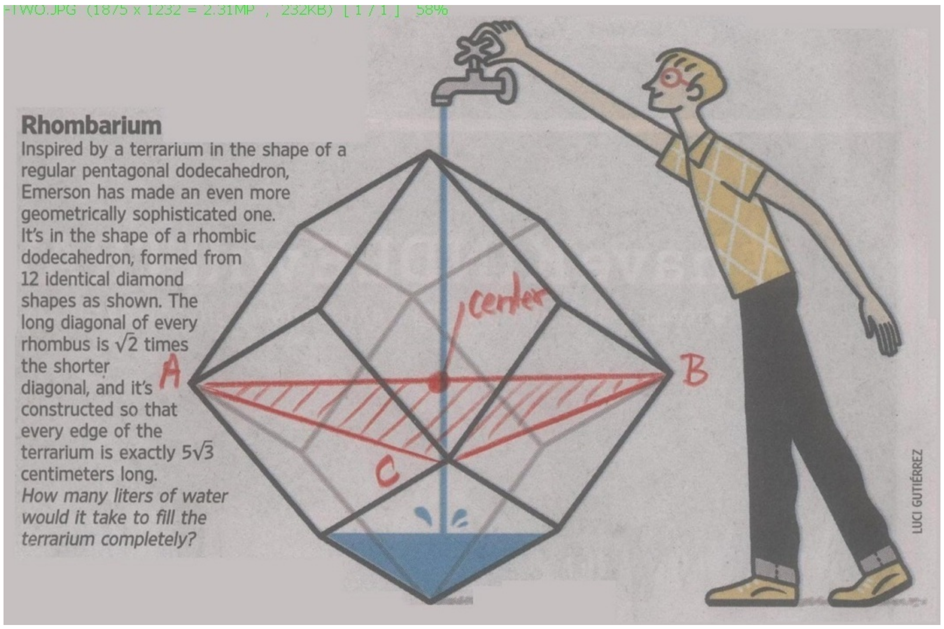
VOLUME OF A TWELVE-SIDED DODECAHEDRON WITH DIAMOND SHAPED FACES

A math problem posed in the latest weekend puzzle page of the Wall Street Journal (Sept., 3-4, 2016) is to find the number of liters of water it takes to fill a twelve-faced aquarium. A scan of their question and the associated geometry follows-



In the picture I have added a triangle ABC in red pencil. Its utility will become obvious below. The twelve faces shown all are diamond shaped with the ratio of the longer diagonal 2a to the shorter diagonal 2b equal to sqrt(2) . Each of the four sides of the diamond have the same length of s= 5 sqrt(3) . A little algebra then shows that-

$$a^2+b^2=75 \quad \text{and} \quad a/b=\text{sqrt}(2)$$

From these follow that-

$$a=5\text{sqrt}(2) \quad \text{and} \quad b=5$$

This last result implies that the area of each of the twelve diamond faces is-

$$\text{Area}=2ab=50\text{sqrt}(2)$$

Next we go back to the schematic and notice that the area of each of the twelve faces just forms the base of a pyramid of height H. To find H we first draw in red pencil the triangle

ABC shown in the figure. The length of the longest side of this triangle represents the distance between opposite vertexes and is taken as 2c. The Pythagorean Theorem then shows that-

$$c = a\sqrt{2} = 10\ cm$$

Now c , which represents the longer slant side of a given pyramid, relates to H as-

$$c^2 = H^2 + a^2$$

Hence H=5 sqrt(2). We recall from solid geometry and calculus that the volume of any pyramid equals one third of the product of its base times its height. This yields our answer for the total volume inside the aquarium of--

