PROPERTIES OF THE POSITIVE INTEGERS

The first introduction to mathematics occurs at the pre-school level and consists of essentially counting out the first ten integers with one's fingers. This allows the individuals to slowly recognize the abstract character of whole numbers given by the sequence-

R={1,2,3,4,5,6,7,8,9,10,11,….2n,2n+1}   with n=1,2,3,etc

The kinder-gardeners soon recognize that two apples plus three apples makes five apples and that apples can be replaced by anything else and the answer will still be five of the anything else. As the years pass the elementary school children learn about the concepts of addition, subtraction, division, and multiplication of integers and non-integers. Certainly by the time they reach middle school they will be able to manipulate collection of integers to obtain equivalent expressions. They will also recognize the difference between even (2n) and odd (2n+1) integers and the concept of zero. By the time high school rolls around students will have mastered the concepts of algebra and introductory calculus all arising from their early experience of number counting by fingers. In addition many of the high school students will have noted that the integers can be broken into prime numbers and composite number. A prime number refers to any integer which can be divided only by itself and one while a composite number consists of the product of several primes taken to specified powers. The above grouping of integers N breaks into the prime sequence-

P={2,3,5,7,11,13,19,23,29,31,37,41,43,47,…}

and the composite sequence-

C={0,4,6,8,9,12,14,15,16,18,20,21,22,24,25,26,…}

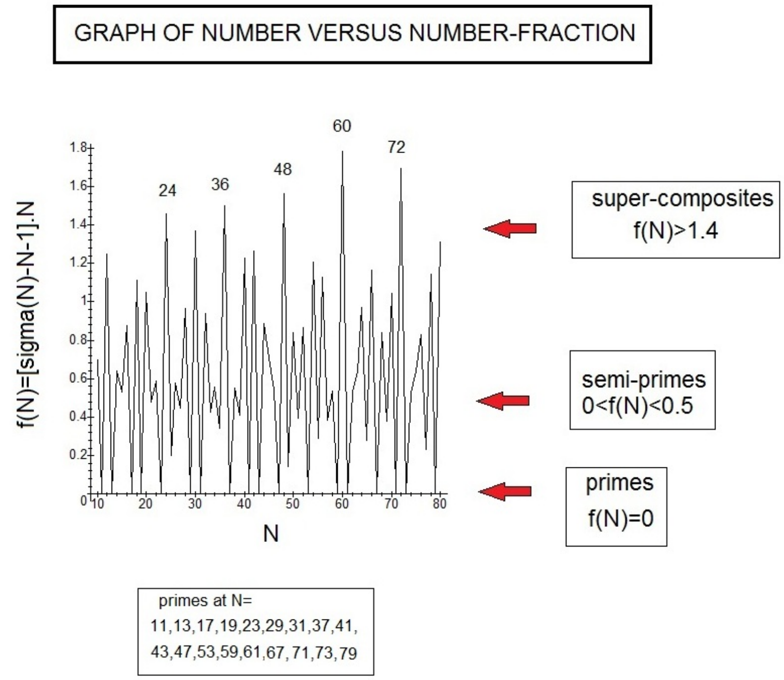
If we neglect the first two integers in the P sequence, we note that all the remaining integers are odd and have the generic form 6n+1 or 6n-1, so that for instance 41=6(7)-1 and 19=6(3)+1. Thus one can state that –

A necessary but not sufficient condition that a number N>3 be a prime is that 6n±1

The reason this law is not sufficient is that certain composite numbers such as 25 also satisfies 6(4)+1. An improved way to identify primes is by means of the number fraction-

$$f(N)=\frac{\{sum\ of\ all\ divisors\ of\ N-excluding\ N\ and\ 1]\}}{N}=\frac{[\sigma(N)-N-1]}{N}$$

We first found this point function about a decade ago. It is easy to evaluate since the sigma function σ(n) of number theory is a well known quantity in most mathematics computer programs. The interesting property of f(N) is that it vanishes only when N is a prime , but remains positive when N is a compound number. Because of the presence of N in the denominator of its definition its value remains on average below f(N)=1. The few f(N)s which exceed f(N) of about 1.4 I have termed super-composites . They typically contain products of 2 and 3 taken to specified powers. A graph of the number-fraction in the range 10<N<80 follows-



We see that only the prime numbers have vanishing f(N). The remaining points are either super-composites such as N=48, 60, and 72, or they are composites composed of two(semi-primes), three, etc prime products making up N. The value of f(35)=12/35 indicates that 35=5 x7 is a composite(semi-prime) consisting of the product of two primes 5 and 7 taken to the first power each. The number $N=2^n$ is always a composite since $f(2^n)=[1-1/2^{n-1}]$. It will never become a super-composite since it approaches one as N goes to infinity. A slight variation on this number produces the Mersenne Numbers-

$$N = 2^n - 1 \quad which \quad yields \quad f(N) = \frac{[\sigma(2^n - 1) - 2^n]}{(2^n - 1)}$$

On setting n=17 we get N=131071 and f(N)=0 . So $2^{17}-1$ is a prime. Going to the much larger number –

$2^{127}-1=170141183460469231731687303715884105727$

we again find f(N)=0 so it is also a prime number.

