analysis FE codes are displacement based
- In this chapter we discuss interpolation methods and elements based on displacement interpolations
- Stiffness matrix formulations will be presented
- Shortcomings and restrictions of the elements due to the interpolations used will be discussed
- We will review the governing equations (for solids elastic bodies) to help us understand the solution methods and accuracy

Review of Solid Mechanics

- The analysis of any solid elastic body has to define and develop the following quantities and/or relations
  - Stress
  - Strain (strain-displacement relations)
  - Constitutive Properties (Stress-Strain relations)
  - Compatibility
  - Equilibrium Equations
  - Boundary Conditions

Stress

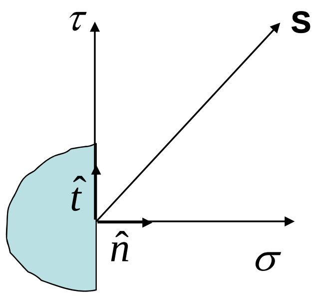
- Stresses are distributed internal forces that result from externally applied forces



Note: There are two types of forces: Surface forces that act on an area of external surface and body force that acts on the volume of the body

Stress/Force acting on a surface

- A force acting on a surface can be resolved into two components: One tangential to the surface (shear force) and the other normal to the surface



Force is a 1st order tensor (vector)
Stress is a 2nd order tensor.

Why is it a tensor?

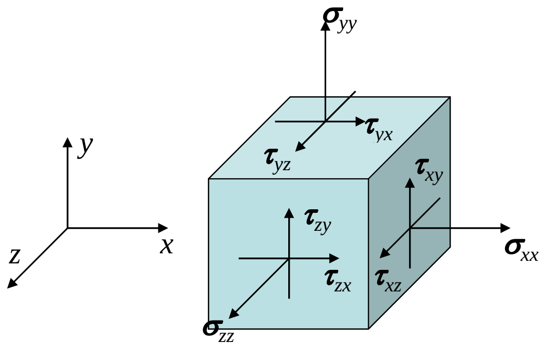
Needs two vectors to specify it. One is the vector of the internal force and the other is the outward normal of the defining area

Representation of stress at a point

- In 3-D space the stress at a point is denoted by the stresses acting on three mutually perpendicular planes at that point
- Often we use a simple infinitesimal rectangular solid and indicate the stresses on the faces of that solid
- Stress terms have two indices.
  - The first indicating the plane on which it acts
  - The second indicating the direction in which it acts

Stress at a point

- State of stress at a point in 3-D Cartesian Coordinates



Matrix representation of state of stress

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix}$$

Sign Conventions:
Normal stresses are positive when acting outward from a surface (tension)
Shear stresses are positive when they act in the +ve direction on a positive face and –ve direction on a –ve face

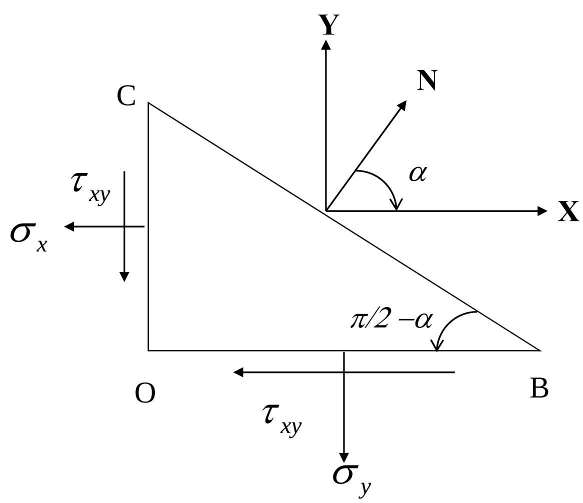
Stress on an arbitrary plane (2-D)

We often need to enforce stress boundary conditions on surfaces that are not always rectangular

Let cos(θ) = l and sin(θ) = m

If length of side BC = A, then length of sides OC=Al and OB=Am

If we write the force equilibrium in X and Y-directions, we have



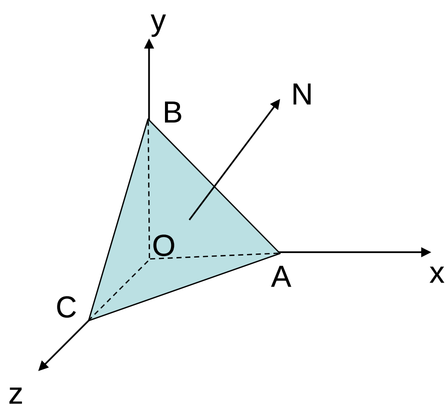
$$\mathbf{X}A = \sigma_x(Al) + \tau_{xy}(Am)$$

Which simplifies to

$$\mathbf{X} = \sigma_x l + \tau_{xy} m$$
$$\mathbf{Y} = \sigma_y m + \tau_{xy} l$$

Stress on an arbitrary plane (3-D)

The forces (per unit area) in X, Y and Z-directions on an arbitrary plane defined by its normal vector N is given by



$$\bar{X} = l\sigma_x + m\tau_{xy} + n\tau_{zx}$$
$$\bar{Y} = l\tau_{xy} + m\sigma_y + n\tau_{yz}$$
$$\bar{Z} = l\tau_{xz} + m\tau_{yz} + n\sigma_z$$

Where *l*, *m* and *n* are the direction cosines of the normal vector of the arbitrary plane

$$l = \cos Nx,\ \ m = \cos Ny,\ \ n = \cos Nz$$

Stress transformations in 3D

Stress transformation in 3D space can be defined using the directions cosines as follows.

$$\sigma_{x'} = l_1^2\sigma_x + m_1^2\sigma_y + n_1^2\sigma_z + 2m_1l_1\tau_{xy} + 2m_1n_1\tau_{yz} + 2n_1l_1\tau_{zx}$$
$$\sigma_{y'} = l_2^2\sigma_x + m_2^2\sigma_y + n_2^2\sigma_z + 2m_2l_2\tau_{xz} + 2m_2n_2\tau_{yz} + 2n_2l_2\tau_{zx}$$
$$\sigma_{z'} = l_3^2\sigma_x + m_3^2\sigma_y + n_3^2\sigma_z + 2m_3l_3\tau_{xy} + 2m_3n_3\tau_{yz} + 2n_3l_3\tau_{zx}$$
$$\tau_{x'y'} = l_1l_2\sigma_x + m_1m_2\sigma_y + n_1n_2\sigma_z + (l_1m_2 - m_1l_2)\tau_{xy} + (m_1n_2 - n_1m_2)\tau_{yz} + (n_1l_2 - l_1n_2)\tau_{zx}$$
$$\tau_{y'z'} = l_2l_3\sigma_x + m_2m_3\sigma_y + n_2n_3\sigma_z + (l_2m_3 - m_2l_3)\tau_{xy} + (m_2n_3 - n_2m_3)\tau_{yz} + (n_2l_3 - l_2n_3)\tau_{zx}$$
$$\tau_{x'z'} = l_3l_1\sigma_x + m_3m_1\sigma_y + n_3n_1\sigma_z + (l_3m_1 - m_3l_1)\tau_{xy} + (m_3n_1 - n_3m_1)\tau_{yz} + (n_3l_1 - l_3n_1)\tau_{zx}$$

Direction cosines in 3D

The direction cosines $l.m$ and n between the new coordinate axes x', y' and z' and the original coordinate system x, y and z are defined as follows

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_2	n_3

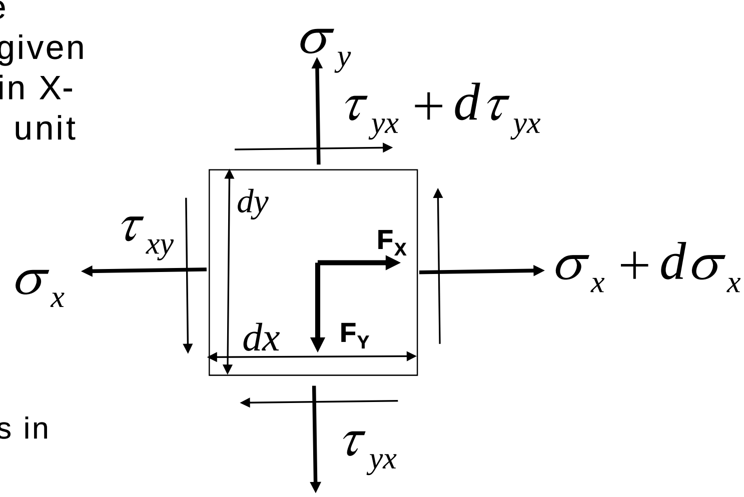
Since the transformation is orthogonal, the direction cosines must satisfy the following properties

$$l_1^2 + m_1^2 + n_1^2 = 1$$
$$l_1^2 + l_2^2 + l_3^2 = 1$$
$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$
$$l_1 m_1 + l_2 m_2 + l_3 m_3 = 0$$

Where, $l_1 = \cos xx'$, $m_2 = \cos yy'$

Equilibrium Equations (2-D)

The 2-D stresses are shown on a volume given of length *dx* and *dy* in X- and Y-directions and unit thickness in the Z-direction



Summation of forces in X-direction

$$(\sigma_x + d\sigma_x)dy - \sigma_x dy + (\tau_{xy} + d\tau_{xy})dx - \tau_{xy} dx + F_{\mathbf{X}} = 0$$

Equilibrium of forces in X-direction

$$(\sigma_x + d\sigma_x)dy - \sigma_x dy + (\tau_{xy} + d\tau_{xy})dx - \tau_{xy}dx + F_{\mathbf{X}}dxdy = 0$$

$$(\cancelto{1}{\sigma_x} + d\sigma_x)dy - \cancelto{1}{\sigma_x} dy + (\cancelto{2}{\tau_{xy}} + d\tau_{xy})dx - \cancelto{2}{\tau_{xy}}dx + F_X dxdy = 0$$

$$d\sigma_x = \frac{\partial \sigma_x}{\partial x}dx$$

$$d\tau_{xy} = \frac{\partial \tau_{xy}}{\partial y}dy$$

$$\frac{d\sigma_x}{dx}dxdy + \frac{d\tau_{xy}}{dy}dxdy + F_X dxdy = 0$$

$$\boxed{\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_X\right)dxdy = 0}$$

2-D Equilibrium Equations

The force equilibrium provide the relations shown below referred to as differential equation of equilibrium

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \mathbf{X} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \mathbf{Y} = 0$$

Establishing moment equilibrium by the same method will provide the condition for symmetry of the stress tensor

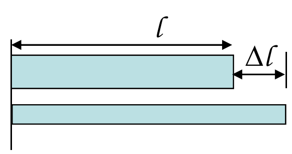
$$\tau_{xy} = \tau_{yx}$$

Strain

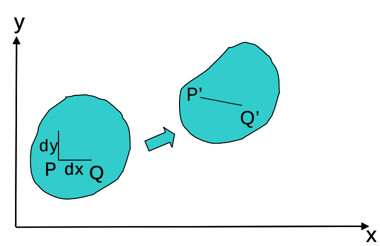
- Why do we need the strain measures? Will displacement not suffice?
- Strain better quantifies the deformation of the body and eliminates rigid body motion/ rotation
- Strain in very general terms is a measure of relative deformation
  - Relative to what?
    - Undeformed body : Lagrangian strain
    - Deformed body: Eulerian strain

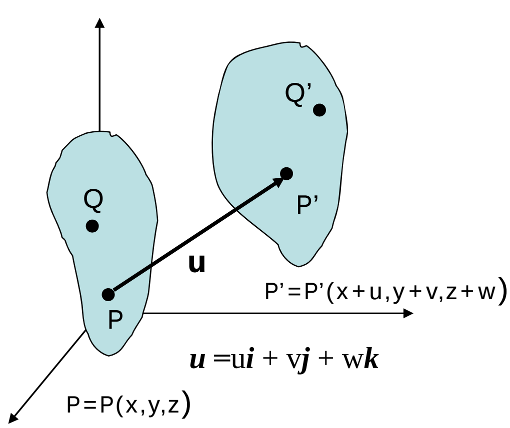
Strain-Displacement Relations

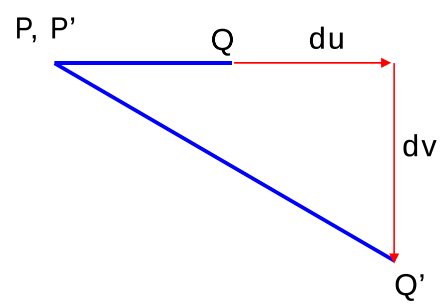
For uniaxial (1-D) case:



