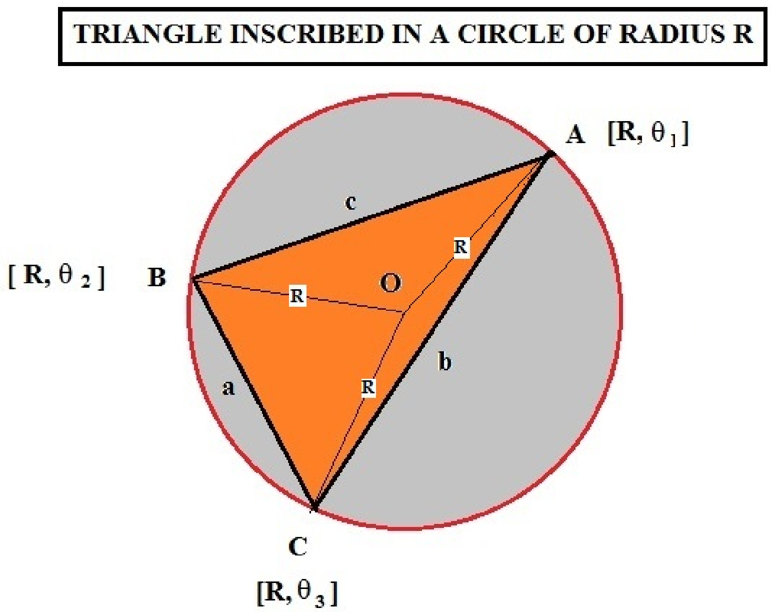
PROPERTIES OF CIRCLES AND TRIANGLES

Circles and triangles form two of the simplest 2D geometrical figures. In terms of formulas, the circle is defined as-

$$(x-\alpha)^2+(y-\beta)^2=R^2$$

,where [x,y]=[α,β] is the circle center and R its radius. If we center the circle on the origin such that α=β=0 and choose three points A, B, and C on the circle and then connect these points with straight lines, the following triangle will result. The length of the triangle sides, which can also be interpreted as chords of the circle, are designated by a, b, and c as shown-



From the geometry and the Law of Cosines one sees at once that the three chords have length–

$$a=2R\sin(\frac{\theta_3-\theta_2}{2})\quad,\quad b=2R\sin(\frac{\theta_1-\theta_3}{2})\quad,\quad c=2R\sin(\frac{\theta_2-\theta_1}{2})$$

The area of the triangle will be the sum of these three isosceles sub-triangles. It is given by-

$$A_T=\frac{1}{4}\left\{a\sqrt{4R^2-a^2}+b\sqrt{4R^2-b^2}+c\sqrt{4R^2-c^2}\right\}$$

The presence of R in this expression is a little inconvenient. So one looks at an alternative way to obtain the area of triangle ABC. We define a new angle ψ as the angle opposite side c. Then from the Law of Cosines and the area of any full oblique triangle, we get-

$$\cos\psi = \frac{a^2 + b^2 - c^2}{2ab} \quad and \quad \sin\psi = \frac{2A_T}{ab}$$

On squaring both equations and then adding, we find-

$$A_T = (\frac{1}{4})\sqrt{(2ab)^2 - (a^2 + b^2 - c^2)^2}$$

This area formula is independent of R and simpler in form than the classic Heron Formula. It will give the area of any triangle knowing only the length of its three sides. By equating the two forms of A_T given, it is also possible to find the value of R for the circle inscribing any triangle. Notice that if vertexes A and B are located along the same diameter of the circumscribing circle then ψ will always be π/2rad=90deg. This is a well known result one learns about in elementary trigonometry.

