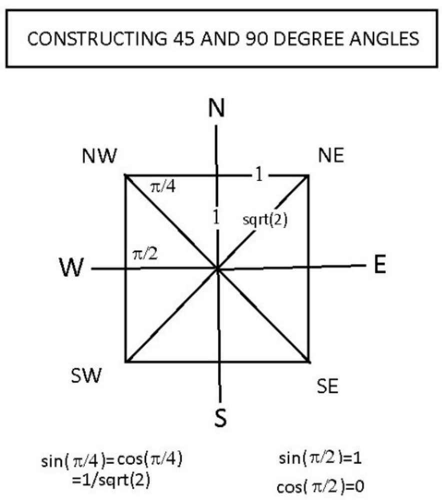
CONSTRUCTION OF ANGLES USING ONLY RULER AND COMPASS

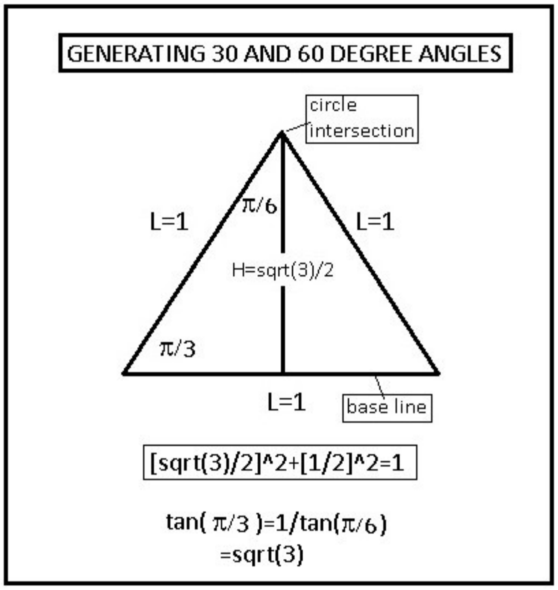
It is well known that that angles θ=π/2, π/3, π/4, and π/6 radians can easily be constructed using only ruler and compass. We want in this note to show how these and other angle values can also be generated using only length measures provided by a simple compass.

(1)-π/2 AND π/4 ANGLES: Here we start by drawing a square plus its two diagonal lines. If the diagonals meet exactly at the square center then we know we have a true square with π/2 and π/4 given as shown in the following-



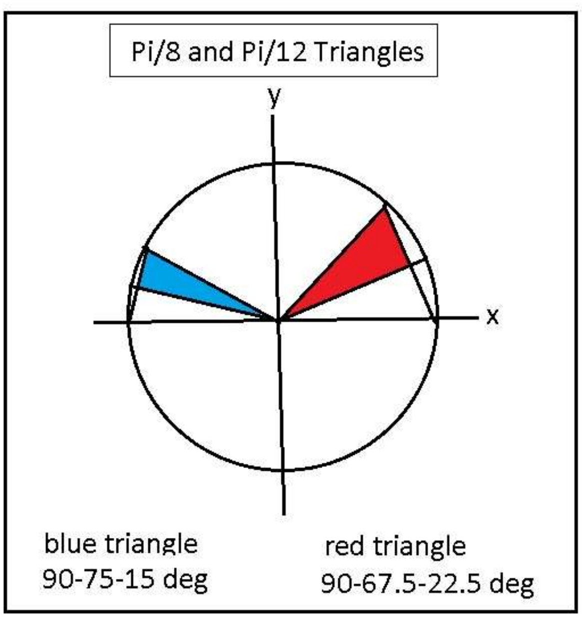
In my woodworking shop I check the accuracy of the right angles of a square by showing the length of the two diagonals to be exactly equal. By bisecting the large square by a vertical and horizontal line into four equal smaller squares, we get the eight compass directions N-NE-E-SE-S-SW-W-NW. We also see that any of the eight sub-triangles shown satisfy the Pythagorean Theorem 1^2+1^2=[sqrt(2)]^2

(2)-π/3 AND π/6 ANGLES: Here we begin with a straight horizontal line of unit length L=1. Next we draw two circles of radius R=L=1 centered at the end points of horizontal line L=1 . Connecting the circle crossing point with straight lines to the L ends produces the following exact equilateral triangle-

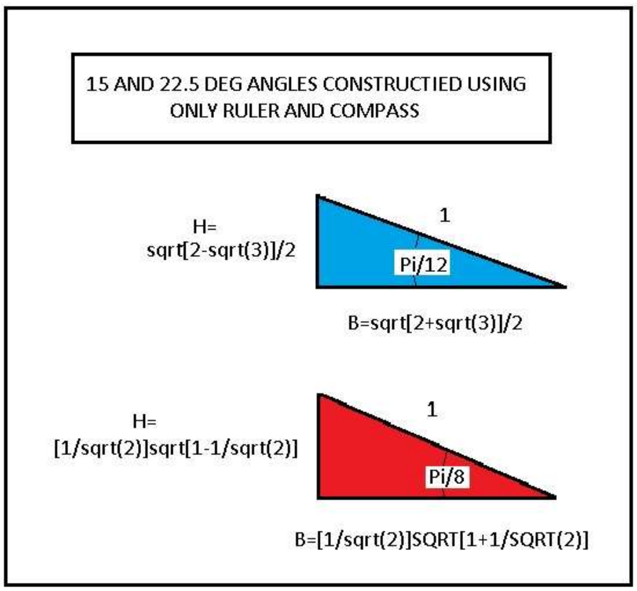


By bisecting the equilateral triangle by a vertical dividing line we get two identical sub-triangles of sides sqrt(3)/2-1/2-1. The three angles of either of the smaller triangles are 90-60-30 degrees or π/2-π/3-π/6 in terms of radians. The center of the large equilateral triangle lies at [x,y]=[0,1/(2*sqrt(3))].This point is also the location of the centroid of a piece of cardboard of this equilateral triangle shape.