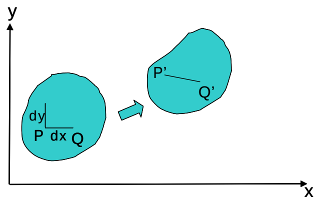
$$\varepsilon = \frac{\Delta l}{l}$$

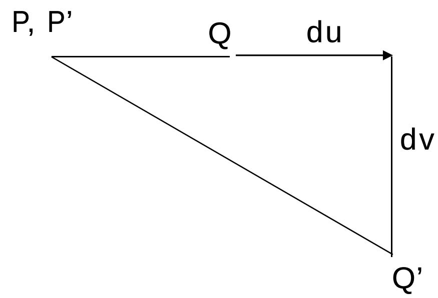






Strain 2-D : Normal strain





$$du = \frac{\partial u}{\partial x} dx$$

$$dv = \frac{\partial v}{\partial x} dx$$

$$\varepsilon_x = \frac{dx' - dx}{dx}$$

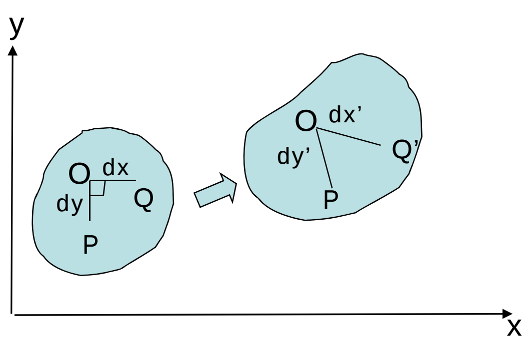
$$\varepsilon_x = \frac{\sqrt{\left(dx + \frac{\partial u}{\partial x} dx\right)^2 + \left(\frac{\partial v}{\partial x} dx\right)^2} - dx^2}{dx}$$

$$\varepsilon_x = 1 + \frac{1}{2}\left(2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right) - 1$$

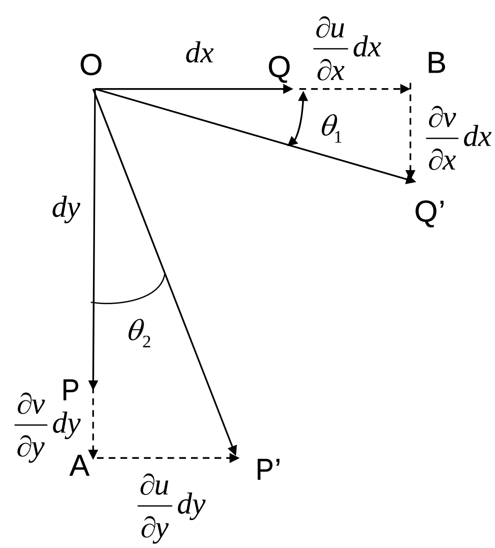
$$\varepsilon_x = \sqrt{1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2} - 1$$

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right)$$

STRAIN: Shear Strain



$$\gamma = \theta_1 + \theta_2$$



Shear strain

- With a bit of trigonometry (see for example, Allen and Haisler, Introduction to Aerospace Structural Analysis, p.60)

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial y}$$

For small displacements

The normal and shear strains are expressed as

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$ Engineering Strains

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$ Tensor strains

- Shear strain definitions are of two forms.
- The above form is referred to as engineering strains.
- The alternate form referred as tensorial strains have a factor of ½ applied to engineering strains.
- To apply coordinate transformations need the tensor form.

Compatibility

- Deformation must be such that the pieces fit together without any gaps or overlap.
- Why is this an issue?
- In 2-D we require only 2 displacements u, and v to describe deformation, but have three strain quantities ε_x,ε_y, and γ_{xy}. This implies only two of the three strain terms are independent.

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\frac{\partial^2}{\partial x \partial y}\left(\gamma_{xy}\right) = \frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial v}{\partial y}\right) \Rightarrow \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2}$$

Stress-Strain Relations

- The stress-strain relations in solid mechanics is often referred to as the Hooke's Law
- Hooke's law of proportionality stated as "*extension is proportional to the force*" refers to the axial extension of a bar under an axial force
- This can be extended to 3-D stress/strain state referred to as the *Generalized Hooke's Law* relates the components of the 3-D stress state to 3-D strains as follows.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

Generalized Hooke's Law

- In the most general form the *generalized Hooke's Law* requires 36 constants to relate the terms of a 3-D Stress state to its corresponding 3-D strain state for an elastic material
- However, from symmetry of the strain energy terms, it can be shown that $c_{ij} = c_{ji}$
- This reduces the number of unknown constants to 21

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

Hooke's Law: Orthotropy and Isotropy

- If we assume the x, y and x coordinates provide the planes of symmetry we can further reduce the number of constants to 9.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

This corresponds to an fully orthotropic material

Isotropy assumes that there is no directional variation on property. Using this argument we can obtain

$$c_{11} = c_{22} = c_{33}$$
$$c_{12} = c_{13} = c_{23}$$
$$c_{44} = c_{55} = c_{66}$$

Hooke's law : Engineering Elastic Constants

The two engineering elastic constants used to relate stress to strain for isotropic materials are the Elastic modulus, *E* and the Poisson's ration ν.

For uniaxial loading, strain in the loading direction obtained from Hooke's law, states

$$\varepsilon_x = \frac{\sigma_x}{E}$$

Transverse to loading direction $\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = -\nu\frac{\sigma_x}{E}$

The relation between the shear stress component and its corresponding shear-strain component is called the *modulus of rigidity* or *modulus of elasticity in shear* and is denoted by the letter G.

$$G = \frac{\tau}{\gamma} = \mu \qquad\qquad G = \frac{E}{2(1+\nu)}$$

Generalized Hooke's Law from Isotropic Materials

The generalized Hooke's law expressed in engineering elastic constants

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] \qquad \gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2(1+\nu)}{E}\tau_{xy}$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right] \qquad \gamma_{yz} = \frac{\tau_{yz}}{G} = \frac{2(1+\nu)}{E}\tau_{yz}$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] \qquad \gamma_{zx} = \frac{\tau_{zx}}{G} = \frac{2(1+\nu)}{E}\tau_{zx}$$

Plane state of stress

There are a large class of problems for which the stresses normal to the plane of the solid are absent or negligibly small. If we assume that the stresses are restricted to the x-y plane, then

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0$$

This simplifies the stress strain relationship to the form as shown below.

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\sigma_y\right] \qquad \sigma_x = \frac{E}{1-\nu^2}\left[\varepsilon_x + \nu\varepsilon_y\right] \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\sigma_x\right] \qquad \sigma_y = \frac{E}{1-\nu^2}\left[\varepsilon_y + \nu\varepsilon_x\right] \qquad \tau_{xy} = G\gamma_{xy}$$

$$\varepsilon_z = \frac{-\nu}{E}\left(\sigma_x + \sigma_y\right)$$

Plane strain

Strains that deform the body normal to the reference plane are absent or are negligible

$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

This indicates that the stress normal to the plane of strain is dependent on the stresses in the plane of the strain $\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] = 0 \Rightarrow \sigma_z = \nu\left(\sigma_x + \sigma_y\right)$

Substituting σ_z into other strain expressions we obtain

$$\varepsilon_x = \frac{1+\nu}{E}\left[(1-\nu)\sigma_x - \nu\sigma_y\right] \qquad \sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu\varepsilon_y\right] \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\varepsilon_y = \frac{1+\nu}{E}\left[(1-\nu)\sigma_y - \nu\sigma_x\right] \qquad \sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_y + \nu\varepsilon_x\right] \qquad \tau_{xy} = G\gamma_{xy}$$

$$\sigma_z = \frac{\nu E}{(1+\nu)(1-2\nu)}\left[\varepsilon_x + \varepsilon_y\right]$$

Conversion from plane strain to plane stress and vice-versa

The solution obtained for the stress and strains in plane stress and plane strain states are qualitatively similar.

To use a plane strain solution for a plane stress or vice versa, we simply interchange the appropriate constants as shown below

For plane stress the expressions in E, ν
For plane stress the expressions in E^*, ν^*

$$E = \frac{E^*}{1-\nu^*} \;,\quad \nu = \frac{\nu^*}{1-\nu^*} \quad or \;\; E^* = E\frac{1+2\nu}{(1+\nu)^2} \quad \text{and} \quad \nu = \frac{\nu}{1+\nu}$$

Solution of 3D Elasticity Problems

3 Equilibrium equations
6 Strain-displacement equations
6 Hooke's law equations
-----
15 $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$
$\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
u, v, w

3 Equilibrium equations
3 Compatibility
6 Hooke's law equations
-----
12 $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$
$\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$

3 Equilibrium equations in displacements
-----
3 u, v, w

3 Equilibrium equations
6 Strain displacement equations
-----
9 $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$
u, v, w

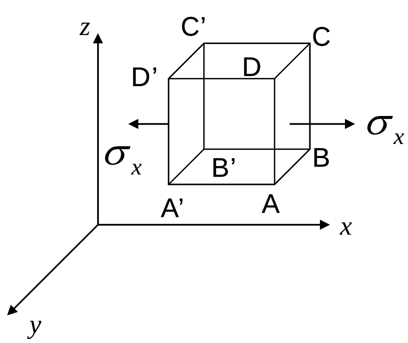
3 Equilibrium equations
3 Compatibility equations expressed in stress
-----
6 $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$

Principle of superposition

- Effects of several forces acting together are equal to the combined effect of the forces acting separately. This is valid only when
  - The stresses and displacements are directly proportional to the load
  - The geometry and loading of the deformed object does not differ significantly from the undeformed configuration

Energy Principles

- Strain Energy Density:
  When an elastic body is under the action of external forces, the body deforms and work is done by these forces. The work done by the forces is stored internally by the body and is called the strain energy.
- Let us consider the unit element of volume dxdydz with only the normal stress σ_x acting on it. The work done, or work stored in the element is



$$\int_{\sigma_x=0}^{\sigma_x=\sigma_x} \sigma_x d\left(u+\frac{\partial u}{\partial x}dx\right)dydz-\int_{\sigma_x=0}^{\sigma_x=\sigma_x} \sigma_x d(u)dydz$$

$$=\int_{\sigma_x=0}^{\sigma_x=\sigma_x} \sigma_x \frac{\partial u}{\partial x}dxdydz$$

Strain Energy

Using Hooke's law $\dfrac{\partial u}{\partial x} = \varepsilon_x = \dfrac{\sigma_x}{E}$

$$\text{Work done} = \int_{\sigma_x=0}^{\sigma_x=\sigma_x} \frac{\sigma_x}{E} d\sigma_x dxdydz = \frac{1}{2}\frac{\sigma_x{}^2}{E} dxdydz$$

For shear stresses, it can be similarly shown that the work done is

$$\frac{1}{2}\frac{\tau^2}{G} dxdydz$$

The strain energy stored in an element $dxdydz$ under a general three dimensional stress system is calculated as

$$dU = \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\right)dxdydz$$

Strain Energy Density

The strain energy density refers to strain energy per unit volume

$$dU_0 = \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\right)$$

Using principal stresses and strains, this can be expressed as

$$dU_0 = \frac{1}{2}\left(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3\right)$$

$$dU_0 = \frac{1}{2E}\left(I_1^2 - 2(1+\nu)I_2\right)$$

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3$$

3.2 Interpolation Functions

Introduction to Interpolation Functions

- *Interpolation* is to devise a continuous function that satisfies prescribed conditions at a finite number of points.
- In FEM, the points are the nodes of the elements & the prescribed conditions are the nodal values of the field variable
- Polynomials are the usual choice for FEM

Polynomial interpolation

The polynomial function $\phi(x)$ is used to interpolate a field variable based on its values at n-points

$$\phi(x) = \sum_{i=0}^{n} a_i x^i \qquad \text{or} \qquad \phi = \lfloor \mathbf{X} \rfloor \{\mathbf{a}\}^T$$

$$\lfloor \mathbf{X} \rfloor = \lfloor 1 \quad x \quad x^2 \quad \ldots \quad x^n \rfloor \text{ and } \{\mathbf{a}\} = \lfloor a_0 \quad a_1 \quad a_2 \quad \ldots \quad a_n \rfloor$$

The number of terms in the polynomial is chosen to match the number of given quantities at the nodes.