Let us look at an equilateral triangle of side length s=a=b=c. In this case –

$$A_T = (\frac{\sqrt{3}}{4})s^2 = (\frac{3s}{4})\sqrt{4R^2 - s^2}$$

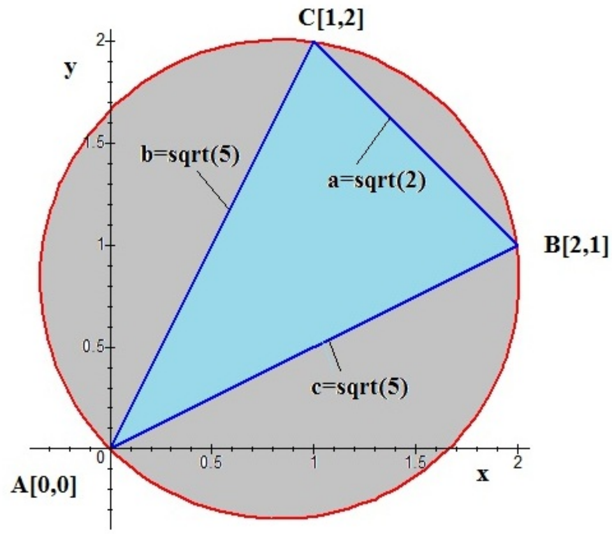
On solving, we find R=s/sqrt(3). Note that only for this case does the triangle centroid coincide with the circle center.

Take next a triangle with corners at A[0,0], B[2,1], and C[1,2]. Let us find the circle which inscribes it and the area of the triangle. In this case the side-length of the triangle are sqrt(2), sqrt(5), and sqrt(5). So its area is-

$$A_T = (\frac{1}{4})\sqrt{40 - (5 - 2 - 5)^2} = \frac{3}{2}$$

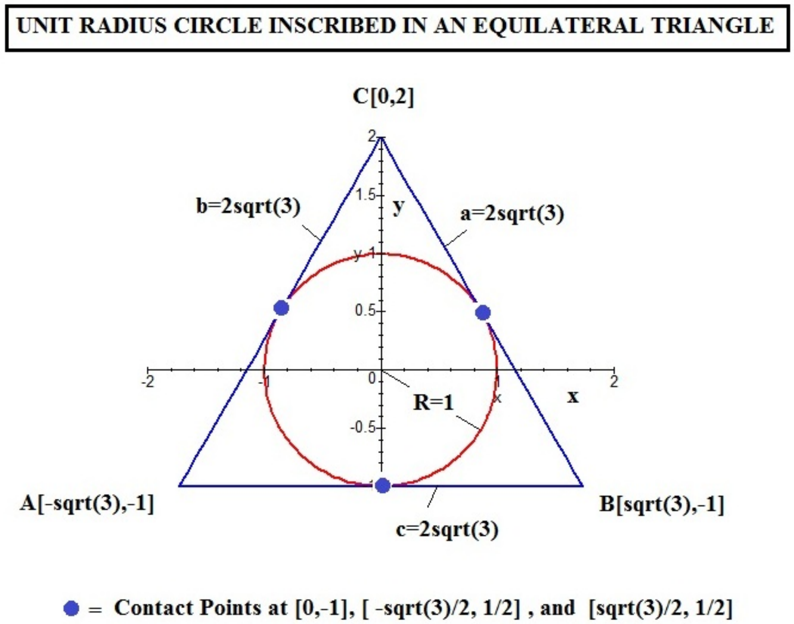
Also the three points are sufficient to define a unique circle with α=β=5/6 and R=5sqrt(2)/6. A graph of this triangle and the circumscribing circle look like this-

CIRCLE (x-5/6)^2+(y-5/6)^2=50/36 INSCRIBING A TRIANGLE OF AREA 3/2



Note here that the centroid of the inscribed isosceles triangle is at [1, 1] while the circle center is at [5/6 ,5/6]. The centroid was here determined by recalling that the centroid of this type of triangle lies at 2/3 of the triangle height H along the symmetry line at π/4. By the Pythagorean Theorem we have H=3/sqrt(2). Hence for the centroid we find x=sqrt(2)cos(π/4)=1 and y=sqrt(2)sin(π/4)=1.

