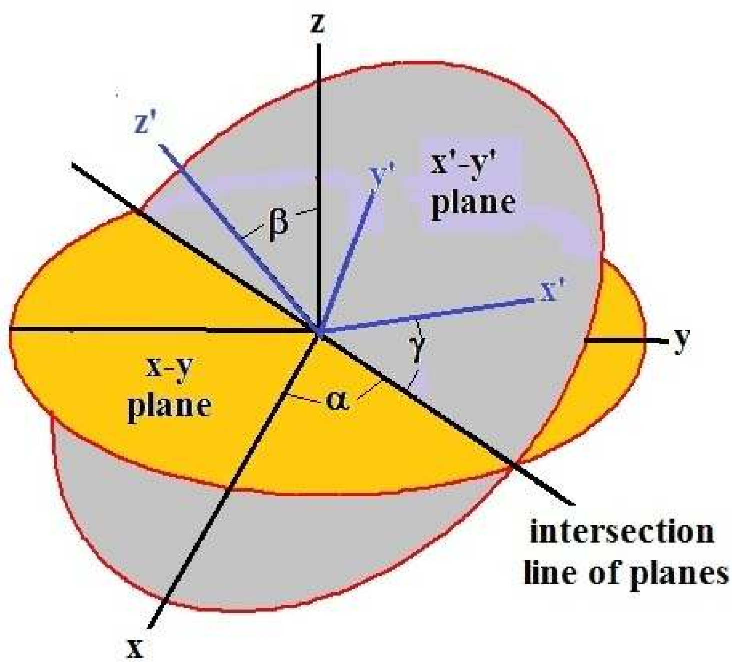
EULER ANGLES AND 3D ROTATIONS BY MATRIX METHODS

In the previous note we discussed how to rotate figures in 2D using a standard 2x2 Rotation Matrix involving just a single angle θ. In 3D the rotation problem becomes more complicated since it will now generally involve three independent angles α, β, and γ in order to uniquely specify how two orthogonal Cartesian coordinate systems[x,y,z] and [x',y',z'] with a common origin relate to each other. One way to relate the two coordinate systems is by the use of Euler Angles which are denoted in the literature by either[φ, θ,ψ] or [α, β, γ]. Their definitions are as shown in the following graph-

DEFINITION OF EULER ANGLES



The first Euler Angle α is measured by a counterclockwise rotation about the z axis of the x axis. This produces an intersection line between the old x-y plane and the new x'-y' plane. Next we define the second angle β as the angle produced by a counterclockwise rotation about the intersection line of the z axis. Finally a third Euler Angle γ is the angle between the intersection line and the new x' coordinate.

Although these Euler Angles can always be used to find the image of point of P(x,y,z) in the new coordinate system, it is often easier to just simply use a bit of mental visualization and make one or two successive rotations using standard 3x3 Rotation Matrixes. This second approach (which is really a disguised form of Euler Angles) involves the three Rotation Matrixes-

$$M_x(\theta)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos(\theta) & -\sin(\theta)\\ 0 & \sin(\theta) & \cos(\theta)\end{bmatrix},\ M_y(\theta)=\begin{bmatrix}\cos(\theta) & 0 & \sin(\theta)\\ 0 & 1 & 0\\ -\sin(\theta & 0 & \cos(\theta)\end{bmatrix}\ and\ M_z(\theta)=\begin{bmatrix}\cos(\theta) & -\sin(\theta) & 0\\ \sin(\theta) & \cos(\theta) & 0\\ 0 & 0 & 1\end{bmatrix}$$

Here the subscript indicates the axis about which one is rotating. The rotation angle θ is considered positive when measured in a counterclockwise manner when looking toward the coordinate origin. Note the sign change in the sin(θ) term in the $M_y(\theta)$ matrix.