With one quantity per node, we calculate a_i's using the n-equations resulting from the expressions for ϕ_i at each of the n-known points

$$\phi(x_j) = \sum_{i=0}^{n} a_i x_j^i \qquad \{\phi_e\} = [\mathbf{A}]\{\mathbf{a}\} \qquad \{\mathbf{a}\} = [\mathbf{A}]^{-1}\{\phi_e\}$$

Shape Functions or Basis Functions

Traditional interpolation takes the following steps

1. Choose a interpolation function
2. Evaluate interpolation function at known points
3. Solve equations to determine unknown constants

$$\phi = \lfloor \mathbf{X} \rfloor \{\mathbf{a}\} \longrightarrow \{\phi_e\} = [\mathbf{A}]\{\mathbf{a}\} \longrightarrow \{\mathbf{a}\} = [\mathbf{A}]^{-1}\{\phi_e\} \longrightarrow \phi = \lfloor \mathbf{X} \rfloor \{\mathbf{a}\}$$

In FEM we are more interested in writing ϕ in terms of the nodal values

$$\phi = \lfloor \mathbf{X} \rfloor \{\mathbf{a}\} \longrightarrow \{\phi_e\} = [\mathbf{A}]\{\mathbf{a}\} \longrightarrow \{\mathbf{a}\} = [\mathbf{A}]^{-1}\{\phi_e\} \longrightarrow$$

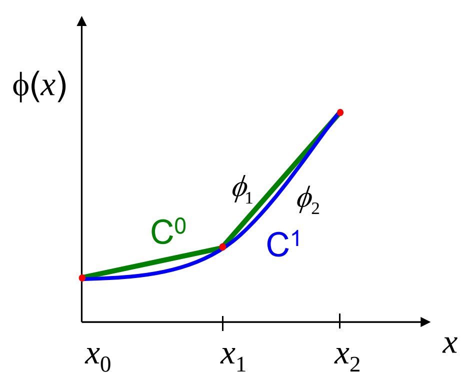
$$\longrightarrow \varphi = \lfloor \mathbf{X} \rfloor [\mathbf{A}]^{-1}\{\phi_e\} \longrightarrow \varphi = \lfloor \mathbf{N} \rfloor \{\phi_e\} \longrightarrow \lfloor \mathbf{N} \rfloor = \lfloor \mathbf{X} \rfloor [A]^{-1}$$

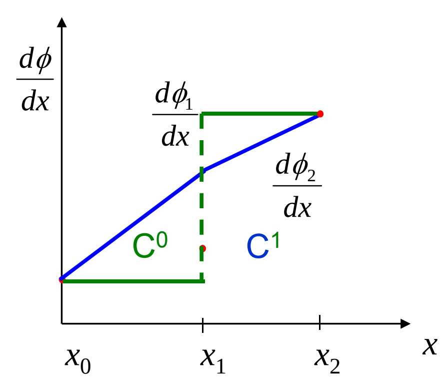
Degree of Continuity

- In FEM field quantities ϕ are interpolated in piecewise fashion over each element
- This implies that ϕ is continuous and smooth within the element
- However, ϕ may not be smooth between elements
- An interpolation function with C^m continuity provides a continuous variation of the function and up to the m-derivatives at the nodes
  - For example in a 1-D interpolation of f(x) C^0 continuity indicates that f is continuous at the nodes and $f,_x$ is not continuous.
  - If the displacement u(x) is C^0 then displacements are continuous between elements, but the strains are not (bar elements)

Degree of Continuity

Function ϕ_1 is C^0 continuous while ϕ_2 is C^1 continuous

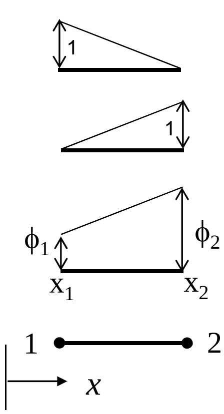




Example: Deriving a 1D linear interpolation shape function

- From $\phi = \lfloor N \rfloor \{\phi_e\}$ each interpolation function is zero at all the dofs except one.
- This can allow us to derive interpolation functions one at a time.
- For linear interpolation between x_1 and x_2, $N_1(x_1)=1$, $N_1(x_2)=0$, $N_1=a_1x+a_2$. So obviously, $N_1=1-(x-x_1)/L$, $L=x_2-x_1$.

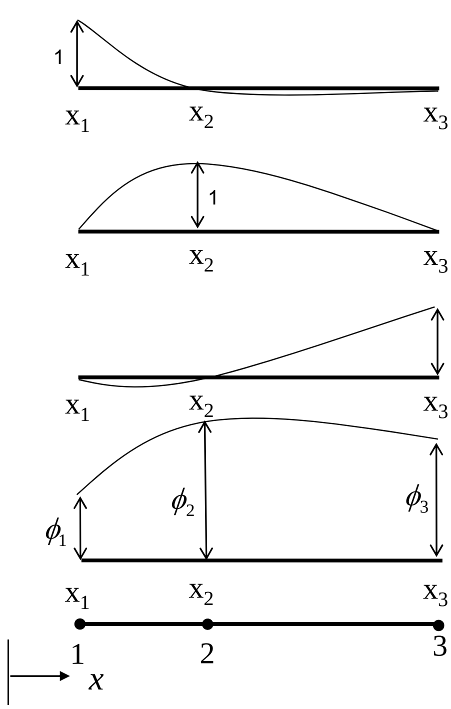
C^0 Interpolation -1D Element



$$N_1 = \frac{x_2 - x}{x_2 - x_1}$$

$$N_2 = \frac{x - x_1}{x_2 - x_1}$$

$$\phi = \lfloor \mathbf{N} \rfloor \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix}$$



$$N_1 = \frac{(x_2 - x)(x_3 - x)}{(x_2 - x_1)(x_3 - x_1)}$$

$$N_2 = \frac{(x_1 - x)(x_3 - x)}{(x_1 - x_2)(x_3 - x_2)}$$

$$N_3 = \frac{(x_1 - x)(x_2 - x)}{(x_1 - x_3)(x_2 - x_3)}$$

$$\phi = \lfloor \mathbf{N} \rfloor \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}$$

Lagrange Interpolation Formula

- Shape functions shown for the C^0 interpolations are special forms of the Lagrangian interpolation functions

$$f(x) = \sum_{k=1}^{n} N_k f_k$$

$$N_k = \frac{(x_1 - x)(x_2 - x)\ldots[x_k - x]\ldots(x_n - x)}{(x_1 - x_k)(x_2 - x_k)\ldots[x_k - x_k]\ldots(x_n - x_k)}$$

In above expressions for N_k the terms in square brackets are omitted

Properties of C^0 Interpolation Shape Functions

- All shape functions N_i, along with function ϕ are polynomials of the same degree

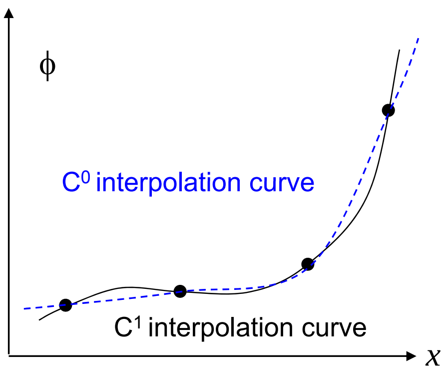
- For any shape function N_i, $N_i = 1$ at node i ($x=x_i$) and zero at all other nodes j, ($x_j \neq x_i$)

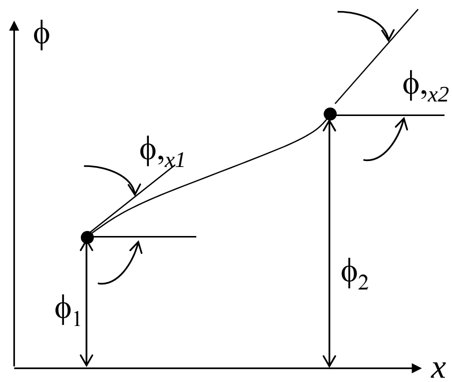
- C^0 shape functions sum to one

$$\sum_{k=1}^{n} N_k = 1$$

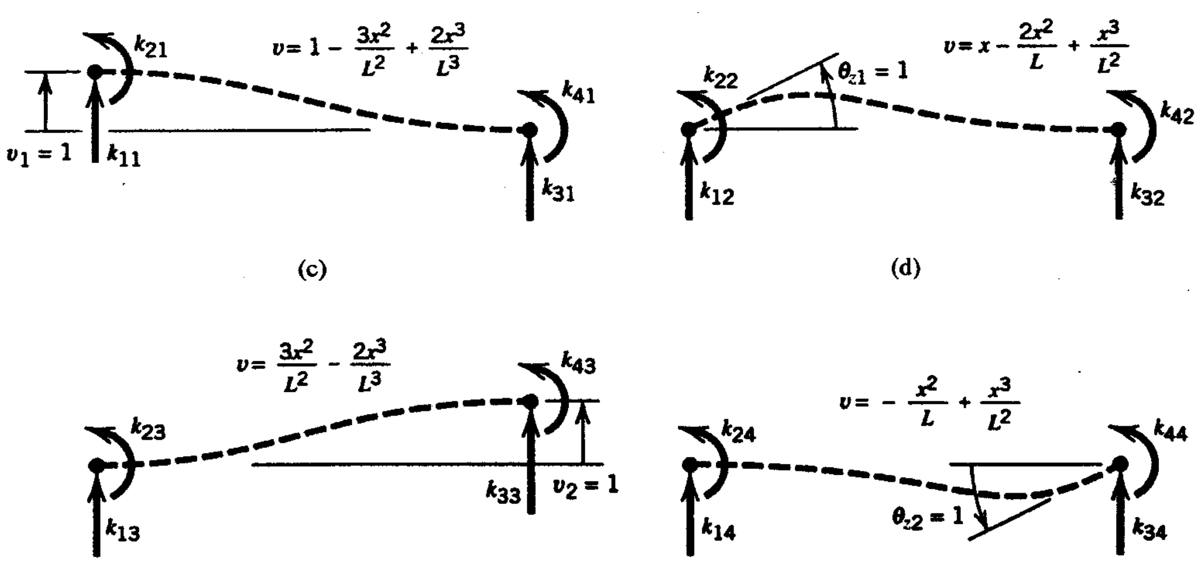
C^1 Interpolation

- Also called Hermitian interpolation (Hermite polynomials)
- Use the ordinate and slope information at the nodes to interpolate





Hermitian interpolation used for beam elements



2-D and 3-D Interpolation

- The 2-D and 3-D shape functions follow the same procedure as for 1-D
- We now have to start with shape functions that have two or more independent terms.
- For example a linear interpolation in 2-D from 3 nodes will require an interpolation function

$$f(x,y) = \lfloor 1 \quad x \quad y \rfloor \lfloor a_1 \quad a_2 \quad a_3 \rfloor^T$$

- If there are two or more components (e.g., u, v and w displacements) then the same interpolation function is used for all components

Principle of Virtual Work

- The principle of virtual work states that <u>at equilibrium</u> the strain energy change due to a small virtual displacement is equal to the work done by the forces in moving through the virtual displacement.
- A virtual displacement is a small imaginary change in configuration that is also a admissible displacement
- An admissible displacement satisfies kinematic boundary conditions
- Note: Neither loads nor stresses are altered by the virtual displacement.

