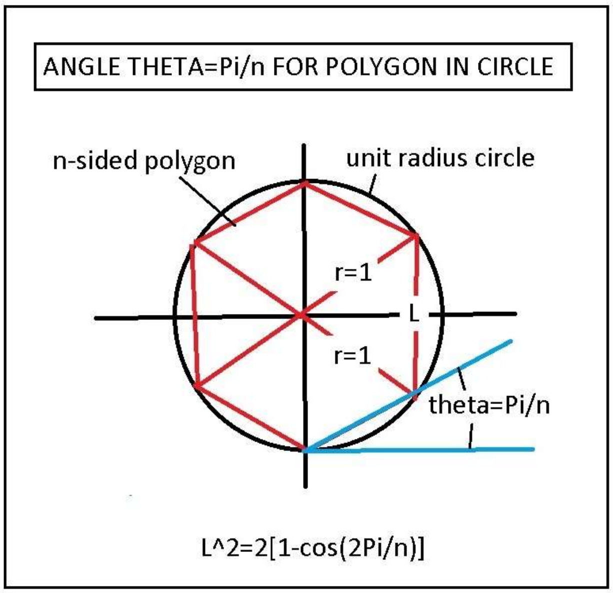
EXPRESSING ANGLES AS INTEGERS AND THEIR ROOTS

I remember back some seventy years ago when first learning about the properties of trigonometric functions tan(θ), sin(θ), and cos(θ), that many of the simpler ones where expressible as combinations of integers and roots. So, for example, sin(π/3)=sqrt(3)/2 and cos(π/4)=1/sqrt(2). This suggested to me that there must be an infinite number of other angles of the form θ=π/n rad whose trigonometric forms are also expressible as combinations of roots and integers. We want in this note to show how one can find these.

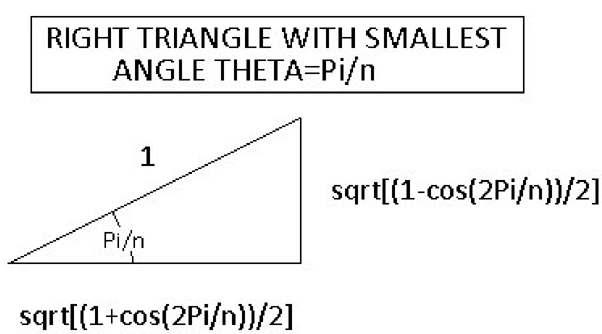
To do so we start with a unit circle intersected by two orthogonal radial lines as shown-



Next we place a regular n-sided polygon into this circle and mark off the exterior angle θ=π/n. Using the Law of Cosines we have-

$$L^2=2[1-\cos(2\pi/n)]$$

, where L is the side-length of any n sided regular polygon. We also find the smallest triangles inside the polygon to have the sides shown-



From this triangle we can read-off the three trigonometric functions-sin(π/n)=$\sqrt{\left[1-\cos\left(\frac{2\pi}{n}\right)\right]/2}$, cos(π/n)=$\sqrt{\left[1+\cos\left(\frac{2\pi}{n}\right)\right]/2}$, and

tan(π/n)=$\sqrt{[1-\cos(\frac{2\pi}{n})]/[1+\cos\left(\frac{2\pi}{n}\right)]}$

These results mean that if cos(2π/n) is expressible as a combination of integers and roots, so will other trig functions for 2n,4n,8n, etc.

Let us demonstrate for n=5. Here cos(2π/5)=[sqrt(5)-1]/4=0.3090169.. . So plugging into the above trigonometric forms, we find-

sin(π/5)=sqrt[(5-sqrt(5))/8] =0.587785… , cos(π/5)=[sqrt(5)+1]/4=0.809016.., an

tan(π/5)=sqrt(5-2sqrt(5))=0.726542…

Note that these results are directly related to a regular pentagon. We also note that the Golden Ratio f=(1+sqrt(5))/2=1.618033.. equals just twice cos($\pi/5$). Also, as seen, the trig functions are , as expected, all expressible as integers and roots of integers.Going on to θ=π/10 we find cos(π/10)=sqrt[(5+sqrt(5))/8]. Next on to θ=π/20 to find-

Cos(π/20)=sqrt[1/2+(1/4)sqrt((5+sqrt(5))/2)]

It is clear from these results that cos(π/5k) for k=1,2,3,.. is presented as a function of integers and roots of sqrt(5). Sin(π/5k and tan(π/5k) follow by the Pythagorean Theorem for right triangles with hypotenuse of length one.

Consider next n=6 where θ=π/6 rad=30 deg. We can use the known result that cos(π/3)=sqrt(3). Substituting into the above trig results, we get-

sin(π/6)= 1/2 , cos(π/6= sqrt(3)/2 , and tan(π/6)=1/sqrt(3)

Going on to θ=π/12, we find-

sin(π/12)=sqrt(2)(sqrt(3)-1)/4 , cos(π/12)=sqrt(2)(sqrt(3)+1)/4

tan(π/12)=[sqrt(3)-1]/[sqrt(3)+1]=2-sqrt(3)

So we can state that trigfunctions for angle θ=π/k6 will always be expressible as function of integers and sqrt(3)

Next consider n=4 where cos(π/4)=sin(π/4)=1/sqrt(2) and tan(π/4)=1. Halving the angle to θ=π/8 produces-

sin(π/8)=sqrt[(sqrt(2)-1)/(2sqrt(2))] , cos(π/8)=sqrt[(sqrt(2)+1)/2sqrt(2))]

tan(π/8)=sqrt[(sqrt(2)-1)/(sqrt(2)+1)]=sqrt(2)-1.

Thus the trig functions for θ=π/k4 are all expressible as integers and the roots sqrt(2).

We were unable to find values for cos(2π/n) when n=7,9,11,13 etc on our PC. This suggests that there are no trig formulas involving the radical sqrt(n) for these cases. So if we have θ=π/64 continued divisions bring one to n=4 and hence produces a closed form integer-sqrt(2) combination solution. However, θ=π/56 does not have a closed form solution since cos(2π/7)does not yield a radical solution involving sqrt(7).

In conclusion we can state that there are an infinite number of integer-root solutions for angles θ=π/kn k=1,2,3,.. provided n=3,4,5. We have found no other values of theta which have this property. It also says that if a solution involves a radical this must be sqrt(2), sqrt(3), or sqrt(5).

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