ALL ABOUT PYRAMIDS

A regular pyramid can be defined as the solid lying above an n sided regular polygon and confined by n slanted isosceles triangles connected to the base edges s and intersecting at a fixed height H above the base center. Simple geometry shows that the area of the polygon base equals-

$$A_{base} = \frac{ns^2}{4}\cot(\frac{\pi}{n})$$

So that the cross-sectional area of the pyramid at any height z<H above the base has the lower value-

$$A(z) = (1-\frac{z}{H})^2 \frac{ns^2}{4}\cot(\frac{\pi}{n}) = A_{base}(1-\frac{z}{H})^2$$

By calculus, we then have that the pyramid volume is-

$$V = A_{base}\int_{z=0}^{H}(1-\frac{z}{H})^2 dz = \frac{1}{3}(A_{base}H)$$

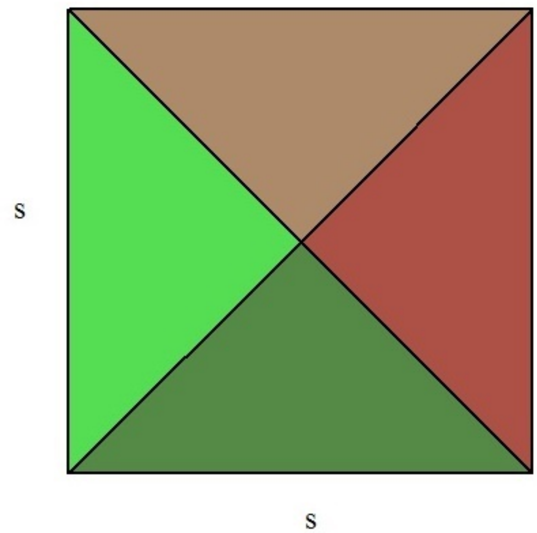
That is, the volume of any regular pyramid equals one third of the product of its base area times the pyramid height. Since a cone can be thought of as having an infinite sided polygon base, the same one third law continuous to hold.

The best known pyramids are the ones with a square base of side-length s and height H=s/sqrt(2). The Egyptian pyramids at Giza closely resemble this type. The volume of such a pyramid equals-

$$V = \frac{s^3}{3\sqrt{2}}$$

and their sides are essentially equilateral triangles of side length s each. Here is a 3D graph of this type of pyramid-

TOP VIEW OF A SQUARE BASE PYRAMID
OF HEIGHT H=s/sqrt(2)



Side surfaces are eqilateral triangles of area
[sqrt(3)/4]s^2 each

The ancient Egyptian pyramid builders where already quite familiar with pyramid properties over 4000 years ago. They could calculate things such as the partial volume of an unfinished pyramid when a height of ‘h’ above the base was reached. In the Moscow Papyrus(over 4000 years old) it is stated that the fraction of the volume of an unfinished pyramid compared to its final volume is precisely-

$$f = \frac{V_h}{V_{total}} = [a^2 + ab + b^2]\frac{(a-b)}{a^3}$$

where ‘a’ is the side-length of the pyramid base and ‘b’ the length of each side of the square cross-section at height h. Thus they knew that when the truncated pyramid had reached half its final height((b/a=1/2) that already seven-eigths of the required stones were in place. How did they discover such a formula prior to calculus? The answer is pretty much straight forward, but it certainly was not at the time of its formulation. My guess is that the architect- mathematician who formulated this law knew the volume of any regular pyramid via sand measurements. He then recognized that the volume of a truncated pyramid just equals the volume of a finished pyramid of base area A_{base} minus the volume of the small pyramid lying above the square of base length ‘b’. That is, the volume fraction of the truncated pyramid to the volume of the full pyramid is-

$$f = \frac{(1/3)a^2H - (1/3)b^2(H-h)}{(1/3)a^2H} = 1 - (1 - \frac{h}{H})(\frac{b}{a})^2$$

But from the geometry he knew that –

$$\frac{H}{H-h} = \frac{a}{b} \quad \text{so that} \quad \frac{h}{H} = \frac{(a-b)}{a}$$

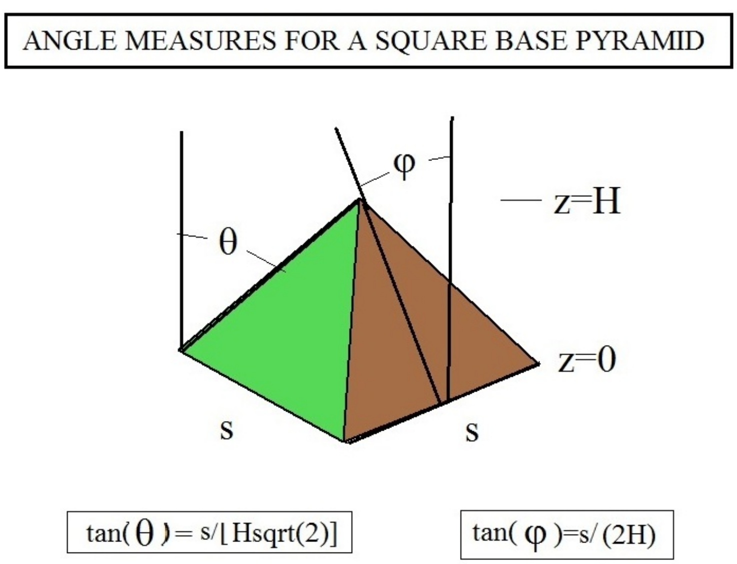
Plugging this h/H into f then produces the compact form-

$$f = \frac{a^3 - b^3}{a^3}$$

On expansion this yields the result stated above. I don’t quite understand why the formulator didn’t leave things in the above more compact form.