To demonstrate the rotation procedure consider a standard cube of side-length 2 centered on the origin and whose faces are parallel to the coordinate axes. Suppose we wish to rotate this cube in such a way that the front vertex at [1,1,1] ends up along the vertical z axis and the z axis coincides with the diagonal through this rotated cube. We should be able to accomplish this by two successive rotations as we now show. First rotate the cube about the x axis by π/4 rad. The matrix procedure is as follows-

$$M_x X=\begin{bmatrix}1 & 0 & 0\\ 0 & 1/\sqrt{2} & -1/\sqrt{2}\\ 0 & 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}1\\0\\\sqrt{2}\end{bmatrix}=X'$$

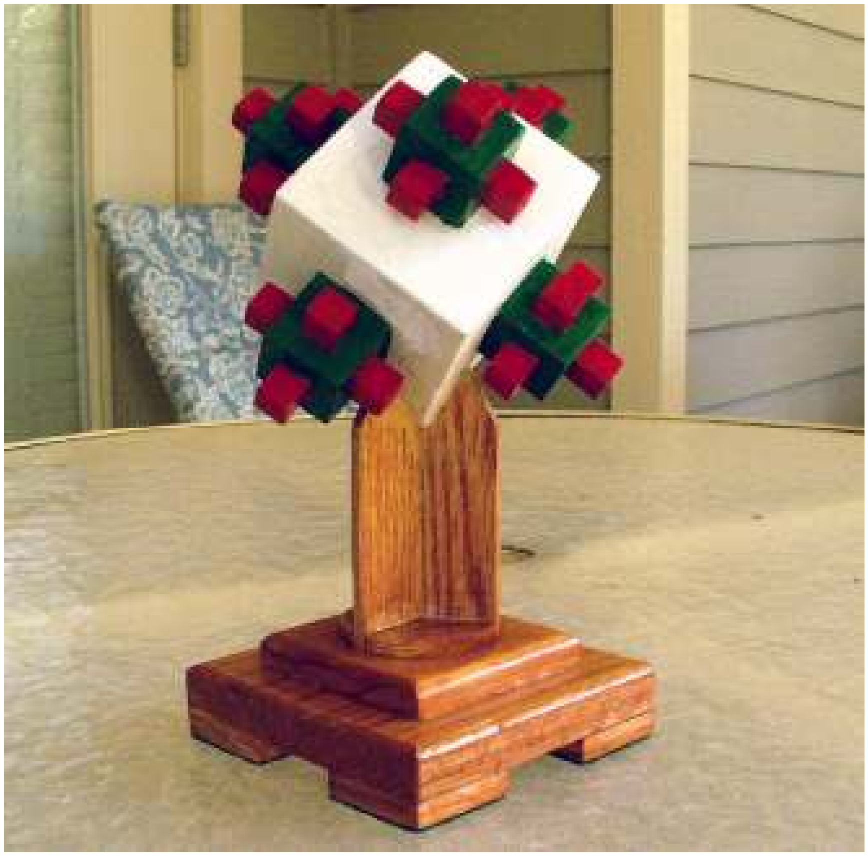
It converts point [1,1,1] to [1,0, sqrt(2)]. Next we perform a second rotation of θ= -arctan(1/sqrt(2) about the y axis which should bring the vertex point onto the z axis. The second rotation, this time about the y axis, yields-

$$M_y X'=\begin{bmatrix}\sqrt{2/3} & 0 & -1/\sqrt{3}\\ 0 & 1 & 0\\ 1/\sqrt{3} & 0 & \sqrt{2/3}\end{bmatrix}\begin{bmatrix}1\\0\\\sqrt{2}\end{bmatrix}=\begin{bmatrix}0\\0\\\sqrt{3}\end{bmatrix}=X''$$

This result means that the original vertex of the cube at [1,1,1] is now located at [0,0,sqrt(3)]. By replacing the matrix [1,1,1] by [x,y,z] we can readily find the transformations for the remaining seven vertexes of the cube. We summarize things in the following table-