Principle of Virtual Work

- The principle of virtual work can be written as follows
$$\int \{\delta\varepsilon\}^T \{\sigma\} dV = \int \{\delta u\}^T \{F\} dV + \int \{\delta u\}^T \{\Phi\} dS$$
- The same can be obtained by the Principle of Stationary Potential Energy

- The total potential energy of a system Π is given by

$$\delta\Pi = \delta U - \delta W \;\; = \delta U + \delta V = 0$$

  – U is strain energy, W is work done, or V is potential of the forces

$$\Pi = U - W$$

Element and load derivation

- Interpolation $\{\mathbf{u}\} = [N]\{\mathbf{d}\}$ $\{\mathbf{u}\} = \lfloor u \quad v \quad w \rfloor$
- Strain displacement $\{\varepsilon\} = [B]\{\mathbf{d}\}$ $[B] = [\partial][N]$
- Virtual $\{\boldsymbol{\delta}\mathbf{u}\}^T = \{\boldsymbol{\delta}\mathbf{d}\}^T [N]^T$ *and* $\{\boldsymbol{\delta}\varepsilon\}^T = \{\boldsymbol{\delta}\mathbf{d}\}^T [B]^T$
- Constitutive law $\{\sigma\} = [E]\{\varepsilon\}$
- Altogether

$$\{\boldsymbol{\delta}\mathbf{d}\}^T \Big(\int [B]^T [E][B] dV \{\mathbf{d}\} - \int [B]^T [E]\{\varepsilon_0\} dV + \int [B]^T \{\sigma_0\} dV$$
$$- \int \; [N]^T \{\mathbf{F}\} dV - \int \; [N]^T \{\phi\} dS \;\; \Big) = 0$$

Stiffness matrix and load vector

- Equations of equilibrium

$$[k]\{\mathbf{d}\} = \{\mathbf{r}_e\}$$

- Element stiffness matrix

$$[k] = \int [B]^T [E][B]\, dV$$

- Element load vector

$$\{r_e\} = \int [B]^T [E]\{\varepsilon_0\}\, dV - \int [B]^T \{\sigma_0\}\, dV + \int [N]^T \{\mathbf{F}\}\, dV + \int [N]^T \{\phi\}\, dS$$

- Loads due to initial strain, initial stress, body forces and surface tractions

Plane Problems: Constitutive Equations

- Constitutive equations for a **linearly elastic and isotropic material** in ***plane stress*** (i.e., $\sigma_z = \tau_{xz} = \tau_{yz} = 0$):

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & 0 \\ -\nu/E & 1/E & 0 \\ 0 & 0 & 1/G \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} + \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix}$$

Initial thermal strains $\varepsilon_{x0} = \varepsilon_{y0} = \alpha \Delta T\,, \quad \gamma_{xy0} = 0$

- In matrix form,

$$\boldsymbol{\varepsilon} = \mathbf{E}^{-1}\boldsymbol{\sigma} + \boldsymbol{\varepsilon}_0 \qquad \boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon} + \boldsymbol{\sigma}_0 \qquad \text{in which } \boldsymbol{\sigma}_0 = -\mathbf{E}\boldsymbol{\varepsilon}_0$$

where

$$\mathbf{E} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \quad \text{for plane stress}$$

Plane Problems: Strain-Displacement Relations

•

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

or, in alternative matrix formats,

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \quad \text{or} \quad \boldsymbol{\varepsilon} = \boldsymbol{\partial}\mathbf{u}$$

Plane Problems: Displacement Field Interpolated

•

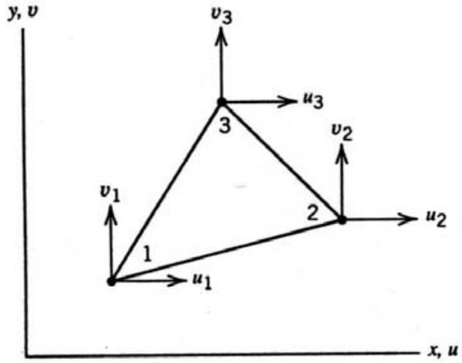
$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \\ 0 & N_1 & 0 & N_2 & \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \end{Bmatrix} \quad \text{or} \quad \mathbf{u} = \mathbf{N}\mathbf{d}$$

- From the previous two equations,

$$\boldsymbol{\varepsilon} = \boldsymbol{\partial}\mathbf{N}\mathbf{d} \quad \text{or} \quad \boldsymbol{\varepsilon} = \mathbf{B}\mathbf{d} \qquad \text{where} \quad \mathbf{B} = \boldsymbol{\partial}\mathbf{N}$$

where **B** is the ***strain-displacement matrix***.

Constant Strain Triangle (CST)



$$u = \beta_1 + \beta_2 x + \beta_3 y$$
$$v = \beta_4 + \beta_5 x + \beta_6 y$$

- The node numbers sequence must go counter clockwise
- Linear displacement field so strains are ***constant!***

$$\varepsilon_x = \beta_2 \qquad \varepsilon_y = \beta_6 \qquad \gamma_{xy} = \beta_3 + \beta_5$$

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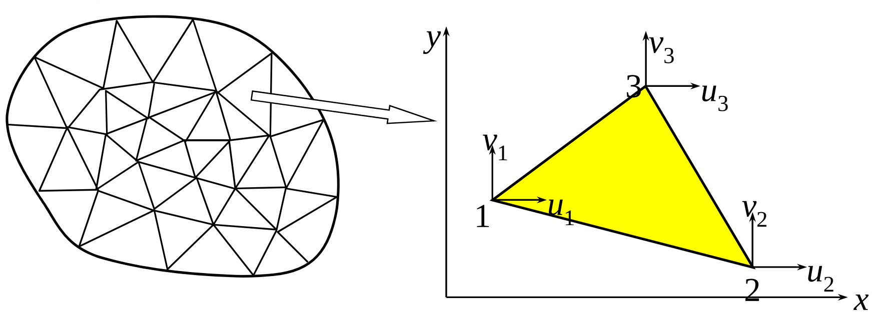
CST ELEMENT

- **Constant Strain Triangular** Element
  - Decompose two-dimensional domain by a set of triangles.
  - Each triangular element is composed by **three corner nodes**.
  - Each element shares its edge and two corner nodes with an adjacent element
  - Counter-clockwise or clockwise node numbering
  - **Each node has two DOFs: *u* and *v***
  - displacements interpolation using the shape functions and nodal displacements.
  - Displacement is linear because three nodal data are available.
  - Stress & strain are constant.



CST ELEMENT *cont.*

- Displacement Interpolation
  - Since two-coordinates are perpendicular, $u(x,y)$ and $v(x,y)$ are separated.
  - $u(x,y)$ needs to be interpolated in terms of u_1, u_2, and u_3, and $v(x,y)$ in terms of v_1, v_2, and v_3.
  - interpolation function must be a three term polynomial in x and y.
  - Since we must have rigid body displacements and constant strain terms in the interpolation function, the displacement interpolation must be of the form

$$\begin{cases} u(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y \\ v(x,y) = \beta_1 + \beta_2 x + \beta_3 y \end{cases}$$

  - The goal is how to calculate unknown coefficients α_i and β_i, i = 1, 2, 3, in terms of nodal displacements.

$$u(x,y) = N_1(x,y)u_1 + N_2(x,y)u_2 + N_3(x,y)u_3$$

CST ELEMENT *cont.*

- Displacement Interpolation
  - x-displacement: Evaluate displacement at each node

$$\begin{cases} u(x_1,y_1) \equiv u_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1 \\ u(x_2,y_2) \equiv u_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2 \\ u(x_3,y_3) \equiv u_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3 \end{cases}$$

  - In matrix notation

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

  - Is the coefficient matrix singular?

CST ELEMENT *cont.*

- Displacement Interpolation

$$\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{Bmatrix}=\begin{bmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\\ 1 & x_3 & y_3\end{bmatrix}^{-1}\begin{Bmatrix}u_1\\ u_2\\ u_3\end{Bmatrix}=\frac{1}{2A}\begin{bmatrix}f_1 & f_2 & f_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\\ u_3\end{Bmatrix}$$

  – where

$$\begin{cases} f_1 = x_2y_3 - x_3y_2, & b_1 = y_2 - y_3, \quad c_1 = x_3 - x_2\\ f_2 = x_3y_1 - x_1y_3, & b_2 = y_3 - y_1, \quad c_2 = x_1 - x_3\\ f_3 = x_1y_2 - x_2y_1, & b_3 = y_1 - y_2, \quad c_3 = x_2 - x_1\end{cases}$$

  – Area:

$$A=\frac{1}{2}\det\begin{vmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\\ 1 & x_3 & y_3\end{vmatrix}$$

CST ELEMENT *cont.*

$$\alpha_1=\frac{1}{2A}(f_1u_1+f_2u_2+f_3u_3)$$

$$\alpha_2=\frac{1}{2A}(b_1u_1+b_2u_2+b_3u_3)$$

$$\alpha_3=\frac{1}{2A}(c_1u_1+c_2u_2+c_3u_3)$$

- Insert to the interpolation equation

$$\begin{aligned}u(x,y)&=\alpha_1+\alpha_2x+\alpha_3y\\ &=\frac{1}{2A}\left[(f_1u_1+f_2u_2+f_3u_3)+(b_1u_1+b_2u_2+b_3u_3)x+(c_1u_1+c_2u_2+c_3u_3)y\right]\\ &=\boxed{\frac{1}{2A}(f_1+b_1x+c_1y)}u_1 \qquad N_1(x,y)\\ &\quad+\boxed{\frac{1}{2A}(f_2+b_2x+c_2y)}u_2 \qquad N_2(x,y)\\ &\quad+\boxed{\frac{1}{2A}(f_3+b_3x+c_3y)}u_3 \qquad N_3(x,y)\end{aligned}$$

CST ELEMENT *cont.*

- Displacement Interpolation
  - A similar procedure can be applied for *y*-displacement *v*(*x*, *y*).

$$u(x,y) = [N_1 \quad N_2 \quad N_3]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$v(x,y) = [N_1 \quad N_2 \quad N_3]\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

$$\begin{cases} N_1(x,y) = \dfrac{1}{2A}(f_1 + b_1 x + c_1 y) \\ N_2(x,y) = \dfrac{1}{2A}(f_2 + b_2 x + c_2 y) \\ N_3(x,y) = \dfrac{1}{2A}(f_3 + b_3 x + c_3 y) \end{cases}$$

Shape Function

  - N_1, N_2, and N_3 are linear functions of *x*- and *y*-coordinates.
  - Interpolated displacement changes linearly along the each coordinate direction.

CST ELEMENT *cont.*

- Displacement Interpolation
  - Matrix Notation

$$\{\mathbf{u}\} \equiv \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\boxed{\{\mathbf{u}(x,y)\} = [\mathbf{N}(x,y)]\{\mathbf{q}\}}$$

  - [**N**]: 2×6 matrix, {**q**}: 6×1 vector.
  - For a given point (x,y) within element, calculate [N] and multiply it with {q} to evaluate displacement at the point (x,y).

CST ELEMENT *cont.*

- Strain Interpolation
  - differentiating the displacement in x- and y-directions.
  - differentiating shape function [**N**] because {**q**} is constant.

$$\varepsilon_{xx} \equiv \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\sum_{i=1}^{3} N_i(x,y)u_i\right) = \sum_{i=1}^{3}\frac{\partial N_i}{\partial x}u_i = \sum_{i=1}^{3}\frac{b_i}{2A}u_i$$

$$\varepsilon_{yy} \equiv \frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(\sum_{i=1}^{3} N_i(x,y)v_i\right) = \sum_{i=1}^{3}\frac{\partial N_i}{\partial y}v_i = \sum_{i=1}^{3}\frac{c_i}{2A}v_i$$

$$\gamma_{xy} \equiv \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \sum_{i=1}^{3}\frac{c_i}{2A}u_i + \sum_{i=1}^{3}\frac{b_i}{2A}v_i$$

CST ELEMENT *cont.*

- Strain Interpolation

$$\{\varepsilon\} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \equiv [\mathbf{B}]\{\mathbf{q}\}$$

  - [**B**] matrix is a constant matrix and depends only on the coordinates of the three nodes of the triangular element.
  - the strains will be constant over a given element

Constant Strain Triangle (CST): Stiffness Matrix

- Strain-displacement relation, **ε=Bd**

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

A is the area of the triangle and $x_{ij}=x_i- x_j$. (textbook has results for a coordinate system with x aligned with side 1-2

- From the general formula $k = B^T EBtA$

  where *t:* element thickness (constant)

CST ELEMENT *cont.*

- Strain Energy: $U^{(e)} = \dfrac{h}{2}\iint\limits_A \{\varepsilon\}^T[\mathbf{C}]\{\varepsilon\}dA^{(e)}$

$$= \frac{h}{2}\{\mathbf{q}^{(e)}\}^T \iint\limits_A [\mathbf{B}]^T_{6\times3}[\mathbf{C}]_{3\times3}[\mathbf{B}]_{3\times6}\, dA\{\mathbf{q}^{(e)}\}$$

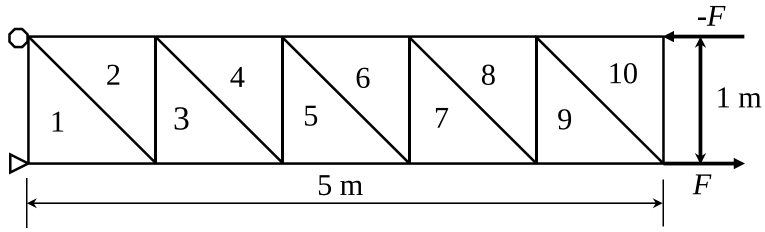
$$\equiv \frac{1}{2}\{\mathbf{q}^{(e)}\}^T[\mathbf{k}^{(e)}]_{6\times6}\{\mathbf{q}^{(e)}\}$$

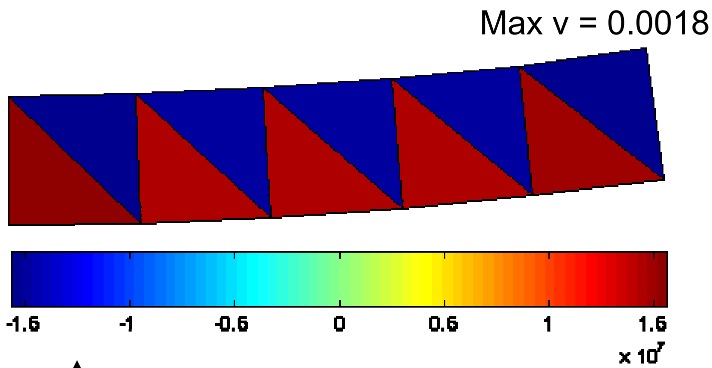
  – Element Stiffness Matrix: $\boxed{[\mathbf{k}^{(e)}] = hA[\mathbf{B}]^T[\mathbf{C}][\mathbf{B}]}$
  – Different from the truss and beam elements, transformation matrix **[T]** is not required in the two-dimensional element because **[k]** is constructed in the global coordinates.

