# ALPHA AND BETA FUNCTIONS IN NUMBER THEORY

One can combine any positive integer N in a variety of ways including summing 1+2+3+4+…+N=N(N+1)/2, multiplying 1x2x3x4x…xN=N!, and also as the integer point functions sigma and tau. The definitions of the last two are given in standard number theory as-

$$\sigma(N) = \textit{sum of all divisors of } N$$

and-

$$\tau(N) = \textit{number of divisors of } N$$

So, for example, $\tau(6)=4$ and $\sigma(8)=15$. Also we note that $\tau(\text{prime})$ is always 2 since a prime is only divisible by the number and one. Likewise $\sigma(N)-(N+1)$ will always equal to zero when N is a prime. These last two facts allow us to define the more convenient variations of the sigma and tau functions, namely-

$$\alpha(N) = \frac{[\sigma(N) - N - 1]}{N}$$

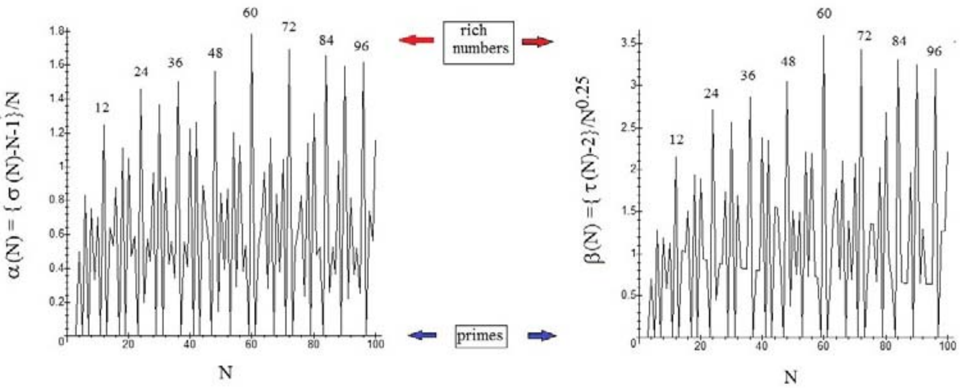
and-

$$\beta(N) = \frac{[\tau(N) - 2]}{N^{1/4}}$$

We have defined and used these functions in earlier notes in this web page. Note that they are generally non-integer rational numbers which become integers when multiplied by N and $N^{1/4}$, respectively. The powers of N present in the denominators were chosen in order to keep the quantities near unity. When N is a prime number the quantities $\alpha(N)$ and $\beta(N)$ will always be zero.

It is our purpose here to look at some of the more interesting properties of the functions $\alpha(N)$ and $\beta(N)$ . Let us begin with a point plot for both over the range $2<N<100$. They look like this-

VALUES OF ALPHA(N) AND BETA(N) IN THE RANGE 2<N<100



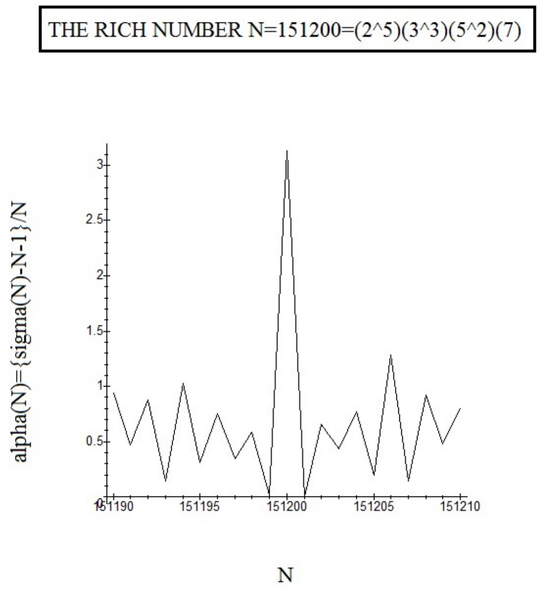

Both functions clearly indicate the location of the prime numbers. Also they indicate local maxima at the same Ns. We call these Ns rich numbers and identity them by noting that they tower above their immediate neighbors. Expressing three of these rich numbers within the range shown, we find-