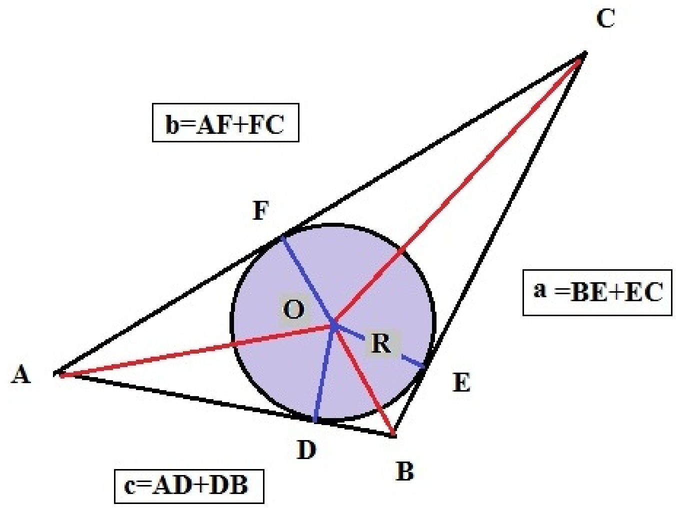
Let us m next look at the problem of a circle centered on the origin and radius R inscribed in any triangle. Beginning with the simplest example , let the triangle be an equilateral triangle of side-length s=3sqrt(3) each. A unit radius circle just encloses this triangle as shown-



We notice that this time the centroid of the triangle and circle coincide and that the distance from circle center to any of the triangle vertexes is just 2 units. One has a total of six , equal area, sub-triangles connecting a vertex with the circle center and with the contact point where the circle touches the triangle. The area of each sub-triangle is just sqrt(3)/2 so that the total area of the equilateral triangle is A_T=3sqrt(3). The fraction of the triangle area covered by the circle is f=π/[3sqrt(3)] .

We can also consider inscribing a circle in an oblique triangle. This time we get the following picture-

OBLIQUE TRIANGLE CIRCUMSCRIBING A CIRCLE



The circle is tangent to the triangle at points D, E, and F and there are three pair of equal area sub-triangles. We find the total triangle area equal to-

$$A_T = (\frac{R}{2})(a+b+c)$$

Comparing this result with the area determined earlier, we find-

$$R = \frac{\sqrt{(2ab)^2 - (a^2+b^2-c^2)^2}}{2(a+b+c)}$$

This means that the radius of the inscribed circle will be known once the sides a, b, and c of the triangle are specified. For the equilateral triangle discussed earlier we had the sides equal to a=b=c=2sqrt(3). So the radius of the inscribed circle is –

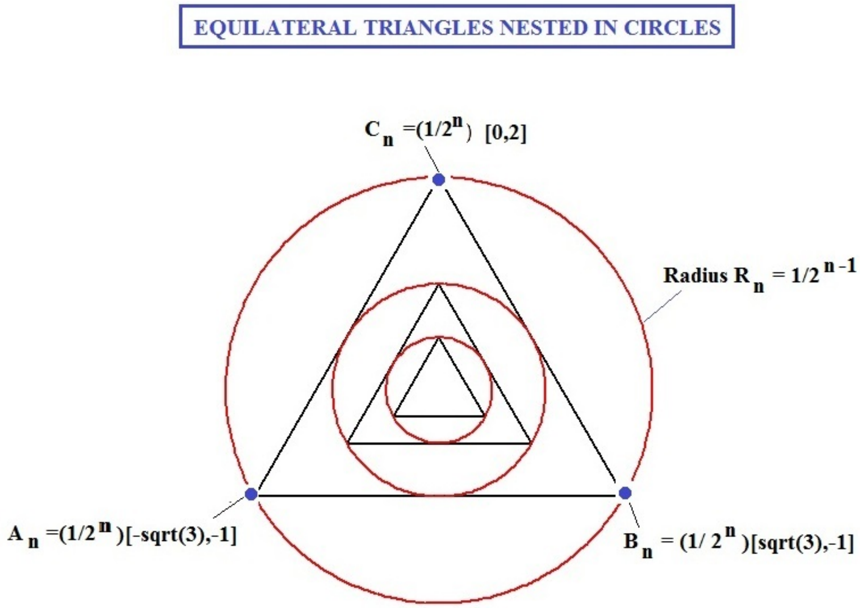
$$R = \frac{\sqrt{4a^4 - a^4}}{2(3a)} = \frac{\sqrt{3}}{6}a = 1$$

as expected. For a right triangle, where $a^2+b^2=c^2$, we get –

$$R = \frac{ab}{a+b+c}$$

If angles A and B of a right triangle are both π/4 radians, then R=c{1/[2(sqrt(2)+1)]}.

