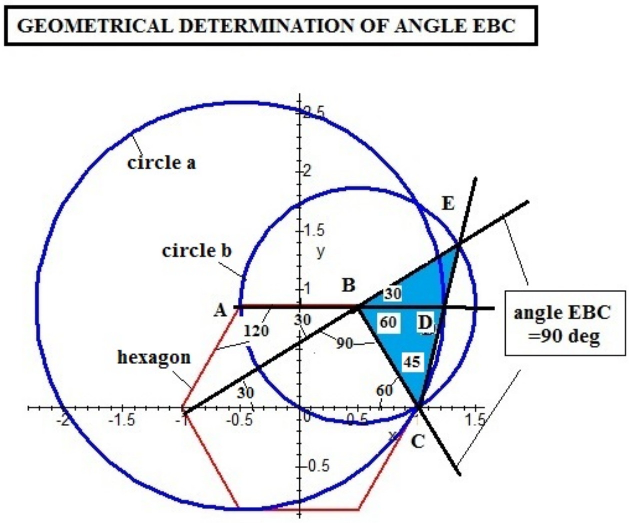
ON THE INTERSECTION OF REGULAR POLYONS WITH CIRCLES

In the November 19, 2016 weekly math puzzle page in the Wall Street Journal the question asked was to find the angle EBC between two intersecting lines constructed from a regular hexagon and two intersecting circles as shown. A sketch of the puzzle follows-



The question was posed as follows-

(1) -Given a regular hexagon of side length one and three neighboring vertex points A, B and C as shown.
(2)-Draw in two circles, the first circle ‘a’ has radius R=sqrt(3) and is centered at [-1/2,sqrt(3)/2] the second smaller circle ‘b’ has radius r=1 and is centered at [1/2,sqrt(3)/2].
(3)-Next draw a straight line through vertexes A and B to intersect circle ‘a’ at D.
(4)- Draw another straight line from vertex C through D to intersect circle ‘b’ at E.
(5)-Now find the angle EBC

The solution is straight forward by noting that each of the six inner angles of the hexagon equal 120 degree. Also if one draws a third straight line going through E and B it will intersect the hexagon at F. Since any of the sub-triangles in the graph add up to a total of 180 deg, we can easily mark the values of the angles shown. Hence angle EBC must equal 30+60=90 deg. The secret to the solution is drawing the third straight line EBF. Alternate routes for finding this answer are possible, but none as simple as this.

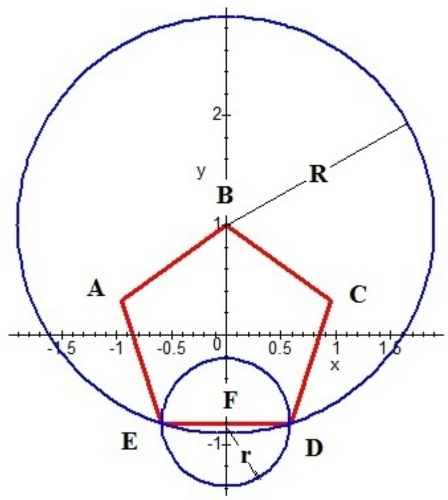
We can generalize the above puszzle by considering any regular polygon centered on the origin and having its n vertexes located at-

$$[r,\theta]=[1, c_o+\frac{2\pi}{n}k] \qquad k=1,2,3,...,n$$

, when expressed in polar coordinates. Here c_o is a constant which determines the rotation angle of the polygon and the distance from the polygon center to any of the vertexes equals 1. This means the sidelengths are s=2 sin(π/n) each. For a square(n=4) where r=1 we have s=sqrt(2).

Consider the following pentagon with radial distances from any of the vertexes A, B ,C, D, and E to the pentagon center of one-

FINDING THE RATIO R/(2r) FOR A PENTAGON AND TWO CIRCLES



We draw two circles one centered at B of radius R large enough to pass through vertexes D and E. A seconf smaller circle of radius r is centered at F and passes through vertexes E and D. The question asked is what is the ratio of R/2r? The solution is found via a little trigonometry using two isosceles triangles. The analysis yields-

$$R=\sqrt{2[1-\cos(\frac{4\pi}{5})]} \quad and \quad r=\sqrt{2[1-\cos(\frac{2\pi}{5})]}$$

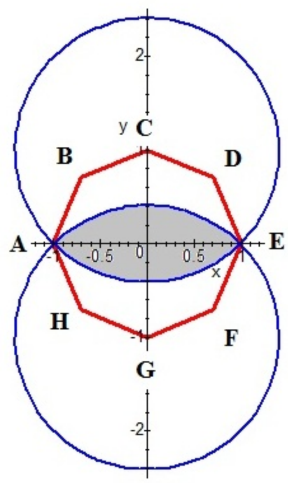
On taking the ratio of R to 2r, we find-

$$\frac{R}{2r} = \frac{1+\sqrt{5}}{2} = 1.6180339887498948482045868343\ldots$$

This value represents the famous Golden Ratio already known to the ancient Greek mathematicians.