$24=2^3(3)$ $60=2^2(3)(5)$ $96=2^5(3)$

Note that they are all divisible by six. What is interesting about this factorization is that rich numbers seem to be constructed by the products of the lowest primes taken to certain positive powers. Thus, for example, we would expect-

$$N=2^5(3^{3)}(5^{2)}(7)=151200$$

to be a rich number. The following graph confirms this point-

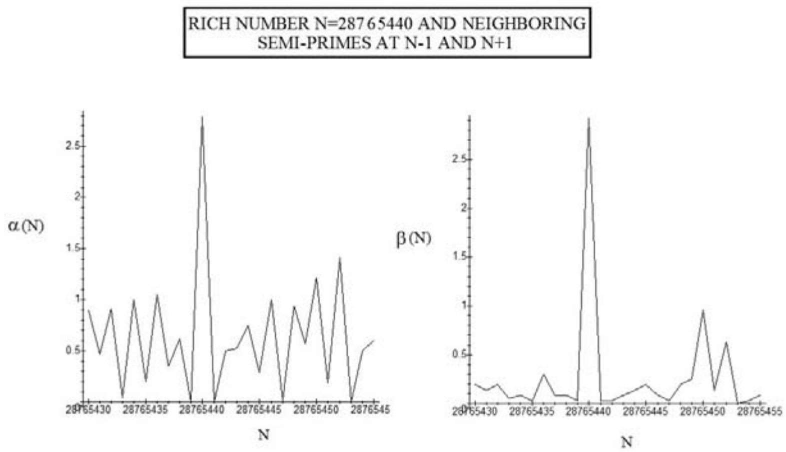



An additional sidelight is that here N+1 is a prime and N-1=47x3217 is a semi-prime. That type of prime and semi-prime will occur quite often about a rich number.

The fact that $\alpha(N)$ and $\beta(N)$ have the same local maximum for the same N continuoes to be valid for larger values of N. For example , the number

$$N=28715440=(2^4)(5)(13)(27611)$$

Both $\alpha(N)$and $\beta(N)$ show it to be a rich number with the immediate neighborhood having very small values corresponding to semi-primes . Here are the corresponding $\alpha(N)$ and $\beta(N)$ graphs-



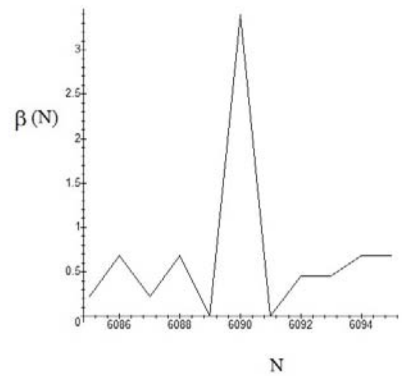

Note that the beta curve is slightly better in distinguishing primes from semi-primes. While the alpha curve suggest the possibility of primes at N-1, N+1, N+6 and N+13 , the beta curve shows that 28765453=N+13 is the only true prime in the indicated range under consideration.

The above graphs have shown that the point functions $\alpha(N)$ and $\beta(N)$ can both be used to identify rich numbers $N_{rich}$ and primes p. The rich numbers take on the form-

$$N_{rich} = \sum_{k=1}^{n} p_k^{\ a_k} = 2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3} \cdot ....$$

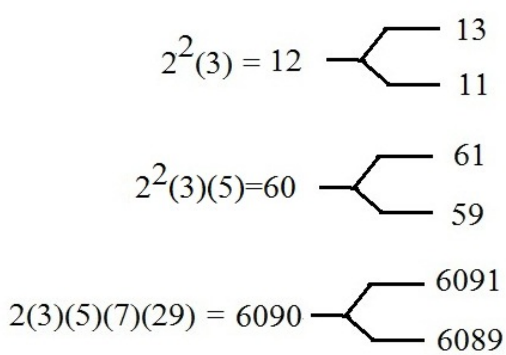
where the a s tend to decrease in value as ks get large. Here $p^k$ is the kth prime. Also we note that often the immediate neighborhood of $N_{rich}$ will equal primes or semi-primes. When both N-1 ans N+1 for any rich number are primes then they must be double primes. Also N will need to be divisible by six. Thus for N=28350 (which is a rich number) we have the double primes N-1=28349 and N+1=28351 and N=6(4725).Another example for the appearance of such double primes is the rich number N=254016000=6(42336000). Here N±1 are both primes. One could also reverse the problem and start with two double primes N-1 and N+1 and then see if N is a rich number. So for instance if p=6089 and q=6091 then N=6090 is indeed a rich number. We show the result in the following graph-