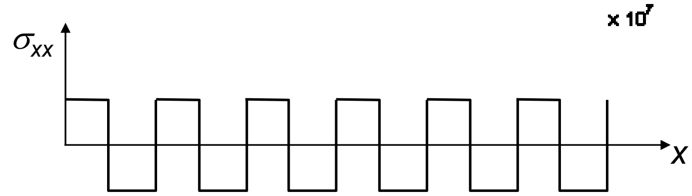
- The strain energy of the entire solid is simply the sum of the element strain energies

$$U = \sum_{e=1}^{NE} U^{(e)} = \frac{1}{2}\sum_{e=1}^{NE}\{\mathbf{q}^{(e)}\}^T[\mathbf{k}^{(e)}]\{\mathbf{q}^{(e)}\} \xRightarrow{\text{assembly}} U = \frac{1}{2}\{\mathbf{Q}_s\}^T[\mathbf{K}_s]\{\mathbf{Q}_s\}$$

BEAM BENDING EXAMPLE







- σ_{xx} is constant along the x-axis and linear along y-axis
- Exact Solution: σ_{xx} = 60 MPa
- Max deflection v_{max} = 0.0075 m

Linear Strain Triangle (LST)

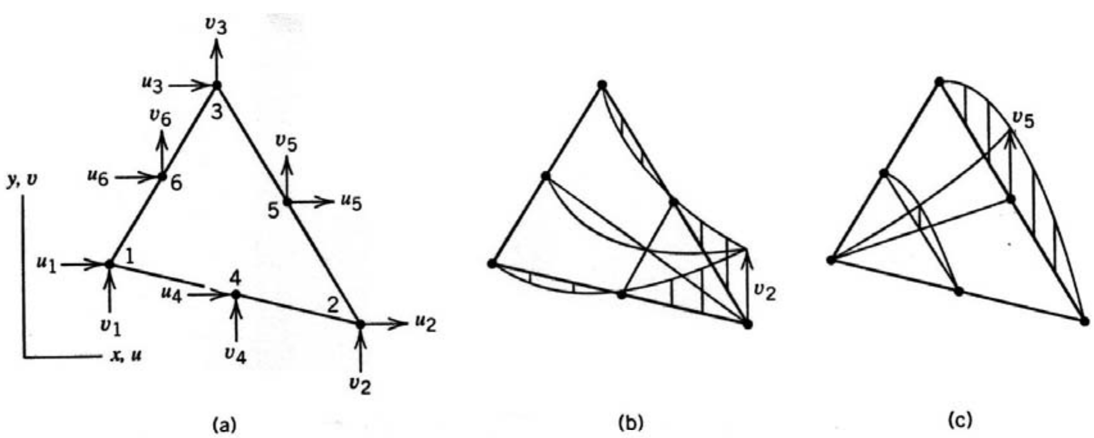


Fig. 3.3-1. (a) A linear strain triangle and its six nodal d.o.f. (b) Displacement mode associated with nodal d.o.f. v_2. (c) Displacement mode associated with nodal d.o.f. v_5. (For visualization only, imagine that displacement occurs normal to the plane of the element.) (*b* and *c* reprinted from [2.2] by permission of John Wiley & Sons, Inc.)

- The element has six nodes and 12 dof.

Linear Strain Triangle (LST)

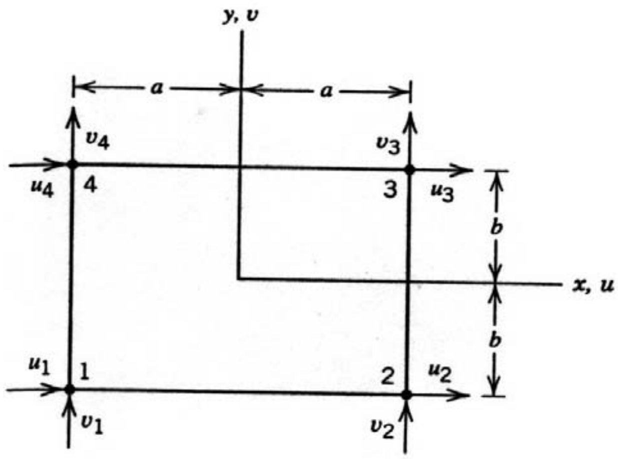
- The quadratic displacement field in terms of generalized coordinates:

$$u = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 x^2 + \beta_5 xy + \beta_6 y^2$$
$$v = \beta_7 + \beta_8 x + \beta_9 y + \beta_{10} x^2 + \beta_{11} xy + \beta_{12} y^2$$

- The linear strain field:

$$\varepsilon_x = \beta_2 + 2\beta_4 x + \beta_5 y$$
$$\varepsilon_y = \beta_9 + \beta_{11} x + 2\beta_{12} y$$
$$\gamma_{xy} = (\beta_3 + \beta_8) + (\beta_5 + 2\beta_{10})x + (2\beta_6 + \beta_{11})y$$

Bilinear Quadrilateral (Q4): CQUAD4 in NASTRAN



- Q4 element has ***four nodes*** and ***eight dof.***
- Can be ***quadrilateral***; but for now ***rectangle***.

- Displacement field:

$$u = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy$$
$$v = \beta_5 + \beta_6 x + \beta_7 y + \beta_8 xy$$

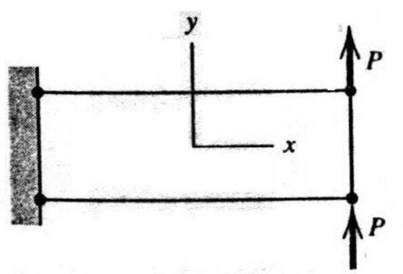
So, u and v are bilinear in x and y. Because of form, sides are stiffer than diagonals-artificial anisotropy!

Q4: The strain fields

- Strain field:

$$\varepsilon_x = \beta_2 + \beta_4 y$$
$$\varepsilon_y = \beta_7 + \beta_8 x$$
$$\gamma_{xy} = (\beta_3 + \beta_6) + \beta_4 x + \beta_8 y$$

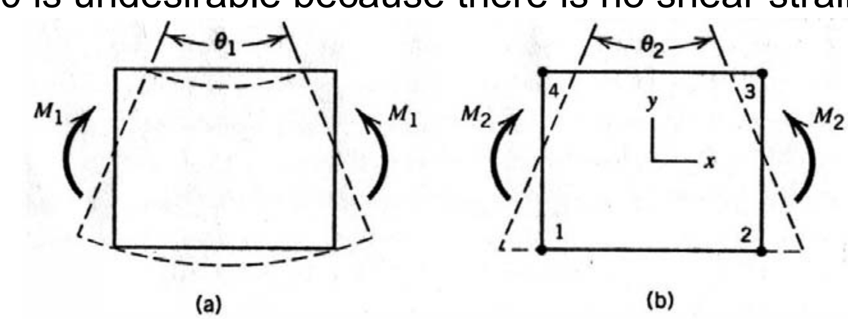
•***Observation 1:*** $\varepsilon_x \neq f(x) \Rightarrow$ Q4 cannot exactly model the beam where $\varepsilon_x \propto x$



A cantilever beam under transverse tip loading.

Q4: Behavior in Pure Bending of a Beam

• ***Observation 2:*** When $\beta_4 \neq 0$, ε_x varies linearly in y - desirable characteristic for a beam in pure bending because normal strain varies linearly along the depth coordinate. But $\gamma_{xy} \neq 0$ is undesirable because there is no shear strain.



• Fig. (a) is the correct deformation in pure bending while (b) is the deformation of Q4 ***(sides remain straight)***.

• Physical interpretation: applied moment is resisted by a spurious shear stress as well as flexural (normal) stresses.

Q4: Interpolation functions

• Easy to obtain interpolation functions

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \\ 0 & N_1 & 0 & N_2 & \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \end{Bmatrix} \quad \text{or} \quad \mathbf{u} = \mathbf{Nd}$$

where matrix **N** is 2x8 and the shape functions are

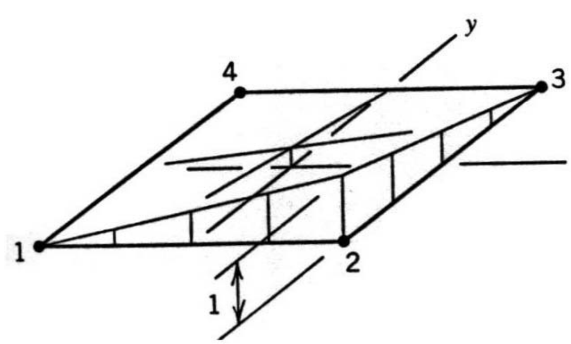
$$N_1 = \frac{(a-x)(b-y)}{4ab} \qquad N_2 = \frac{(a+x)(b-y)}{4ab}$$

$$N_3 = \frac{(a+x)(b+y)}{4ab} \qquad N_4 = \frac{(a-x)(b+y)}{4ab}$$

Q4: The Shape (Interpolation) Functions

• $N_1=1$, $N_2=N_3=N_4=0$ at node 1, x=-a, y=-b, so u= N_1 u_1= u_1 at that node.

• Similarly $N_i=1$ while all other Ns are zero at node i.



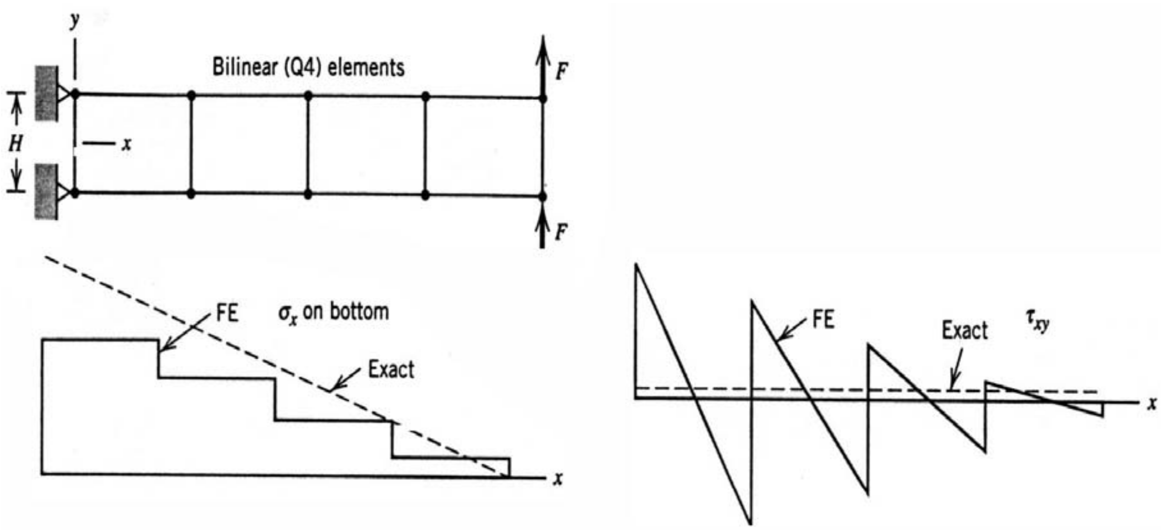
Shape function N_2 of the bilinear quadrilateral.(For visualization only, imagine that displacement occurs normal to the xy plane.)

•See Eqn. 3.6-6 for strain-displacement matrix ($\boldsymbol{\varepsilon}=\partial$ **Nd**=B**d**).

