RELATION BETWEEN VERTEXES, CONNECTORS, AND SUB-AREAS FOR 2D POLYGONS WITH DIAGONALS

A Several hundred years ago the famous Swiss mathematician Leonard Euler (1707-1783) discovered one of the simplest but most important identities in algebraic topology. It states that the number of vertexes(V) and the number of edges(E) of any 3D polyhedron is related to the number of its faces(F) by the formula-

$$F + V - E = 2$$

So for a tetrahedron we have 4+4-6=2 and for a cube we get 6+8-12=2.

The 2D version of this equality (as we have already discussed in an earlier note back in Dec.2017) reads-

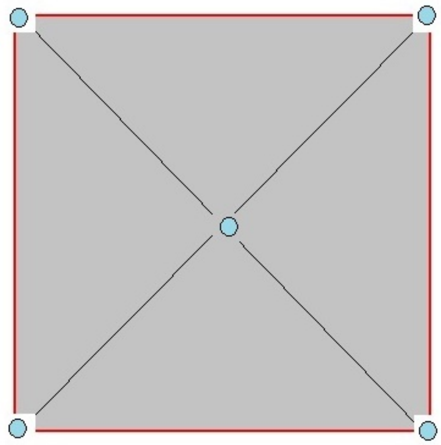
$$A+V-C=1$$

, where we have V vertexes (referred to as nodes in graph theory), C connecting lines between the vertexes, and A the sub-areas created by the connectors. Thus for a square with two diagonals partitioning the square we have V=5, A=4, and C=8. So 4+5=8=1. For a polygon with V vertexes and no diagonal partitions we have C=V and A=1. So 1+V-V=1.

It is our purpose here to discuss the number of sub-areas(A), vertexes(V) and connectors (C) created by an N sided regular polygons when its vertexes are connected a maximum number of allowed diagonals.

We begin with the case of a regular square partitioned by its two possible diagonals as shown-

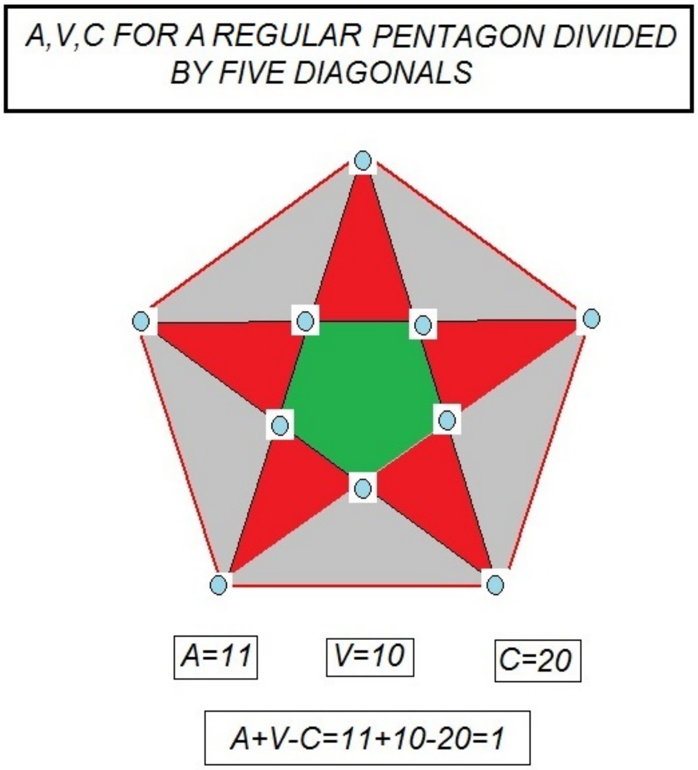
SQUARE SUB-DIVIDED BY TWO DIAGONALS



N=4, D=2, A=4 , V=5, C=8

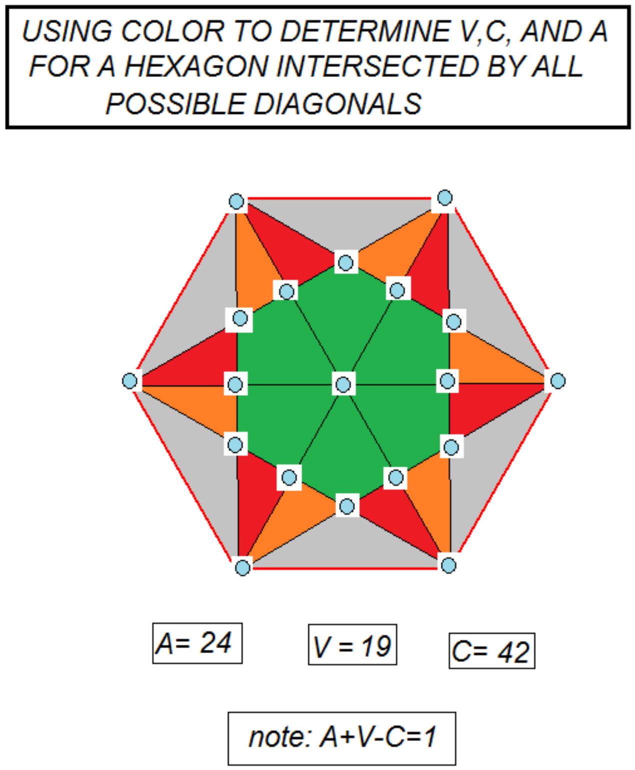
Here N=4 and D=2 and there are five vertexes, eight connectors and four sub-areas. This yields A+V-C=1.