Pierre Fermat proposed in the sixteen hundreds that the number –

$$F=2^{(2^n)}+1$$

is a prime number for all positive n. Leonard Euler however proved him wrong for the value of n=5, although the n=1 through 4 clearly are primes. Euler struggled with this problem for months finally being able to actually factor $F=2^{32}+1$ into its components. Using the number fraction we can find the composite nature of F at n=5 simply by noting that-

$$f(2^{32}+1)=\frac{[\sigma(2^{32}+1)-2^{32}-2]}{(2^{32}+1)}=0.001560211647\neq 0$$

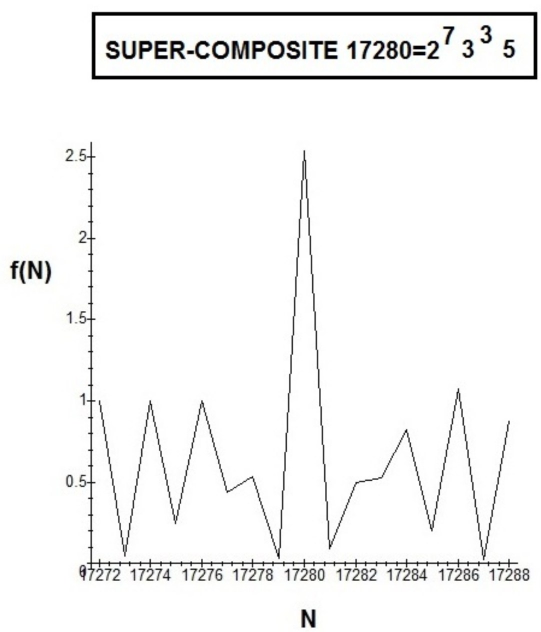
Since f(N) does not vanish it must be a composite. Note to prove this we actually never needed to find the semi-primes components-

4294967297=(641)(6700417)

As already mentioned, super-composites have the form-

$N=2^a3^b5^c\ldots$ where a>b>c

and an f(N)>1.4. Such numbers become particularly conspicuous when N gets large. We show you here an example for the super-composite N=17280-



Note here that this super-composite stands head and shoulders above its immediate neighbors. There are no primes in the range shown, however, there are multiple semi-primes such as 17273=23x751,17279=37x467,17281=11x1571 and 17287=59x293. Also we have composites such as 17286=2x3x43x67 consisting of more than two products of primes.

We next turn to the summation of the first N integers. Starting with N=1 we get the partial sums-

Partial sum:	1		3		6		10		15		21		28		46
First difference:		2		3		4		5		6		7		8	
Second difference:			1		1		1		1		1		1		

This implies that the sum will be a quadratic in N of the form $S(N)=A+BN+CN^2$. Matching the first three partial sums then yields A=0, B=1/2, and C=1/2. Hence we have-

$$S(N)=0+N/2+N^2/2=N(N+1)/2$$

For just the odd numbers we get the partial sums-

Partial sum:	1	4	9	16	25	36	49

But these are recognized at once to be the square of the integers. Hence we find-

$$S(99)=[(99-1)/2]^2=49^2=2401$$

So the sum of all odd integers through 99 is 2401 and thus approximately half of the sum of all integers through 99 which have the higher value of 4950. It leaves us with the sum of the even integers through 99 to be 4950-2401=2549.

An interesting observation is that the sum of the reciprocals of the integers reads-

$$R(N)=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+....\frac{1}{N}$$

For finite N this sum has finite positive values and one would at first glance predict that a finite positive value should remain as N goes to infinity. This turns out however to not be the case for one has-

$$R(\infty)=\sum_{n=1}^{\infty}\frac{1}{n}=\infty$$

This seies is known as the harmonic series. That it diverges follows from the fact that

$$\sum_{n=0}^{N} \frac{1}{n} > \int_{x=1}^{N} \frac{dx}{x} = \ln(N)$$

So since ln(N) is infinite as N goes to infinity so must R(∞) be.