• All in all, Q4 converges properly with mesh refinement and works better than CST in most problems.

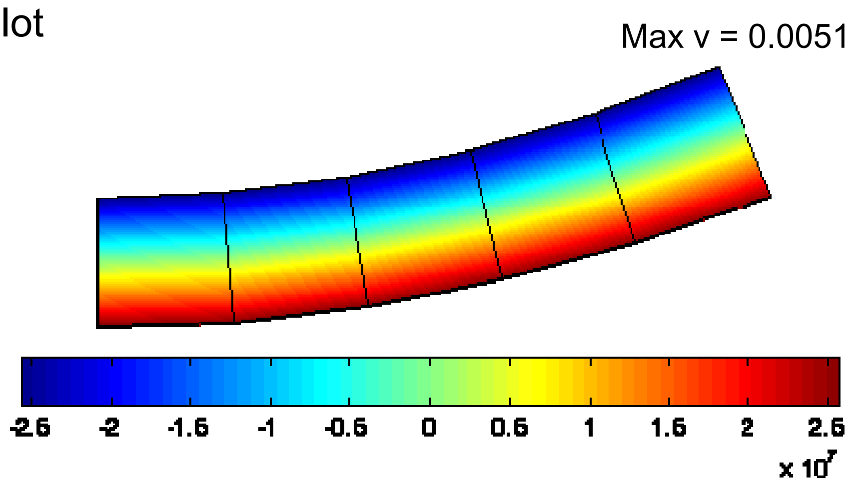
Coarse mesh results

• Q4 element is over-stiff in bending. For the following problem, deflections and flexural stresses are smaller than the exact values and the shear stresses are greatly in error:



BEAM BENDING PROBLEM *cont*.

- Sxx Plot



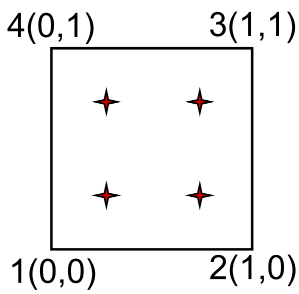
- Stress is constant along the x-axis (pure bending)
- linear through the height of the beam
- Deflection is much higher than CST element. In fact, CST element is too stiff. However, stress is inaccurate.

BEAM BENDING PROBLEM *cont*.

- Caution:
  - In numerical integration, we did not calculate stress at node points. Instead, we calculate stress at integration points.
  - Let's calculate stress at the bottom surface for element 1 in the beam bending problem.
  - Nodal Coordinates:1(0,0), 2(1,0), 3(1,1), 4(0,1)
  - Nodal Displacements:

    u = [0, 0.0002022, -0.0002022, 0]

    v = [0, 0.0002022, 0.0002022, 0]



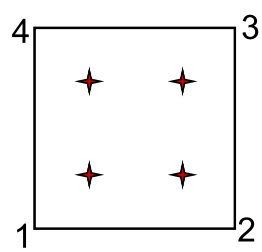
  - Shape functions and derivatives

$$
\begin{array}{lll}
N_1 = (x-1)(y-1) & \partial N_1/\partial x = (y-1) & \partial N_1/\partial y = (x-1) \\
N_2 = -x(y-1) & \partial N_2/\partial x = -(y-1) & \partial N_2/\partial y = -x \\
N_3 = xy & \partial N_3/\partial x = y & \partial N_3/\partial y = x \\
N_4 = -(x-1)y & \partial N_4/\partial x = -y & \partial N_4/\partial y = -(x-1)
\end{array}
$$

BEAM BENDING PROBLEM *cont*.

- At bottom surface, y = 0

$$
\begin{array}{ll}
\partial N_1/\partial x = -1 & \partial N_1/\partial y = x-1 \\
\partial N_2/\partial x = 1 & \partial N_2/\partial y = -x \\
\partial N_3/\partial x = 0 & \partial N_3/\partial y = x \\
\partial N_4/\partial x = 0 & \partial N_4/\partial y = -(x-1)
\end{array}
$$



u = [0, 0.0002022, -0.0002022, 0]

v = [0, 0.0002022, 0.0002022, 0]

- Strain

$$\varepsilon_{xx} = \sum_{I=1}^{4} \frac{\partial N_I}{\partial x} u_I = 1 \times 0.0002022$$

$$\varepsilon_{yy} = \sum_{I=1}^{4} \frac{\partial N_I}{\partial y} v_I = -0.0002022 \times x + 0.0002022 \times x = 0$$

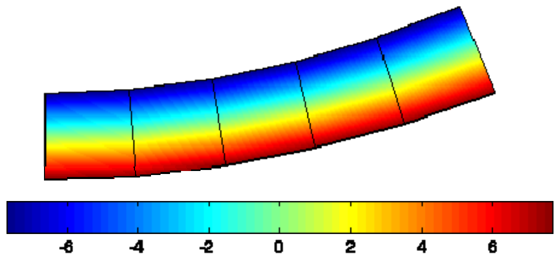
$$\gamma_{xy} = \sum_{I=1}^{4} \left(\frac{\partial N_I}{\partial x} v_I + \frac{\partial N_I}{\partial y} u_I \right) = 0.0002022 - 0.0004044x$$

- Stress:

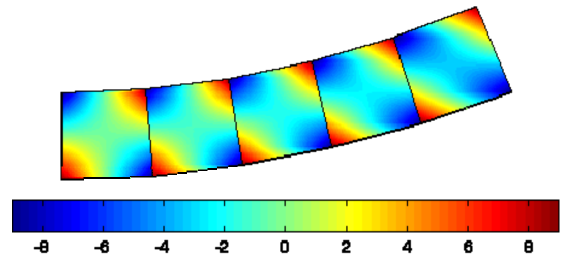
$$\{\sigma\} = [\mathbf{C}]\{\varepsilon\} = \{4.44,\ 1.33,\ 1.55\} \times 10^7$$

RECTANGULAR ELEMENT

- y-normal stress and shear stress are supposed to be zero.



σ_{yy} Plot



τ_{xy} Plot

ε_{xx} is a linear function of y alone

ε_{yy} if a linear function of x alone

γ_{xy} is a linear function of x and y

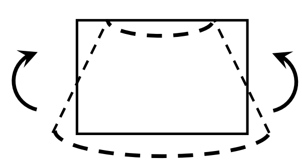
$$\varepsilon_{xx} = \sum_{I=1}^{4} \frac{\partial N_I}{\partial x} u_I \quad \varepsilon_{yy} = \sum_{I=1}^{4} \frac{\partial N_I}{\partial y} v_I$$

$$\partial N_1 / \partial x = (y-1) \qquad \partial N_1 / \partial y = (x-1)$$
$$\partial N_2 / \partial x = -(y-1) \qquad \partial N_2 / \partial y = -x$$
$$\partial N_3 / \partial x = y \qquad \partial N_3 / \partial y = x$$
$$\partial N_4 / \partial x = -y \qquad \partial N_4 / \partial y = -(x-1)$$

RECTANGULAR ELEMENT

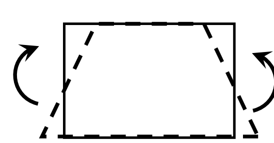
- Discussions
  - Can't represent constant shear force problem because ε_{xx} must be a linear function of *x*.
  - Even if ε_{xx} can represent linear strain in y-direction, the rectangular element can't represent pure bending problem accurately.
  - Spurious shear strain makes the element too stiff.

$$\boxed{\begin{aligned} u &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy \\ v &= \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy \end{aligned}}$$



Exact

$$\varepsilon_{xx} = \alpha_2 + \alpha_4 y$$
$$\varepsilon_{yy} = \beta_3 + \beta_4 x$$
$$\gamma_{xy} = (\alpha_3 + \beta_2) + \alpha_4 x + \beta_4 y$$

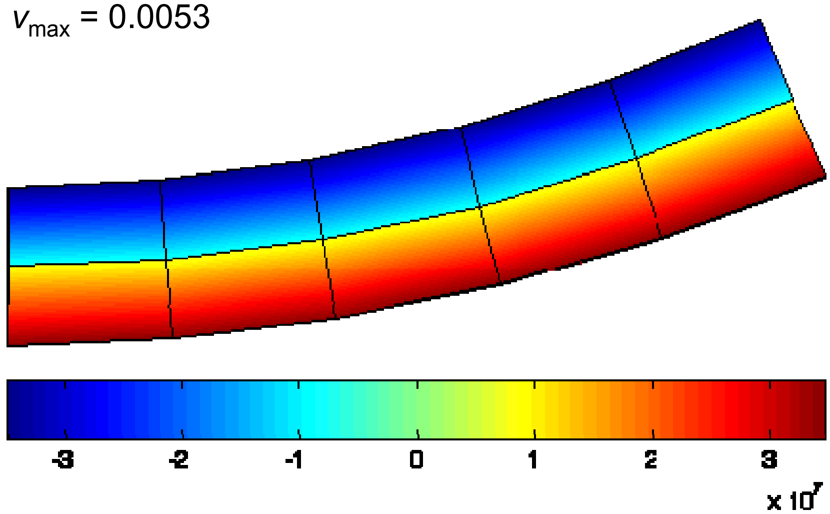


Rectangular element

$\alpha_4 \neq 0$

RECTANGULAR ELEMENT

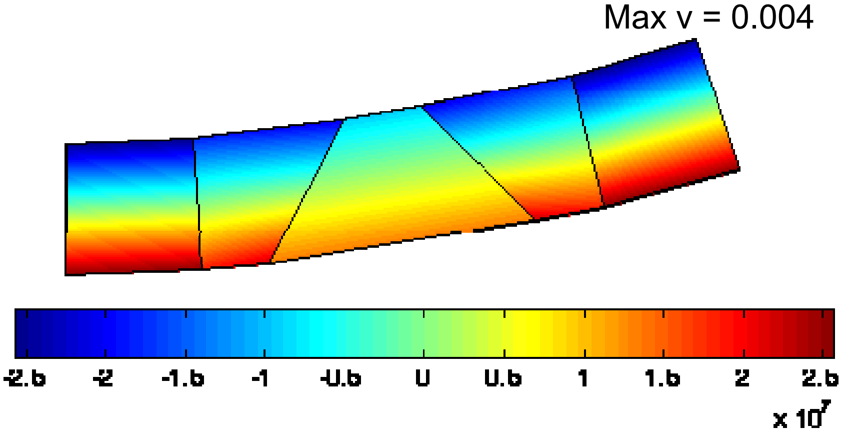
- Two-Layer Model
  - σ_{xx} = 3.48×10^{7}
  - v_{max} = 0.0053



BEAM BENDING PROBLEM *cont.*

- Distorted Element

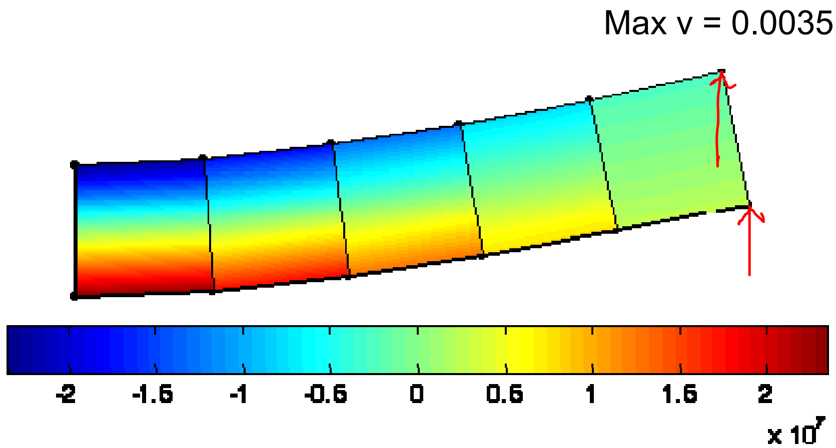
Max v = 0.004



- As element is distorted, the solution is not accurate any more.

BEAM BENDING PROBLEM *cont.*

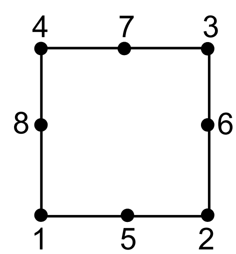
- Constant Shear Force Problem



- Sxx is supposed to change linearly along x-axis. But, the element is unable to represent linear change of stress along x-axis. Why?
- Exact solution: v = 0.005 m and σ_{xx} = 6e7 Pa.

