VOLUME RATIO FOR AN ELLIPSOID INSCRIBED OR CIRCUMSCRIBED ABOUT A RIGHT CYLINDER

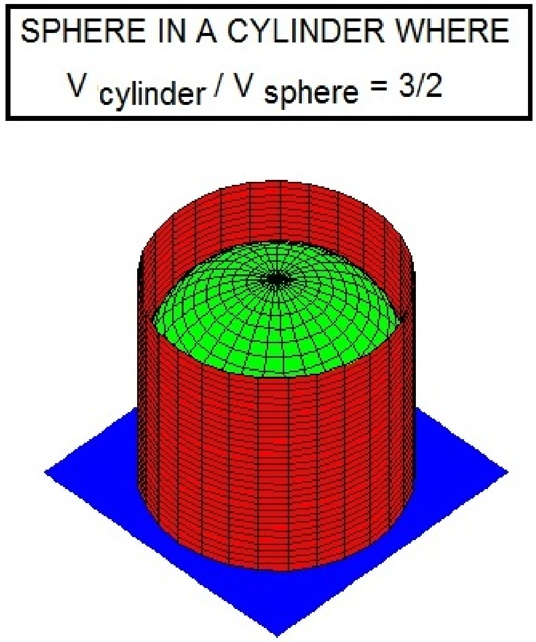
INTRODUCTION:

About 22 hundred years ago the famous Greek mathematicians, physicist and inventor Archimedes of Syracuse (287-212 BC) showed that the volume of a sphere of radius r contained in a right cylinder of the same radius and height 2r just equals 2/3 of the volume of the cylinder. He was so proud of this discovery, which preceded calculus by two millenniums, that he had a grave marker put on his final resting place consisting of a column toped by a cylinder containing a solid sphere. We know about this fact from Markus Cicero the famous Roman orator who came upon Archimedes's grave in Sicily some 172 years after Archimedes's death by a Roman soldier during the second Punic War in 212BC

We want here to extend the Archimedes analysis of a sphere in a cylinder by looking at the more generalized problem of finding the volume ratio of an ellipsoid relative to either an inscribed or circumscribed right cylinder. As will be shown via simple calculus the optimum volume ratio is always 2/3 for any ellipsoid circumscribed by a cylinder and 1/sqrt(3) for any cylinder circumscribed by an ellipsoid regardless of the values of a and b appearing in the ellipsoid definition.

ELLIPSOID IN A CYLINDER:

We begin our analysis with the easier problem of an ellipsoid within a cylinder. Our model for this analysis will be the following-



where we replace the green sphere with a 3D ellipsoid given by-

$$(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$$

The outer cylinder shown in red becomes –

$$radius\ y = b \quad with \quad -a \le x \le a$$

Now the volume of the ellipsoid is-

$$V_{ellipsoid} = 2\pi b^2 \int_{x=0}^{a} [1 - (\frac{x}{a})^2] dx = \frac{4}{3}\pi a b^2$$

Also the volume of the circumscribed cylinder is-

$$V_{cylinder} = 2a\pi b^2$$

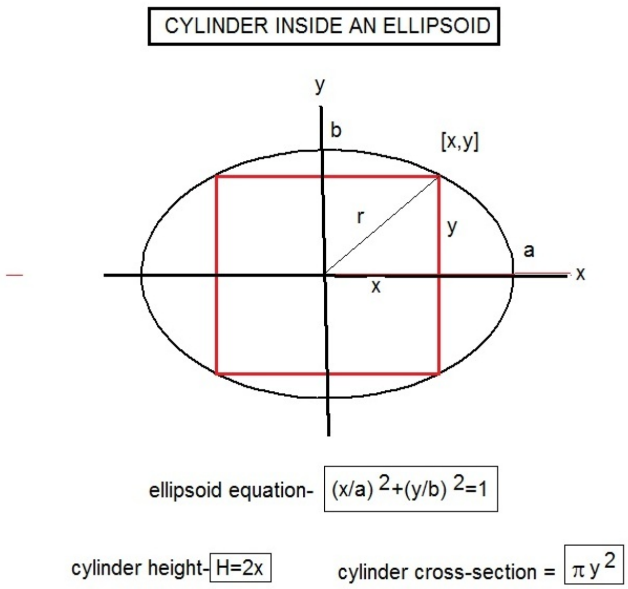
Taking the ratio we get-

$$R = \frac{V_{cylinder}}{V_{ellipsoid}} = \frac{6\pi ab^2}{4\pi ab^2} = \frac{3}{2}$$

This is the famous Archimedes result. It apparently holds for all ratios of a to b not just where they are equal.

CYLINDER IN AN ELLIPSOID:

For this second problem we begin with the following schematic-



Herte we have a cylinder of height H=2x and radius y confined to an ellipsoid with the earlier determined volume $V_{ellipsoid}=4\pi ab^2/3$. The volume of the cylinder is –

$$V_{cylinder} = 2\pi xy^2 = 2\pi ab^2 z(1-z^2) \quad with \quad z = x/a$$

Thus the ratio of the cylinder volume to the ellipsoid volume is-

$$R = \frac{V_{cylinder}}{V_{ellipsoid}} = (\frac{3}{2})z(1-z^2) \quad with \quad 0 \le z \le 1$$

This cubic in z has a maximum at z=1/sqrt(3)=0.57735..and zeros at z=0 and z=1 .The corresponding value of maximum R also equals 1/sqrt(3) regardless of the values of a and b. So things also apply equally well to a right cylinder in a sphere.

CONCLUSION:

We have shown that the maximum volume ratio of an ellipsoid and either an inscribed or circumscribed cylinder are independent of the aspect ratio of the ellipse. When the cylinder surrounds the ellipsoid the volume ratio remains the Archimedian result of 3/2=1.500. However, when the cylinder lies within the ellipsoid one finds the different volume ratio of sqrt(3)=1.732.

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