THE RICH NUMBER N=6090 AND ITS NEIGHBORING DOUBLE PRIMES AT N-1 AND N+1

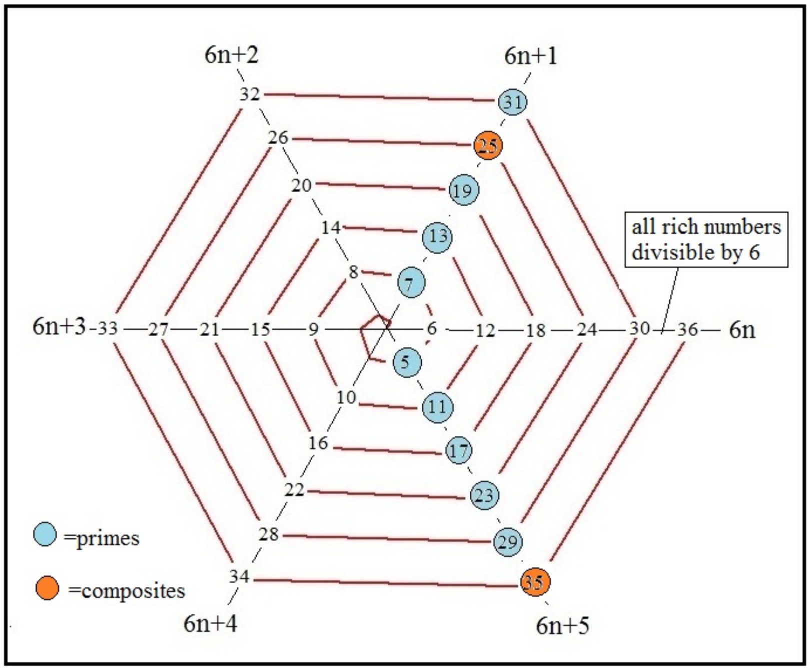



Symbolically one can represent twin(or twin) primes about a rich number in the following forms-

SCHEMATIC OF SOME RICH NUMBERS AND THEIR NEIGHBORING DOUBLE PRIMES




Note that rich numbers are always even and divisible by six if its immediate neighbors are both primes. This fact is clearly supported by the following integer spiral-




We first came up with this hexagonal integral spiral about a decade ago. It represents a much improved version of the standard Ulam Spiral. It beautifully places all primes greater than 3 along just two lines 6n+1 and 6n-1. The corresponding rich number must lie along the 6n line. Note from the graph that the rich numbers 6,12,18,30,.. have 6n±1 primes. But the rich numbers N=24 and 36 do not. The reason that one of the neighbors ends in five. This would also happen if either N+1 or N-1 represents the product of prime numbers. Hence a more accurate statement is that –

A necessary but not sufficient condition that N±1 are twin primes is that N is even and divisible by six.

Here are the first 25 rich numbers N where N+1 and N-1 are primes-

$N_{rich}$={6,12,18,30,42,60,72,102,108,138,154,180,192,198,228,270,282,312,348,420,432,