BEAM BENDING PROBLEM *cont.*

- Higher-Order Element?
  - 8-Node Rectangular Element



$$u(x,y) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 + a_6 x^2 y + a_7 xy^2$$

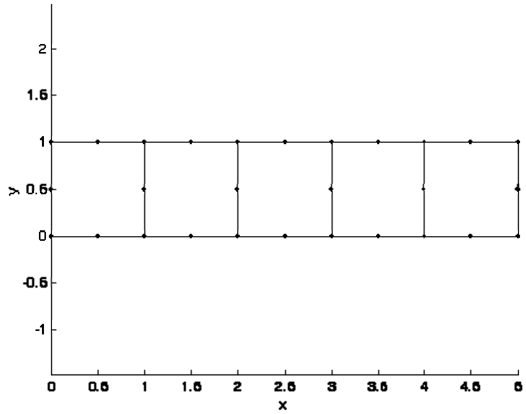
  - Strain

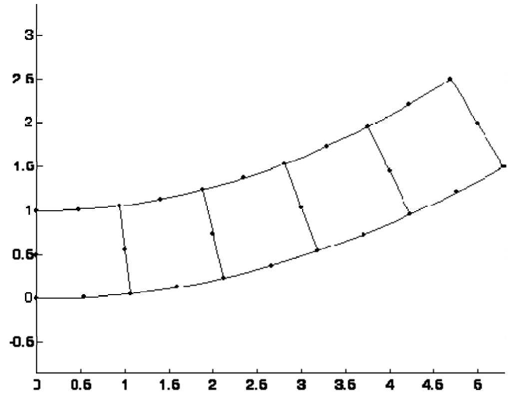
$$\frac{\partial u(x,y)}{\partial x} = a_1 + 2a_3 x + a_4 y + 2a_6 xy + a_7 y^2$$

  - Can this element accurately represent pure bending and constant shear force problem?

BEAM BENDING PROBLEM *cont*.

- 8-Node Rectangular Elements

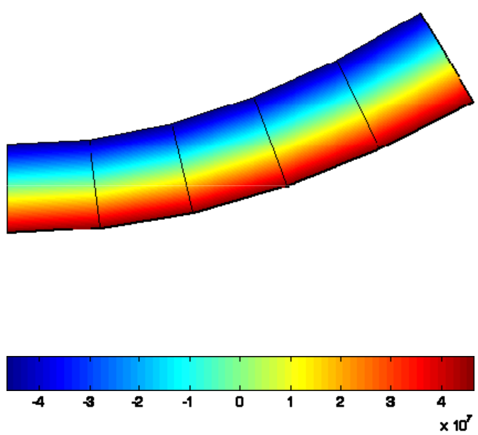




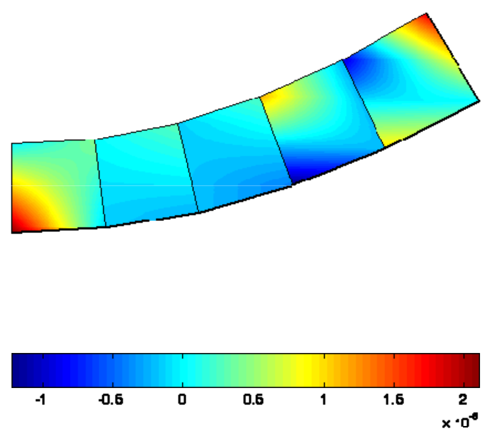
- Tip Displacement = 0.0075 m,   Exact!

BEAM BENDING PROBLEM *cont*.

- If the stress at the bottom surface is calculated, it will be the exact stress value.



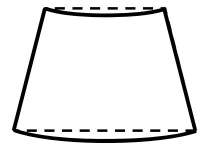
Sxx

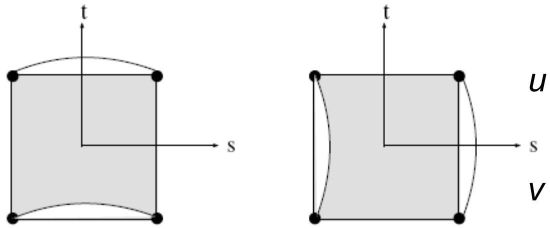


Syy

Q6: Additional Bending Shape Functions

- Q4: Artificial shear deformation under pure bending
- Additional shape functions to solve the issue





$N_5 = 1 - s^2$ $N_6 = 1 - t^2$

$$u = \sum_{i=1}^{4} N_i(s,t)u_i + (1-s^2)a_1 + (1-t^2)a_2$$

$$v = \sum_{i=1}^{4} N_i(s,t)v_i + \underbrace{(1-s^2)a_3 + (1-t^2)a_4}_{\text{Bubble modes}}$$

- Strain ε_{xx} can vary linear along x-dir.
- Shear strain γ_{xy} can vanish for pure bending
- Nodeless DOFs, a_1, a_2, a_3, and a_4, are condensed in the element level (total 12 DOFs)

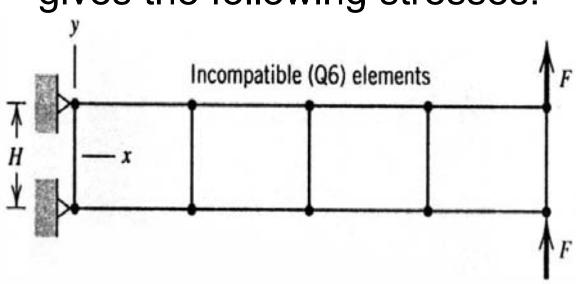
Modeling Bending with the Q6 Element

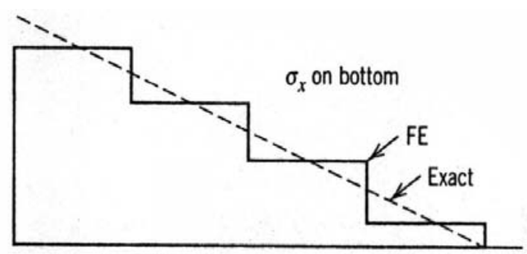
$$\begin{bmatrix} \mathbf{K}_{dd} & \mathbf{K}_{da} \\ \mathbf{K}_{ad} & \mathbf{K}_{aa} \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ \mathbf{a} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_d \\ \mathbf{f}_a \end{Bmatrix}$$

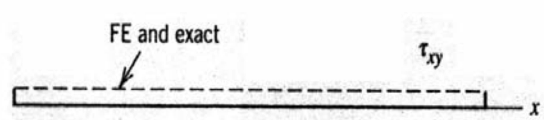
$$\{\mathbf{a}\} = \mathbf{K}_{aa}^{-1}\{\mathbf{f}_a - \mathbf{K}_{ad}\mathbf{d}\}$$

$$[\mathbf{K}_{dd} - \mathbf{K}_{da}\mathbf{K}_{aa}^{-1}\mathbf{K}_{ad}]\{\mathbf{d}\} = \{\mathbf{f}_d - \mathbf{K}_{da}\mathbf{K}_{aa}^{-1}\mathbf{f}_a\}$$

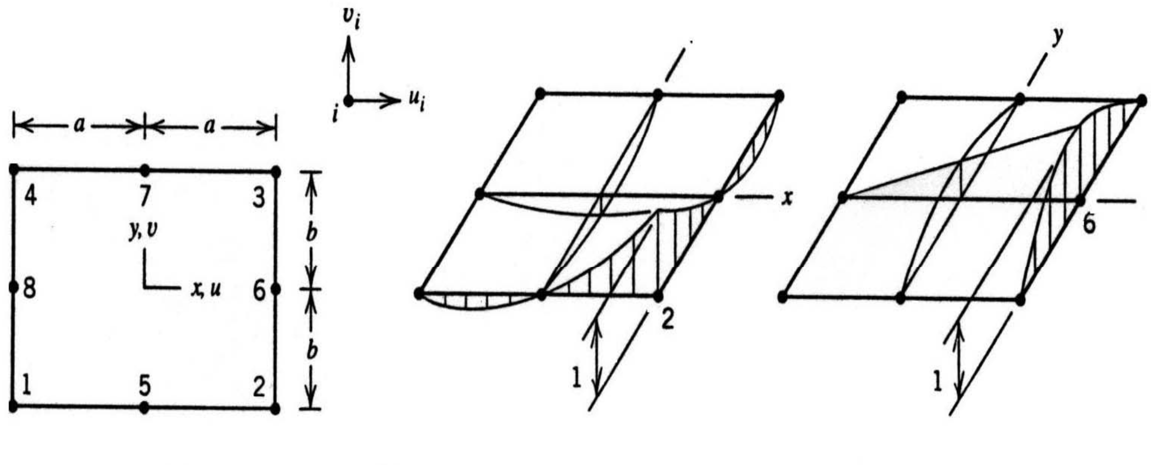
•Modeling the previous bending problem with Q6 elements gives the following stresses:







Quadratic Quadrilateral (Q8)



(a) (b) (c)

Fig. 3.7-1 A quadratic quadrilateral. (b,c) Shape functions N_2 and N_6. (For visualization only, imagine that displacement occurs normal to the *xy* plane.)

•***4 corner nodes*** and ***4 side nodes*** and ***16 nodal dof***.

Quadratic Quadrilateral (Q8): Displacement

•The displacement field, which is quadratic in x and y:

$$u = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 x^2 + \beta_5 xy + \beta_6 y^2 + \beta_7 x^2 y + \beta_8 xy^2$$
$$v = \beta_9 + \beta_{10} x + \beta_{11} y + \beta_{12} x^2 + \beta_{13} xy + \beta_{14} y^2 + \beta_{15} x^2 y + \beta_{16} xy^2$$

• Two types of shape functions (ξ=x/a, η=y/b) :

$$N_2 = \tfrac{1}{4}(1+\xi)(1-\eta) - \tfrac{1}{4}(1-\xi^2)(1-\eta) - \tfrac{1}{4}(1+\xi)(1-\eta^2)$$
$$N_6 = \tfrac{1}{2}(1+\xi)(1-\eta^2)$$

•**The edges** x=±a **deform into a parabola** (i.e., quadratic displacement in y) (same for y=±b)

Quadratic Quadrilateral (Q8): Strains

• The strain field:

$$\varepsilon_x = \beta_2 + 2\beta_4 x + \beta_5 y + 2\beta_7 xy + \beta_8 y^2$$
$$\varepsilon_y = \beta_{11} + \beta_{13} x + 2\beta_{14} y + \beta_{15} x^2 + 2\beta_{16} xy$$
$$\gamma_{xy} = (\beta_3 + \beta_{10}) + (\beta_5 + 2\beta_{12})x + (2\beta_6 + \beta_{13})y + \beta_7 x^2 + 2(\beta_8 + \beta_{15})xy + \beta_{16} y^2$$

• Strains have linear and quadratic terms. Hence, Q8 can represent many strain states exactly.

For example, states of constant strain, bending strain, etc.

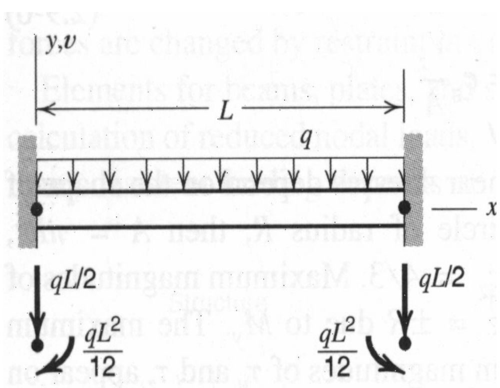
3.11 Nodal loads

- Consistent (work-equivalent) loads

$$W = \mathbf{d}^{\mathrm{T}}\mathbf{r}_e = \int \mathbf{u}^{\mathrm{T}}\mathbf{F}\,dV + \int \mathbf{u}^{\mathrm{T}}\mathbf{\Phi}\,dS$$

- ***Mechanical loads***: concentrated loads, surface traction, body forces.