Another interesting feature of circles and triangles is that they can be made to nest into each other. Let us demonstrate this fact by placing an equilateral triangle of side-length 2sqrt(3) into a circle of radius R=2. Next inscribe a smaller circle of radius R=1 into a new triangle of side-length sqrt(3). Continuing the procedure we get the nested pattern shown-



It is quite clear that the nested circles have radius $R=2^{1-n}$ and so decrease their radius by a factor of two at each step. The vertexes of the triangles are located at-

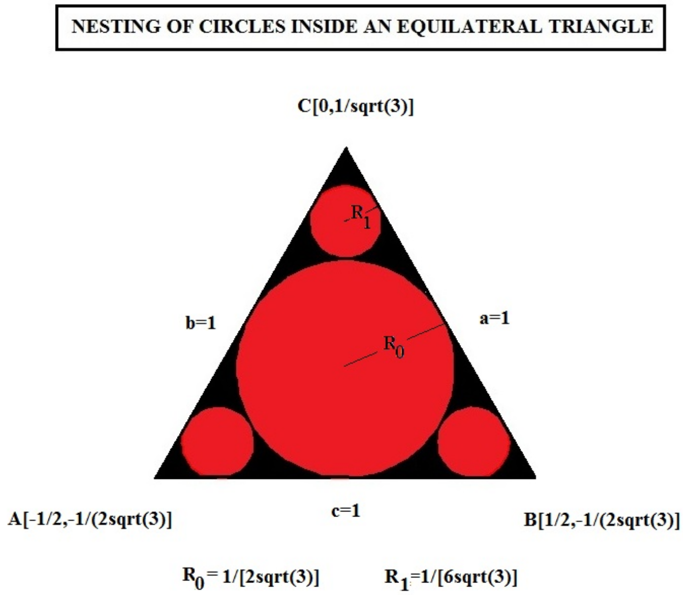
$$A_n = \frac{1}{2^n}[-\sqrt{3},-1] \quad , \quad B_n = \frac{1}{2^n}[\sqrt{3},-1] \quad , and \quad C_n = \frac{1}{2^n}[0,2]$$

At each step, starting with n=0 , the circle area πR^2 and triangle area 3sqrt(3) decease by a factor of four . Also the circles and triangles remain centered on the origin at [0,0].

Another type of nesting involving one equilateral triangle filled with progressively smaller circles until the area is filled. In this case we start with and equilateral triangle of three equal sides of length one each. We then place the largest possible circle into this triangle. It will have a radius of R_0=1/[2sqrt(3) . Next one draws a smaller circle of radius R1=1/[6sqrt(3)] centered at [x.y]=[0,2/(3sqrt(3)]. It will be tangent to both the larger circle and two sides of the triangle. We follow this by rotating this last circle through θ=±2π/3 radians using the rotation transformation-

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

This allows us to write the equations for the two remaining circles of radius R_1. The following picture results-



One can continue the procedure with the next three smaller circles placed near the corners of the triangle at A, B, and C and having a radius of R_2=1/[18 sqrt(3)]. Continuing on, we have the identity-

$$\frac{1}{\sqrt{3}} = R_0 + 2\{R_1 + R_2 + R_3 + \ldots\}$$

So using the geometric series, we see that $R_n=1/[2 \times 3^n]$, since R_0=1/[2sqrt(3)]. One still has some left over gaps bordered by two scalloped sides plus a third straight side. Circles fitting into these gaps can be found but will have rather more complicated radii than the other circles. We will reserve such a calculation for later date. Such a calculations will be related to the classic problem of Appolonius in which one shows that the largest circle which just fits into the gap between three unit diameter circles is R=[2sqrt(3)-3]/3.