The Great Seal of the United States, as found on the back of the dollar bill, contains an image of a pyramid with an H/s ratio of about 1.2. To my eyes this tall pyramid is not as aesthetically pleasing as that of the Great Pyramid of Cheops(alias Khufu) where H/s is about 1/sqrt(2)=0.7071.

There are two basic angles associated with a square base pyramid. These are the angle θ and the angle φ. They are defined as shown in the following graph-



From the geometry we have that the base diagonal has length s sqrt(2), so that the larger angle θ is given by-

$$\theta = \arctan[s/(H\sqrt{2}]$$

The smaller angle φ equals-

$$\varphi = \arctan[s/(2H)]$$

For an Egyptian Pyramid, where H≈s/sqrt(2), we get the angles-

θ = 45deg   with   φ = 35.26deg

The compliment to φ is here 90-35.26=54.74deg. The stone slant at the bottom of the Great Pyramid has been measured as 51.9deg, showing that this pyramid departs slightly from having its four sides being perfect equilateral triangles.

We look next at the total side area of pyramids having an n-sided polygon base and height H. From the geometry, we have that the n isosceles triangles forming the sides each have bisectors of length sqrt[H^2+$(s/2)^2$cot$(\pi/n)^2$]. Thus the total area of all n sides becomes-

$$A_{sides} = \frac{ns}{2}\sqrt{H^2 + (\frac{s}{2})^2 \cot(\frac{\pi}{n})^2}$$

For a pyramid with square base(n=4) and height H=s/sqrt(2) we get the total side surface area to be s^2sqrt(3). Recalling the value of the base area , we get that the pyramid has a total surface area of-

$$A_{total} = \frac{ns}{2}\sqrt{H^2 + (\frac{s}{2})^2 \cot(\frac{\pi}{n})^2} + \frac{ns^2}{4}\cot(\frac{\pi}{n})$$

As n gets large, one has that ns approximately equals the circumference of a circle of radius r, That is, 2πr=ns. In this limit we have essentially a cone of base radius r and height H. The area of the cone side surface is then-

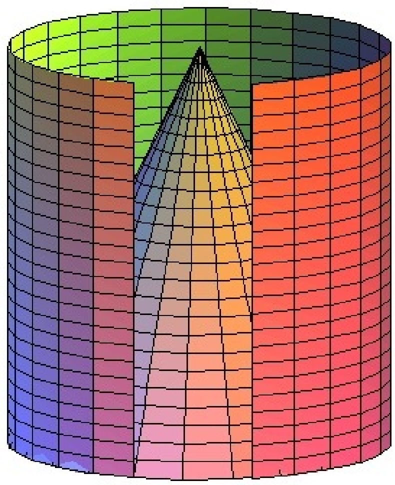
$$A_{sides} = \pi r\sqrt{H^2 + (\pi r)^2 \frac{\lim}{n \to \infty}\{\frac{1}{[n\tan(\pi/n)]^2}\}}$$

Taking the limit, one finds-

$$A_{sides} = \pi r\sqrt{H^2 + r^2}$$

for a cone's side surface. The volume of this cone is simply $\pi r^2 H/3$. It represents essentially one-third of the volume of an enclosing cylinder. Here is a picture of a cone inside such a cylinder-

CONE IN A CYLINDER

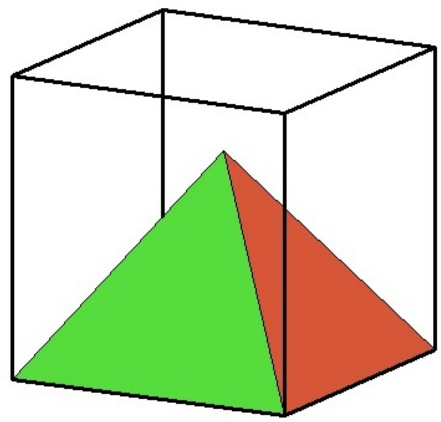


Volume Cone/Volume Cylinder=1/3

I constructed this figure in a few minutes using the math program MAPLE and Microsoft Paint Brush. Some 2000 years ago Archimedes showed that the largest sphere which can fit into a cylinder has 2/3 of the cylinder volume. He was so proud of this discovery that he had a model of a sphere in a cylinder placed on his grave.

One can also place any regular pyramid into an ' n' sided rectangular box. For n=4 we get the following picture-

PYRAMID CONFINED TO A RECTANGULAR BOX



$$V_{pyramid} / V_{box\ below\ pyramid\ vertex} = 1/3$$

The volume of the pyramid, regardless of height, remains one third of the minimum box volume which just encloses the entire pyramid.

U.H.Kurzweg
Sept.16, 2017
My 81st Birthday