There is another type of infinite series known as the geometric series. It reads-

$$S(r)=1+r+r^2+r^3+\ldots=\sum_{n=0}^{\infty} r^n$$

where r is a fraction lying in o<r<1. This series can be summed in closed form by subtracting rS(r) from S(r) to yield-

$$S(r)=1/(1-r)$$

If the upper limit of the sum is kept at the finite value of n=N, we find that-

$$\sum_{n=0}^{N} r^n = \frac{(1-r^{N+1})}{(1-r)}$$

So for the special case of r=1/2 and N=5 we get $(2^6-1)/2^5=63/32$.

Next let us look at the product of the integers-

1x1=1

1x2=2

1x2x3=6

1x2x3x4=24

1x2x3x4x5-120

The terms on the right of this equality are known as the factorials so that 6=3!, 24-4! , and 120=5!. Furthermore we have the identity that –

$$(N+1)N! =(N+1)!$$

so 6!=6x120=720 . Another interesting identity involving factorials is that-

$$N!=\int_{t=0}^{\infty} t^N \exp(-t)dt$$

This fact allows us to evaluate integrals of the form-

$$\int_{x=0}^{\infty} x^a \exp(-bx)dx = \frac{a!}{b^{a+1}}$$

There are certain algebraic expressions such as-

$y^2=1+k^2x^2$

which have solutions for integer values of x and y provided k^2 is an integer. These are known as Diophantine Equations. Take the special case of k=sqrt(2);This produces the positive integer pair-

[x,y]=[2,3],[12,17],[70,99],[408,577] and so on.

Solving the Diophantine equation we get-

$$\sqrt{2}=\left(\frac{y}{x}\right)\sqrt{1-(\frac{1}{y})^2}=1-\frac{1}{2y^2}-\frac{1}{8y^4}-\frac{1}{16y^6}-O(1/y^8)\,1$$

when expanded as a series. We can take any of the Diophantine solutions given above to have a converging series for sqrt(2). Taking [x,y]=[408,577] we get the very rapidly convergent series-

$$\sqrt{2}=\frac{577}{408}\left\{1-\frac{1}{2(322929)}-\frac{1}{8(322929)^2}-O(1/322929^4\right\}$$

Already the first three terms of the series multiplied by 577/408 yield the approximation-

sqrt(2)≈ 1.4142135623730950512

compared to the exact value of-

sqrt(2)= 1.4142135623730950488

This approximation is good to 16 places after the decimal.

Another property of the positive integers is that there are certain N which match the sum of all its divisors minus N. These are called the perfect numbers. In terms of σ(N) and f(N) we can express them as-

2N=σ(N)   or   f(N)=1-(1/N)

The ancient Greek mathematician Euclid showed that a necessary condition for a number to be perfect is that it satisfy-

$$N = 2^{n-1}(2^n - 1)$$ for certain primes n

It indeed generates such numbers for n=2,3,5,7,13,17,19,31,61,89,… but fails for
for n=11,23,29,37,41, 43,47, etc. . Looking at the values of N where σ(N)-2N=0, we find the first eight perfect numbers to be-

6, 28, 496, 8128, 33550336, 8589869056, 137438691328, 2305843008139952128

Note that each of these perfect numbers are even.

Finally we look at the concept of logarithms for integers. In a decimal integer system we can define any integer N as-

$N=10^a$ with 'a' the exponent of 10 also referred as the logarithm of N to base 10

Each integer will have associated with it a unique logarithm which most of the time will be a non-integer. Here is a brief table of N versus a=log(N)-

N	1	2	3	4	5	6	7	8	9	10
a=log(N)	0	0.3010	0.4771	0.6020	0.6989	0.7781	0.8450	0.9030	0.9542	1

The advantage of logarithms in the past was that they could speed up the multiplication and division of large numbers. Today they are no longer necessary since direct calculations with electronic calculators can do the job faster. This is also the reason slide rules have been replaced by hand calculators. A typical multiplication using logarithms is-

$$3x4 = 10^{0.4771}\, x\, 10^{0.6020} = 10^{1.0791} = 11.997752$$

More extemsive tables will bring the result 12. There are two bases commonly used for logarithms. The first of these is the Briggs logarithm which uses the base 10 and is designated by log(N) in the literature. The other is the natural logarithm based on the base e=2.71828… and designated as ln(N). The two logarithms relate to each other by the multiplication factor ln(10)=2.302585093… .Thus-

ln(5)=ln(10) log(5)=2.302585x0.6989=1.6094… .

For any other base b we have-

$$N=\exp(\ln(N)=b^{\log_b(N)}$$

So for b=2 we get $\log_2(8)=3$ or the equivalent $2^3=8$.

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