[x,y,z]	[x',y',z'], θ=π/4	[x",y",z"], θ=arctan[1/sqrt(2)]
[1,1,1]	[1,0,sqrt(2)]	[0,0,sqrt(3)]
[1,1,-1]	[1,sqrt(2),0]	[sqrt(2/3),sqrt(2),1/sqrt(3)]
[-1,1,-1]	[-1,0,sqrt(2)]	[-2sqrt(2/3),0,1/sqrt(3)]
[-1,1,1]	[-1, 0,sqrt(2)]	[0,0,sqrt(3)]
[-1,-1,1]	[1,-sqrt(2),0]	[sqrt(2/3),-sqrt(2),1/sqrt(3)]
[1,-1,1]	[1,-sqrt(2),0]	[sqrt(2/3),-sqrt(2),1/sqrt(3)]
[1,-1,-1]	[1,0,-sqrt(2)]	[2sqrt(2/3),0,-1/ssqrt(3)]
[-1,-1,-1]	[-1,0,sqrt(2)	[0,0,-sqrt(3)]

A practical application of the results in this table is finding the angle a support column should be cut to fit a cube snugly under conditions where the cube's diagonal points straight upward. I ran into this problem several years ago while constructing a fractal cube in my workshop. Here is a photo of the finished wooden configuration-



The angle along an edge can be determined by taking the dot product between vectors V_1=i[sqrt(2/3)]-j[sqrt(2)]+k[1/sqrt(3)-sqrt(3)] and V_2=-k. It yields-

$$\theta_{edge} = \cos^{-1}\left\{\frac{(\sqrt{3}-\frac{1}{\sqrt{3}})}{\sqrt{\frac{2}{3}+2+(\frac{1}{\sqrt{3}}-\sqrt{3})^2}}\right\} = \cos^{-1}(\frac{1}{\sqrt{3}}) = 54.7356 \text{ deg}$$

with respect to the vertical. When determining the column cut for a snug fit to a face, as we have done in the model shown, the angle becomes smaller with a value of-

$$\theta_{face} = \cos^{-1}\left\{\frac{(\sqrt{3}-\frac{1}{\sqrt{3}})}{\sqrt{\frac{2}{3}+(\sqrt{3}-\frac{1}{\sqrt{3}})^2}}\right\} = \cos^{-1}(\frac{\sqrt{2}}{\sqrt{3}}) = 35.2643 \text{ deg}$$

I remember at the time getting this angle by trial and error using a piece of cardboard and scissors.

In the above rotation calculations one can combine the separate matrix operations into a single evaluation which, for three successive rotations about each of the three axes, would read-

$$M_z M_x M_y X = X'''$$

As an example, consider what happens to the point P[2,0,0] upon three successive rotations about the x, y and then z axis. We take the rotation angle to be θ=π/4 for each rotation. This produces-

$$\begin{bmatrix} a & -a & 0 \\ a & a & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & 0 & a \\ 0 & 1 & 0 \\ -a & 0 & a \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & a & -a \\ 0 & a & a \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -\sqrt{2} \end{bmatrix} \; where\ a = 1/sqrt(2)$$

That is, the three rotations move point [2,0,0] to [1,1,-sqrt(2)]. Note that these operations do not commute. If one were to take the different operation order $M_y M_z M_x X$ the final point would be located at [1,sqrt(2),-1].

As a final example, we consider the plane x+y+z=1 and try to cast into a form where the plane becomes oriented parallel to the x-y plane. To accomplish this we first rotate things about the x axis with an angle for which tan(θ)=dz/dy=-1. This rotation produces-

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ (y-z)/\sqrt{2} \\ (y+z)/\sqrt{2} \end{bmatrix} = \begin{bmatrix} x \\ (1-x-2z)/\sqrt{2} \\ (1-x)/\sqrt{2} \end{bmatrix}$$

for any point on the plane. The result tells us that the next rotation should be about the y axis with the angle θ= –arctan(1/sqrt(2). Doing so yields the two rotation result-

$$\begin{bmatrix} \sqrt{2/3} & 0 & 1/\sqrt{3} \\ 0 & 1 & 0 \\ 1/\sqrt{3} & 0 & \sqrt{2/3} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (2x-y-z)/\sqrt{6} \\ (y-z)/\sqrt{2} \\ 1/\sqrt{3} \end{bmatrix}$$

for any point on the plane. The result clearly shows that we have a new plane z=1/sqrt(3) which is indeed parallel to the x-y plane.

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