CONSTRUCTION OF THE GOLDEN RAIO AND SOME OF ITS BETTER KNOWN PROPERTIES

The irrational number-

$$\varphi = [1+\text{sqrt}(5)]/2=1.61803398\ldots$$

Is known as the Golden Ratio. It arises in many areas of mathematical analysis and is closely linked to the numbers in the familiar Fibonacci sequence-

$$F_n=\{0,1,1,2,5,8,13,21,34,55,\}$$

The ratio is also defined by the algebraic equation-

$$\varphi^2=\varphi+1$$

This follows from-

$$\varphi^2=\frac{[1+sqrt(5)]^2}{4}=\varphi+1$$

It is the purpose of this note to present a geometric derivation of the Golden Ratio and also discuss its powers.

We begin by constructing a table for the powers of φ. Simple substitution produces-

$$\varphi\text{^}2=1+\varphi$$

$$\varphi\text{^}3=1+2\varphi$$

$$\varphi\text{^}4=2+3\varphi$$

$$\varphi\text{^}5=3+5\varphi$$

$$\varphi\text{^}6=5+8\varphi$$

$$\varphi\text{^}7=8+13\varphi$$

$$\varphi\text{^}8=13+21\varphi$$

$$\varphi\text{^}9=21+34\varphi$$

$$\varphi\text{^}10=34+55\varphi$$

One of the questions on the recent math Olympiad was to find φ^12. The answer is obvious-

$$\varphi^{12} = 89 + 144\varphi = 161+72\text{sqrt}(5)$$

From this table we also see at once that the nth power of φ equals the sum of the (n-1) plus the (n-2) power. Thus, $\varphi^{11} = \varphi^{10} + \varphi^{9} =$ 55+89φ . You will also notice that the constants in these equations go as a Fibonacci sequence-

$$F_n=\{ 1,1,2,3,5,8,13,21,34,..\}$$

The next number in this sequence goes as the sum of the two previous numbers. So the next term after 34 will be the Fibonacci number 21+34=55.

One can generate a continuous fraction for the Golden Ratio starting with the basic definition divided by φ. That is-

$$\varphi=1+1/\varphi$$

A repeated substitution of φ on the right side using the left definition produces the continuous fraction-

$$\varphi = 1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}}$$

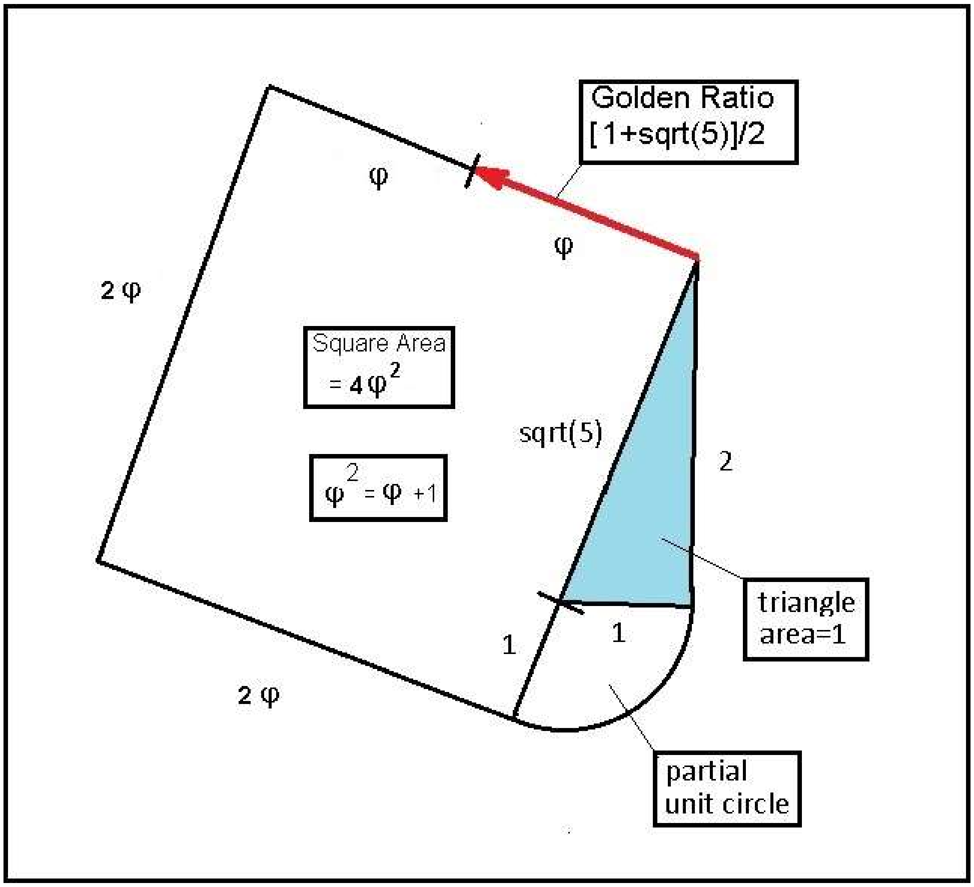
Taking things out to n terms produces the rational φ approximations-

{ 3/2,5/3,8/5,13/8,21/13,34/21,55/34,89/55,..}

The last term shown yields 89/55=1.61818 compared to the exact value of 1.6180339. These results are also obtainable from the iterative form-

$$\varphi[n+1] = 1+1/\varphi[n] \qquad \text{subject to } \varphi[0] = 2$$

We next consider a geometric construction of the Golden Ratio and its powers. One starts with a right triangle of base length one , height two, and hypotenuse sqrt(5). Next one draws a partial circle of radius one centered at the left corner of the triangle as shown-



. Combining the length sqrt(5) plus the radius of unity, yields a line equal to 2φ.

. One next draws a square of side length 2φ . On subtracting φ from the top side of the square, we are left with the Length of a Golden Ratio indicated in red. The area of the square equals 4φ^2 =10.47213.. .

Finally we come to a quick and accurate approximation for the numerical value of the Golden Ratio. To get a rapidly converging answer we start with the Diophantine equation-

Y^2=1+5x^2-

This will have integer solutions-

x	y
4	9

72	161
1292	2889
23184	51841
416920	930249
7465176	1669264159

Re-writing the above equation and using the largest values of x and y in the above table, we find-

$$\mathrm{sqrt}(5)=(y/x)\mathrm{sqrt}\{1-1/(y^2)\}$$

or the final desired form-

$$\varphi = (1/2)\{1+(y/x)\mathrm{sqrt}\ (1-1/y^2)\} =(1+\mathrm{sqrt}(5))/2$$

with x=7465176 and y=16692641 . An approximation to this exact solution is gotten by using the series expansion-

$$\mathrm{sqrt}(1-1/y^2)=1-1/(2y^2)+O(1/y4)$$

That is –

$$\varphi\sim \{ \quad 1+(y/x)(1-1/(2*y^2)) \} /2$$

So we know that here the series for sqrt(1-1/(2y^2)) using only the first two terms produces an estimate for φ good to 32 digits.

Finally let us give the value of the Golden Ratio good to one hundred digits-

φ =1.6180339887498948482045868343656381177203091798057628621354486
227052604628189024497072072041893911 38

The number is clearly irrational although the approximations will be rational.

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