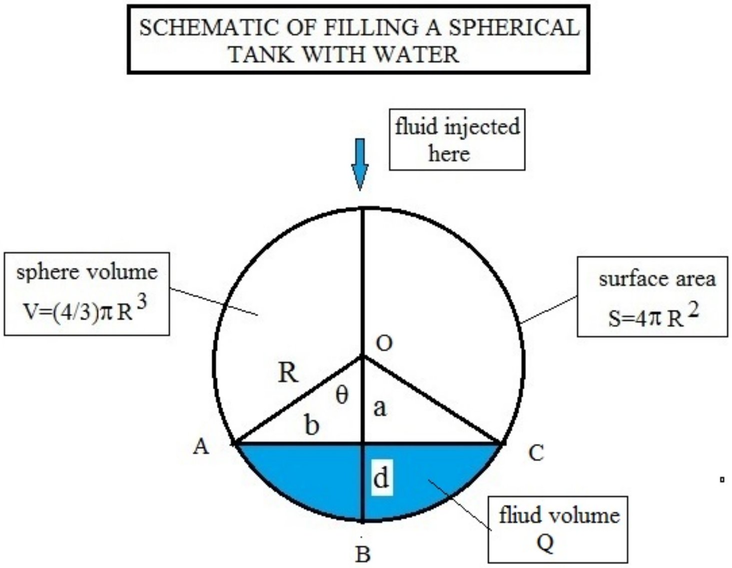
$$Volume = 12\left\{\frac{1}{3}\cdot 50\sqrt{2}\cdot 5\sqrt{2}\right\} = 2000\ cc = 2\ \ liter$$

Thus it takes two liters of water to fill the aquarium. Notice that a sphere of radius R=c=10 just inscribing the aquarium would have about twice the volume .

There are multiple more difficult variations of this problem which may be considered. For example, we could construct a spherical aquarium of radius R. Its volume and surface area are-

$$\mathrm{V} = \frac{4}{3}\pi R^3 \quad \text{and} \quad \mathrm{A}=4\pi \mathrm{R}^2$$

, respectively. The question we can ask is how does the fluid volume Q in the sphere correlate to the depth d of the liquid level? This is the familiar gas-tank problem of estimating volume via a dip stick. The fluid is being poured into the hollow sphere by a small hole in the top as shown in the following schematic-



By the symmetry of the problem it will be sufficient to just find Q for $0<d<R$.

To solve the problem we concentrate on the inverted ice-cream cone defined by OABC and realize that the fluid volume Q shown in blue is just the volume of this cone minus the volume of the cylindrically symmetric pyramid OAC. This pyramid has a volume of-

$$V_{OAC} = \frac{\pi}{3} b^2 a = \frac{\pi}{3} a(R^2 - a^2)$$

Now the volume of the upside-down ice-cream cone described in spherical coordinates is-

$$V_{cone} = 2\pi \int_{R=0}^{R} R^2 dR \int_{\theta=0}^{\theta} \sin(\theta) d\theta = \frac{2\pi R^3}{3}[1 - \cos(\theta)]$$

Also we have from the above picture that-

$a=R-d$ and $\cos(\theta)=a/R=1-(d/R)$

Hence the volume of the fluid in the aquarium is-

$$Q = \frac{\pi R^3}{3}[\frac{d}{R}]\left\{2 - [(1 - \frac{d}{R})(2 - \frac{d}{R})]\right\}$$

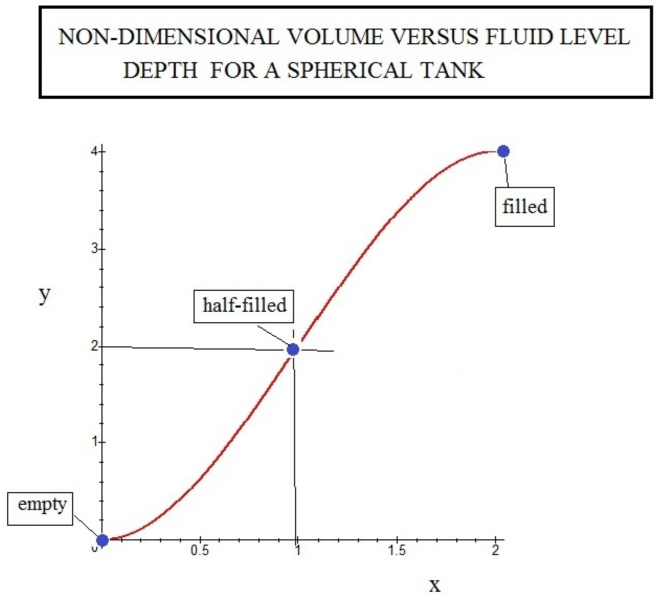
Introducing the non-dimensional variables-

$$y = \frac{3Q}{\pi R^3} \quad and \quad x = \frac{d}{R}$$

then produces the cubic equation result-

$$y = x[2-(1-x)(2-x)]$$

A plot of y versus x follows-



Note the symmetry about the lines y=2 and x=1. As expected the graph shows the level in the tank rises rapidly at the beginning and end of the filling. The level change is slowest when the aquarium is half full.

Sept.4, 2016