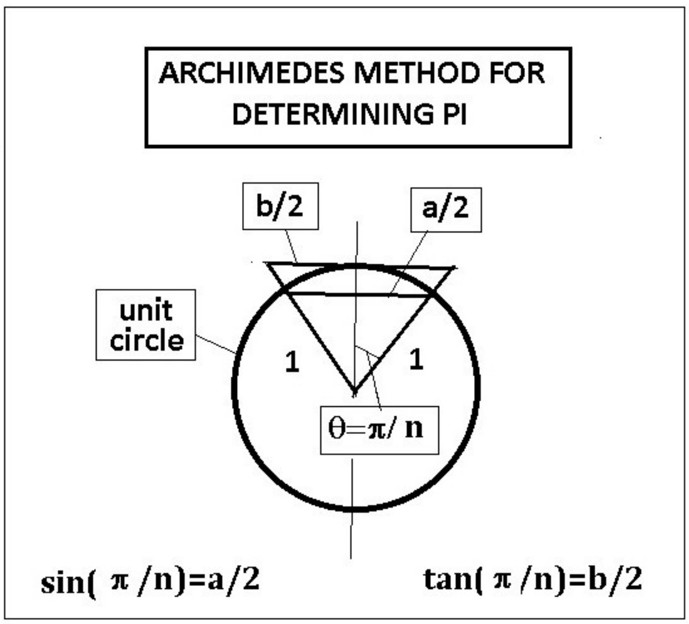
BOUNDS ON PI USING THE ARCHIMEDIAN GEOMETRIC APPROACH

About 2200 years ago the famous Greek mathematician Archimedes of Syracuse(287-212 BC)gave the first accurate bounds for the value of $\pi=3.1415926\ldots$. By using a geometric approach involving regular polygons inside and outside of a unit circle, he was able to come up with the estimate-

$$3+10/71=3.140845< \pi <3+1/7=3.142857$$

using a regular polygon of 96 sides. We want in this article to repeat his calculations for regular polygons of $n=2^m$ sides.

Our starting point is the following sketch-



We have a unit radius circle of area π showing the pie shaped parts of an inscribed and circumscribed regular polygon. Geometry shows that the total area of the inner polygon will be-

$$A_{in}=(n/2)\sin(2\pi/n)$$

The outer polygon has the larger area-

$$A_{out}=n\tan(\pi/n)$$

Since we will be dealing with polygons with a large number of sides it pays to define $n=2^m$. This produces the bound relation-

$$2^{m-1}\sin(45/2^{m-3})< \pi < 2^m\tan(45/2^{m-2})$$

, where the angles are measured infractions of degrees. Starting with m=4, we get-

m=4	n= 16	$3.061<\pi<3.182$
m=6	n=64	$3.136<\pi<3.144$
m=8	n=256	$3.1412<\pi<3.1417$
m=10	n=1024	$3.14157<\pi<3.14160$
m=12	n=4096	$3.141591<\pi<3.141593$
m=14	n=16384	$3.14159257<\pi<3.14159269$

So it takes a polygon of 16384 sides to get a six place accuracy on π. The method works but will require ms much larger than 14 to get accuracies of fifty digits or so. The above m=14 case is about as accurate as the well known Otto ratio of $\pi\approx \frac{355}{113} = 3.1415927$, obtainable via continued fractions. The most accurate value of π obtained by applying the Archimedes method is due to the Dutch-German mathematician Ludolph van Ceulen(1540-1610). He used a polygon of $n=2^{62}$ sides to find π out to 35 places. He was so proud of his achivement that he had the result engraved on his tombstone. The modern way to calculate π to high accuracy is to use arctan formulas, AGM methods, or iteration approaches.

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