Example: Beam under uniform loads



- Normal forces are obvious. For moments

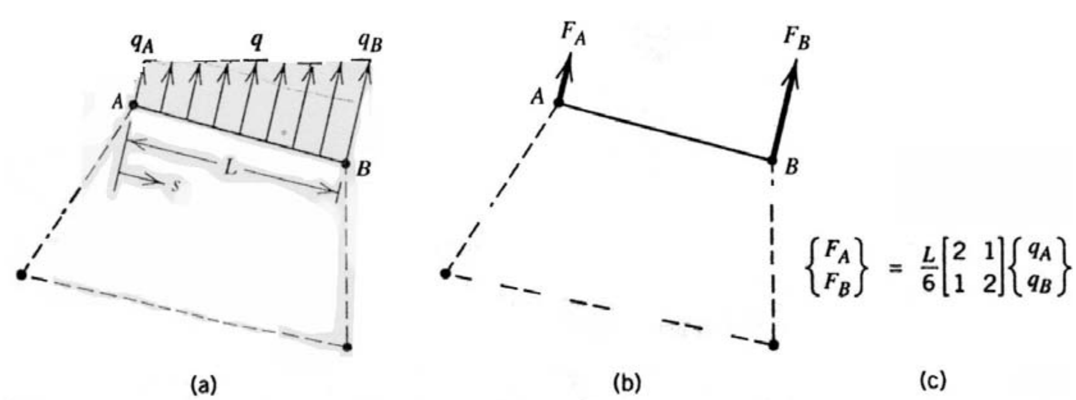
$$M_1\theta_1 =$$

$$\int_0^L qN_2\theta_1 dx = \int_0^L q\left(x - \frac{2x^2}{L} + \frac{x^3}{L^2}\right)dx$$

$$= qL^2\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{qL^2}{12}$$

Work equivalent (consistent) normal loads

•***Normal surface traction*** on a side of a plane element whose sides remain straight (q is force/length):

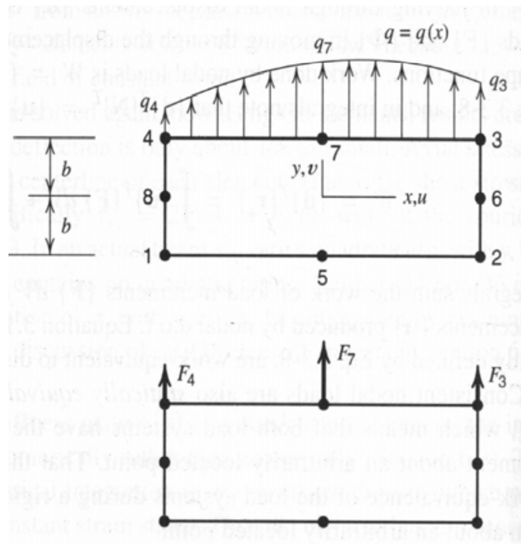


(a) Linearly varying distributed load on a linear-displacement edge.
(b,c) Work-equivalent nodal loads.

Loads on Quadratic sides

*

$$\begin{Bmatrix} F_4 \\ F_7 \\ F_3 \end{Bmatrix} = \int_{-a}^{a} [N]^T [N]\, dx \begin{Bmatrix} q_4 \\ q_7 \\ q_3 \end{Bmatrix} = \frac{a}{15} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} \begin{Bmatrix} q_4 \\ q_7 \\ q_3 \end{Bmatrix}$$



Distributed Shear Traction

- ***Shear traction*** on a side of a plane element whose sides remain straight (q is force/length):

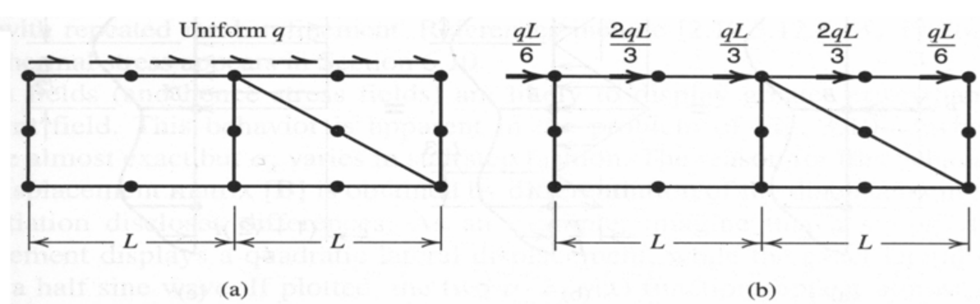
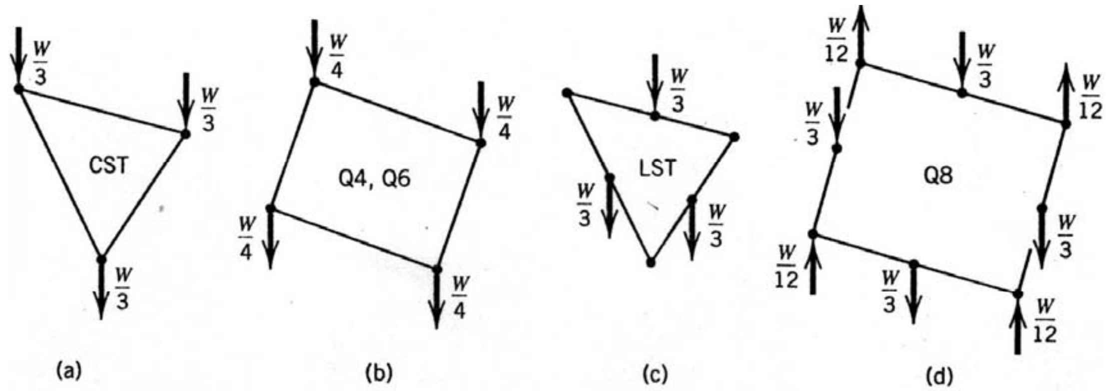


Figure 3.11-2. Allocation of uniformly distributed side-tangent load to uniformly spaced nodes.

- In (b), a Q4 element and two LSTs share the top midnode so that the nodal loads from Q4 and the right LST are combined.

Uniform Body Force

- Work-equivalent nodal forces corresponding to weight as a body force:



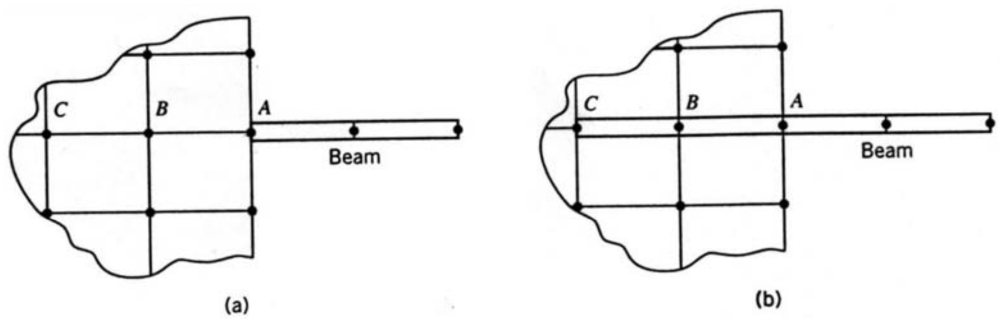
Work-equivalent nodal forces associated with element weight W, for triangular and rectangular quadrilateral elements.

- LST has no vertex loads and vertex loads of Q8 are upwards!
- The resultant in all cases is W, the weight of the element.

Connecting beam and plane elements

• Since the previous plane elements have translational dof only, no moment can be applied to their nodes.

• The connection (a) of a beam and plane elements cannot transmit a moment and the beam can freely rotate. (Singular **K**!)

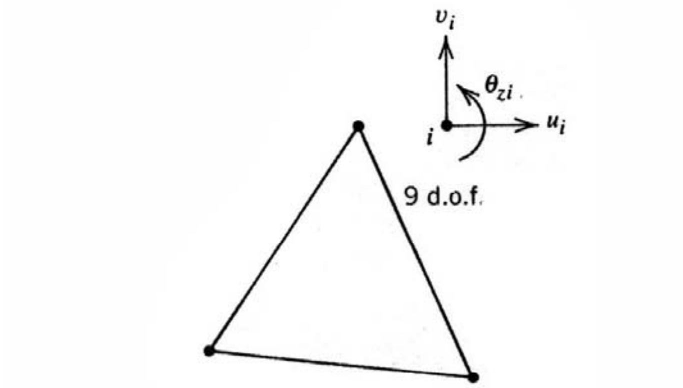


...... Connecting a 2D beam element to plane elements. (a) No moment is transferred. (b) Moment is transferred.

•In (b) the beam is extended. Rotational dof at A, B and C are associated with the beam elements only. A plane element with drilling dof would also work but is not recommended.

Elements with Drilling DOF

- **Drilling dof**: rotational dof about axis normal to the plane.
- A CST with these added to each node has 9 dof.
- This dof allows twisting and bending rotations of shells under some loads to be represented. See Section 3.10



The d.o.f. in a triangular element with drilling d.o.f.

Stress calculation

- Combining Hooke's law with strain-displacement equation

$$\boldsymbol{\sigma} = E\left(B\mathbf{d} - \boldsymbol{\varepsilon}_0\right) \qquad e.g. \quad \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\left(B\mathbf{d} - \begin{Bmatrix} \alpha T \\ \alpha T \\ 0 \end{Bmatrix} \right)$$

- Stresses are most accurate inside elements

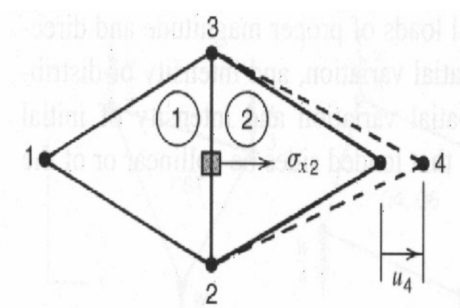


Figure 3.12-1. Stress appears in element 2 but not in element 1. The differential element (shaded) spans the interelement boundary.

Improving stresses at nodes and boundaries

- One common technique is averaging, but

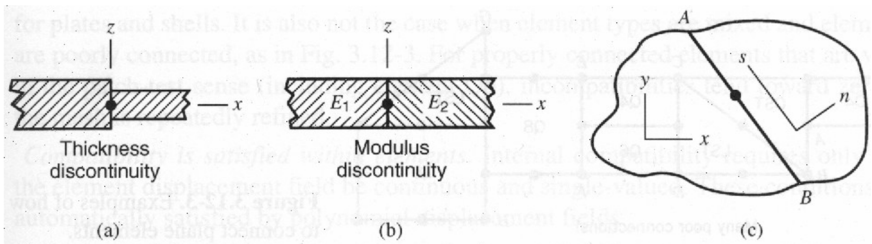


Figure 3.12-2. Some situations in which stresses should not be averaged at a node. (a,b) Plane elements seen in cross section, with Cartesian coordinates *xyz*. (c) Plane elements seen in plan view, with interelement boundary *AB*.

- There are interpolation and extrapolation techniques that we will study later

Examples of poor meshing

- Do not create unnecessary discontinuities!

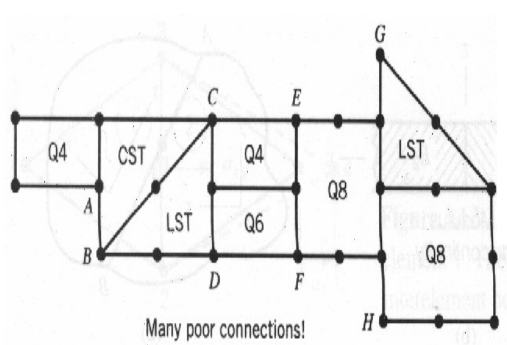


Figure 3.12-3. Examples of how *not* to connect plane elements.

Example of discontinuities in un-averaged stresses

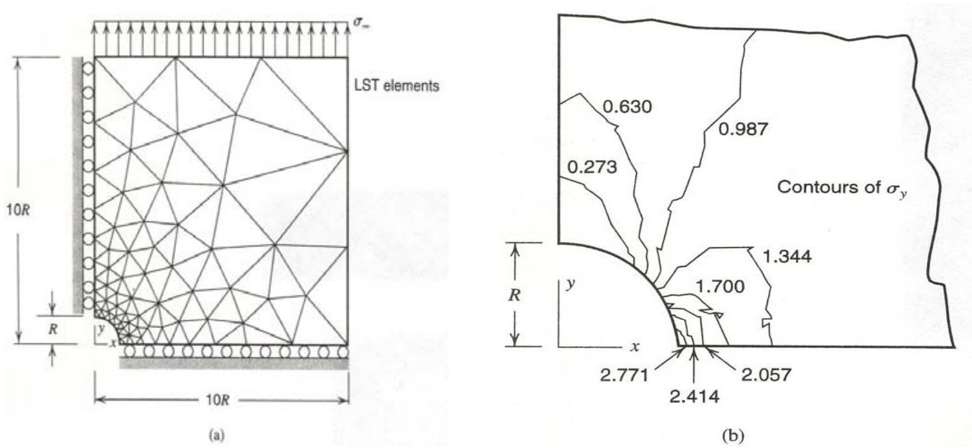


Figure 3.14-1. (a) FE domain, mesh, and boundary conditions for modeling a hole in an infinite plate. (b) Unaveraged contours of σ_y from a portion of the mesh in part (a).