THE CATALAN CONSTANT

Another, less well-known, constant encountered in analysis is the Catalan constant defined as-

$$G=\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-...=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=.91596559417721901505..$$

By looking at our earlier discussion on series summation by Laplace transforms, this series may also be written as the equivalent integral-

$$G=\int_0^{\infty}\frac{t}{2\cosh(t)}dt$$

Also one notes the similarity between the infinite series for $\pi^2/8$ and that for G. Subtracting these two series from each other, one finds-

$$G-\frac{\pi^2}{8}=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}[(-1)^n-1]$$

which can be converted to-

$$G=\frac{\pi^2}{8}-2\sum_{n=0}^{\infty}\frac{1}{(3+4n)^2}=\frac{\pi^2}{8}-\int_0^{\infty}\frac{t\exp(-t)}{\sinh(2t)}dt$$

The last integral is again established via Laplace transforms and resembles one representing the polygamma function Psi(n,z). Indeed, from the **Handbook of Mathematical Functions** by **Abramowitz and Stegun**, one has the Psi function definition-

$$\Psi(n,z)=\frac{d^{n+1}}{dz^{n+1}}[\ln(\Gamma(z))]=(-1)^{n+1}\int_{n=0}^{\infty}\frac{t^n\exp(-zt)}{[1-\exp(-t)]}dt$$

So on setting **z=3/4** and **n=1**, one arrives at the additional identity-

$$G=\frac{[\pi^2-\Psi(1,3/4)]}{8}$$

which clearly indicates a linear relation between Catalan's constant G and the trigamma function Psi(1, z) evaluated at z=3/4.

Many additional identities for G are found in the literature. For example, one has-

$$G=\int_{n=0}^{1}\frac{\arctan(x)}{x}dx$$

which follows by use of the expansion arctan(x)=x -1/3 x^3-1/5x^5+... and also -

$$G=-\int_{n=0}^{1}\frac{\ln(x)}{(1+x^2)}dx$$

which follows by making the substitution x=exp(-t) into the above integral expression for G involving sinh(2t). Many other(and generally more complicated) identities can be found on the internet. One rather lengthy but very rapidly converging series for G is the Broadhurst (1998)formula-

$$\begin{aligned}G=3\sum_{k=0}^{\infty}\frac{1}{2\cdot 16^k}\Big(&\frac{1}{(8k+1)^2}-\frac{1}{(8k+2)^2}+\frac{1}{2(8k+3)^2}\\&-\frac{1}{2^2(8k+5)^2}+\frac{1}{2^2(8k+6)^2}-\frac{1}{2^3(8k+7)^2}\Big)\\-2\sum_{k=0}^{\infty}\frac{1}{8\cdot 16^{3k}}\Big(&\frac{1}{(8k+1)^2}+\frac{1}{2(8k+2)^2}+\frac{1}{2^3(8k+3)^2}\\&-\frac{1}{2^6(8k+5)^2}-\frac{1}{2^7(8k+6)^2}-\frac{1}{2^9(8k+7)^2}\Big)\end{aligned}$$

This formula yields a result accurate to ten decimal places by just taking the first six terms (k=0 through 5) in the two indicated infinite series. Note that G is most likely an irrational number although no formal mathematical proof of this conjecture has been given. Computer runs to several million places have shown no repetitive patterns as would be expected for a rational number.