Next we examine the more complicated case of a regular pentagon (N=5) subdivided by its five possible diagonals(D=5). We get the following picture-



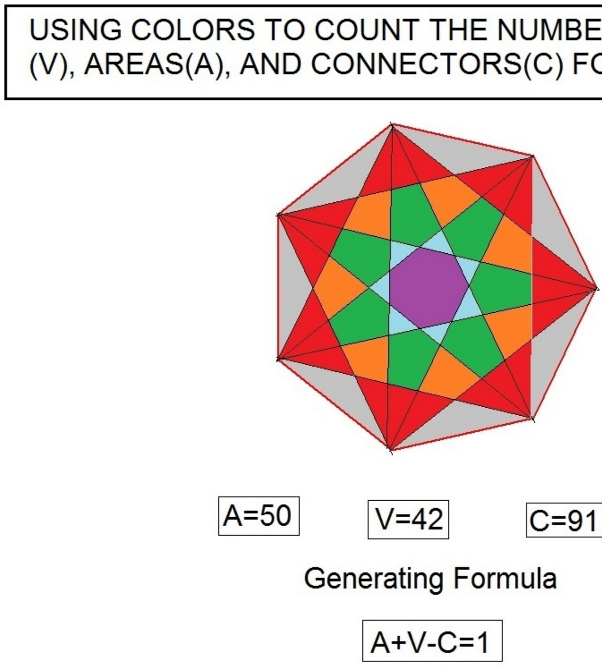
We find A+V-C=11+10-11=1 so the formula is again verified. This time we found it convenient to color the pattern in order to more easily count the values of A,V and C.

Next we look at a regular hexagon (N=6) partitioned by its nine diagonals (D=9). Here are the results -



This time the values A=24 and V=19 were easy to read off of the graph. The third value of C followed from C=A+V-1=24+19-1=42.

Next look at the heptagon N=7 with the maximum number of unique diagonals at D=14. Here we get the colorized picture-



For this case, we count the sub-areas to be A=50 with V=42. This produces C=91.

We can extend things further to still larger N sided polygons sliced into smaller sub-areas by all its possible diagonals. Using our MAPLE math program, we can draw any N sided regular polygon by the program-

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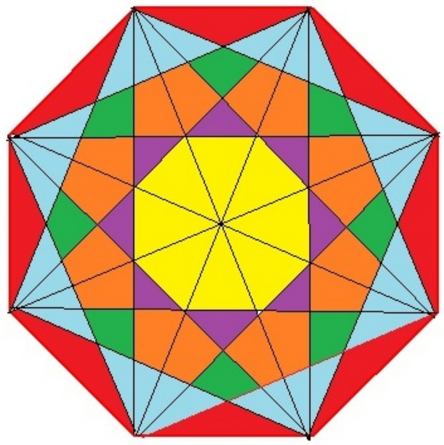
with(plots):
listplot([seq([1,(Pi*(2*n)/N)],n=0..N)],coords=polar,color=red,
thickness=2,scaling=constrained,axes=none);
```

After rotating this result by an appropriate angle and drawing in all possible non-redundant diagonals one often ends up with interesting figures which can be colorized such as in the following octagon-

COLORIZED PATTERN FOR A REGULAR OCTAGON DIVIDED BY TWENTY DIAGONALS



N=8, D=20, A=80,V=71,C=150

We find the relation between the number of possible unique diagonals (D) and the polygons number of sides (N) given by-

$$D=(N/2)[-3+N]$$

This follows from solving the linear equations $2=4A+16B$ and $5=5A+25B$ to get $A=-2/3$ and $B=1/2$. Thus for the octagon we have $D=4(-3+8)=20$ diagonals. For the square it was $D= (4/2)[-3+4]=2$ and for a pentagon it was $D=(5/2)[-3+5]=5$.

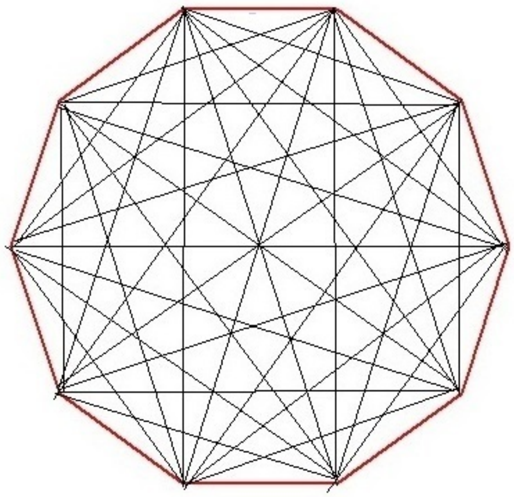
The relation between A,V and C remains as is, namely,-

$$A+V-C=1$$

A color count for the subdivided octagon shown yields A=80 and V=71. Thus C=150.

We have not been able find a unique formula relating N or D to the number of sub-areas A created. One does note, however, that A increases rapidly in a monotonic manner with both increasing N and D. For a decagon, where N=10 and D=35, we find after some rather careful color counting that A=220. Here is its pattern-

DECAGON WITH ITS 10 SIDES, 35 DIAGONALS, AND 220 SUB-AREAS



One can fill in the various sub-areas with colors of your choice to make some pleasing pictures resembling a flower with symmetrically placed petals.

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