462,522,570,600}

Some larger rich numbers with neighboring primes are-

$N_{rich}$={213948, 17787282, 1664157180}

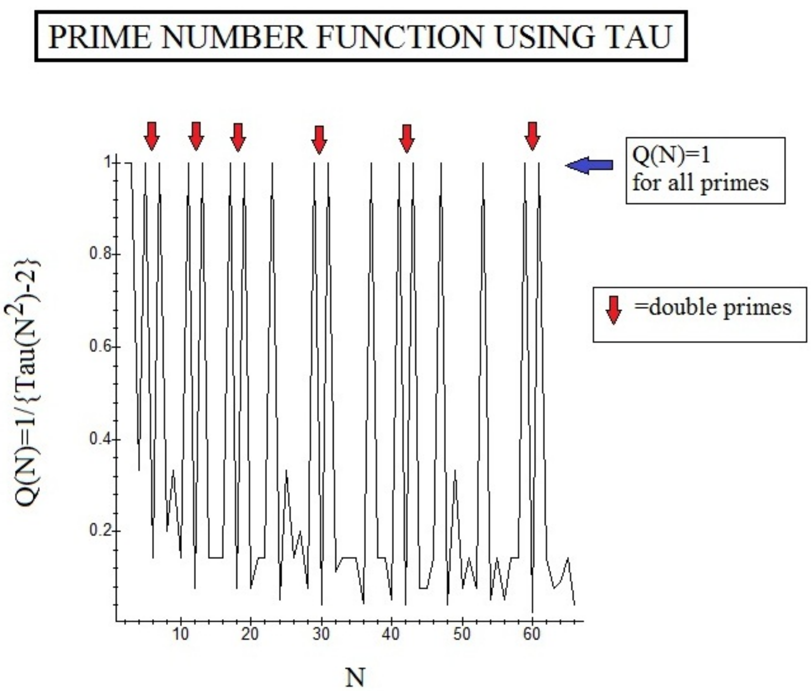
In number theory it is less time consuming to calculate τ than σ. In an earlier note we have already shown that there exists a prime number function given by-

$$P(N)=\frac{\sigma(N^2)-1}{\sigma(N^3)-N^3-1}$$

This point function has P(N)=1 whenever N is a prime and a value less than one when N is a composite. An even easier prime number function may be generated using tau. We find-

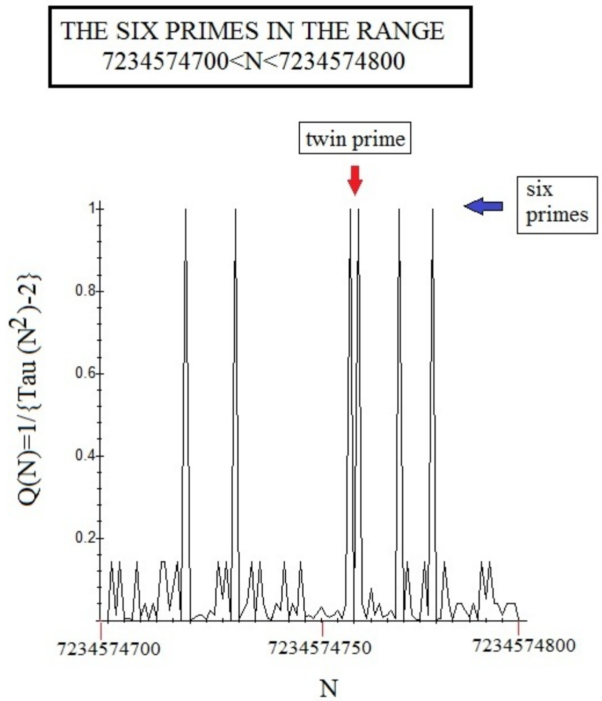
$$Q(N) = \frac{1}{\{\tau(N^2) - 2\}}$$

Thus Q(3)=1/(3-2)=1, Q(4)=1/(5-2)=1/3, and Q(5)=1/(3-2)=1. So prime numbers apparently correspond to Q(N)=1 and composite numbers to Q(N) less than one. A graph of this behavior follows-




The graph clearly shows the location of prime number Ns and offers the additional benefit that double primes ( also referred to as twin primes) are easy to spot by finding two vertical spikes separated from each other by 2 units.

Just to show how quickly one can pick out primes and twin primes for any N, consider the properties of Q(N) in the 100 unit range 7234574700<N<7234574800. The graph in this case looks like this-




There are a total of six primes located at-

7234574719
7234574731
7234574759
7234574761
7234574771
7234574779

The middle pair indicates a twin prime. Its corresponding rich number is $7234574760=2^3(3^2)(5)(7)(2870863)$. There is little doubt that there are an infinite number of double primes just as there are an infinite number of regular primes.

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