We consider next an octagon of unit radial distance from its centewr to any of the eight vertexes. On this octagon we superimpose two equal radius circles of radius R=sqrt(2) centered at [x,y]=[0,1] and [0,-1] as shown-

FINDING THE AREA OF THE FOOTBALL SHAPED REGION LYING WITHIN AN OCTAGON



The questiojn posed is to find the area of the football shaped grey region within the octagon <u>without</u> the use of calculus. Via calculus one simply has-

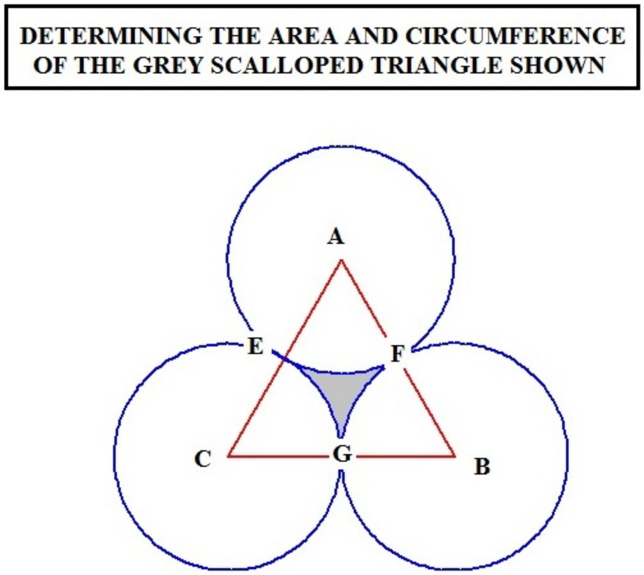
$$Area = 4\int_{x=0}^{1}[-1+\sqrt{2-x^2}\,] = (\pi - 2)$$

To get this answer by geometry we note that the cone shaped sector CAE has area 2π(1/4) and the triangle CAE has area 1. So the entire shaded region yields an area-

Area=2{2π(1/4)-1}=(π-2)

which agrees with the integral result. Note that in either case the area of the octagon was not directly involved.

The next problem deals with an equilateral triangle of side-length s=sqrt(3) plus three circles of radius sqrt(3)/2 each centered on the three vertexes A, B and C as shown-



The question is to find both the area and circumference of the gray scalloped triangle formed by the circles. Since each of the interior triangle angles equals π/3 rad, the three cone-shaped areas within the triangle have a total area of-

$$Area_{cones}=3[\pi\ sqrt(3)^2/24)=3\pi/8$$

While the area of the equilateral triangle is-

$$Area_{triangle}=3sqrt(3)/4$$

This leaves us with the grey scalloped triangle area of

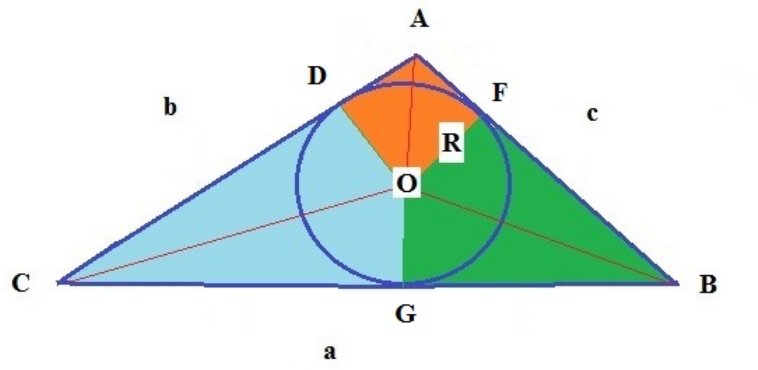
$$A_{gray-triangle}=(\frac{3}{8})\{2\sqrt{3}-\pi\}$$

The circumference of the scalloped grey triangle equals three times one-sixths of the circumference of a circle. That is-

$$C_{gray-triangle}=3(\pi\sqrt{3}/6)=\pi\sqrt{3}$$

As another problem we look at the classic puzzle of finding the area of an oblique triangle ABC by placing a circle of radius R at the center of the triangle large enough to just be tangent to the triangle sides at D, F, G as shown-

DERIVATION OF HERON'S AREA FORMULA FOR A TRIANGLE



By drawing three lines bisecting the angles A, B, and C which meet at the center O and then three more lines from O and perpendicular to the three sides of the triangle, one creates a total of six right triangles marked in orange, light blue, and green. There are two identical triangles for each color. The total area of triangle ABC is then-

$$A_{\text{triangle}}=R(a+b+c)/2=Rs/2$$

, wherere s is the half-perimeter. It is a bit more difficult to find the value of R but this can be done in a round-about manner. Using the Law of Cosines we have from the figure that-

$$\cos(C)=\frac{c^2-a^2-b^2}{2ab}$$

And we know that the area of any triangle may be written as-

$$A_{triangle}=\frac{1}{2}ab\sin(C)$$

Hence we find-

$$A_{\text{triangle}}=(\frac{1}{4})\sqrt{(2ab)^2-(c^2-a^2-b^2)^2}$$

So one finds that-

$$R = (\frac{1}{2})\{\frac{\sqrt{(2ab)^2 - (c^2 - a^2 - b^2)^2}}{(a+b+c)}$$

Thus for an equilateral triangle where a=b=c=sqrt(3), we find R=1/2.
By expanding the radical for $A_{triangle}$ in the above expression we get-

$$4A_{triangle} = \sqrt{[c^2 - (a-b)^2][(a+b)^2 - c^2]}$$
$$= \sqrt{[a+b+c-2a][a+b+c-2b][a+b+c-2c][a+b+c]}$$

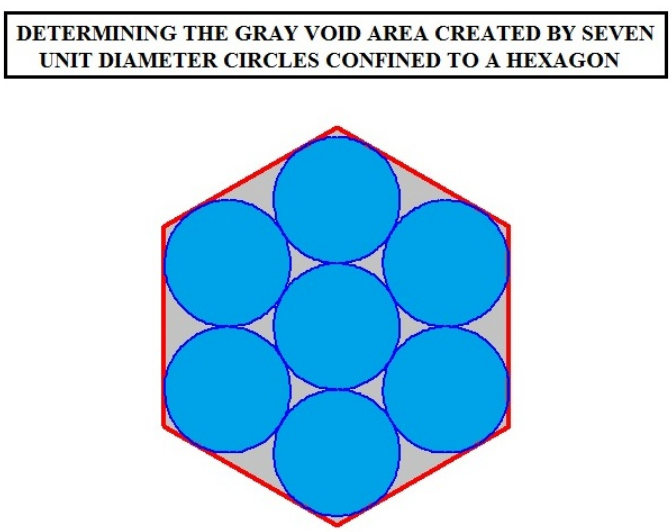
Now recalling that the semi-perimeter is defined as s=(a+b+c)/2, we obtain the famous Heron formula for the area of any triangle-

$$A_{triangle} = \sqrt{s(s-a)(s-b)(s-c)}$$

Heron actually derived his result based purely on geometry. The proof is a bit more messy than that gotten here using just algebra.

I remember over sixty years ago back in my high school geometry class that my teacher had no idea how Heron's Formula is derived. I suspect the same is still true in today's geometry classes.

As a final problem, we have a collection of even unit diameter circles tightly packed into a hexagon as shown-



We then ask the question "What is the area of the sum of the gray shaded spacings between the circles and the hexagon?

A closer inspection indicates that there are three distinct gray areas repeated six times each. We call the areas A_{large}, A_{medium}, and A_{small}. A bit of manipulation then shows the total grey area to be-

$$\begin{aligned} A_{grayarea} &= A_{large} + A_{medium} + A_{small} \\ &= \frac{3}{4}(4-\pi) + \frac{3}{4}(2\sqrt{3}-\pi) + \frac{1}{4}(2\sqrt{3}-\pi) \\ &= (2\sqrt{3}+3-7\pi/4 = 0.966314.. \end{aligned}$$

This means the total grey area equals just 1.2303 times the area of one of the circles.

Black Monday
Nov.25, 2016