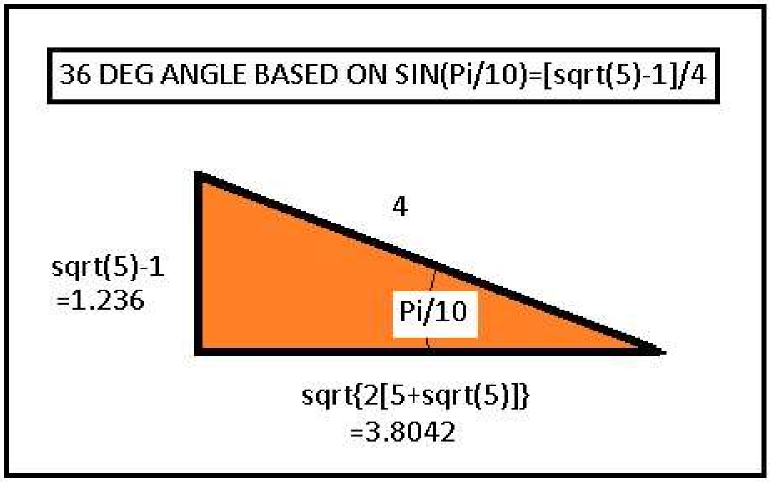
(3)-ANGELS SMALLER THAN π/6 RADIANS: The simplest way to generate angles less than 30 degrees(or π/6 radians) is to start with a compass drawn unit radius circle with a horizontal and a vertical line through its center as shown-



Next draw in the known sectors of angle 30 and 45 deg.. Halve these and you get a blue triangle with Pi/12 and a red triangle with Pi/8 radian as the smallest angles. Using the law of cosines we find the height of the blue triangle to be H=(1/2)sqrt[2-sqrt(3)]=0.2588… and the height of the red triangle to be H=[1/sqrt(2)]sqrt[1-1/sqrt(2)]=0.3827… . Using the Pythagorean Theorem the bases B of the blue and red triangle are sqrt[2+sqrt(3)]/2 and [1/sqrt(2)]{sqrt[(1+1/sqrt(2)]}, respectively. Here are the two triangles associated with the constructed angles of 15 and 22.5 deg..-

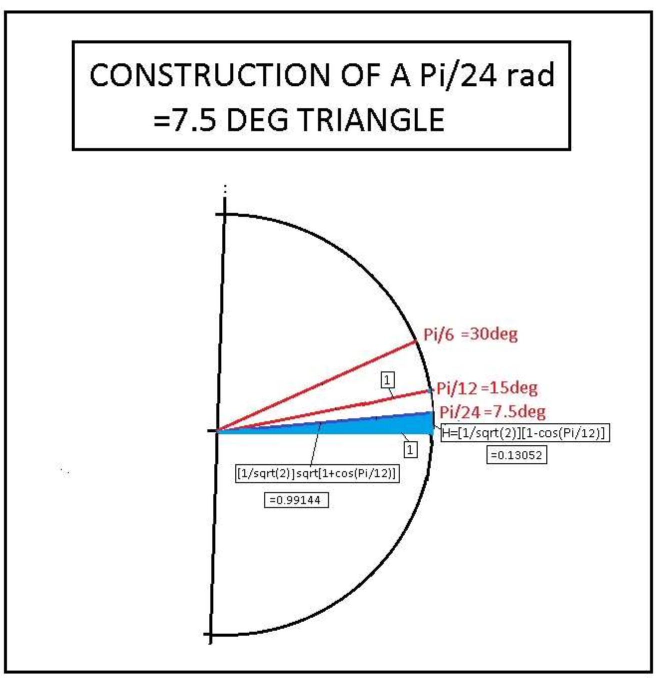


Another way one can construct smaller angles is to start off with a trigonometric form of the angle and then build a triangle from there.Take sin(π/10)=[SQRT(5)-1]/4.This produces a triangle as shown-

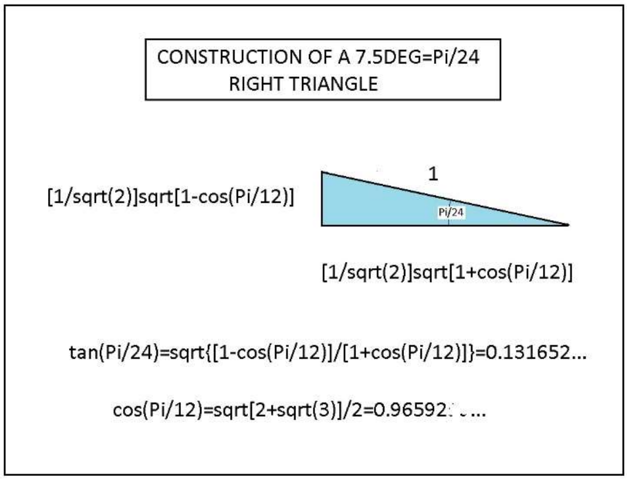


The results tell us that tan(Pi/10)=[sqrt(5)-1] /sqrt[2(5+sqrt(5)] =0.324919… .

We also can write sin(Pi/10)=[sqrt(5)-1]/4 =(1/2)(φ-1),where φ=1.6180..is the golden ratio. Smaller angles of θ=π/n, for n>12 do not lend themselves to simple radical forms using my PC. However continued halving of angle θ=π/n to θ=π/2n, etc.is always possible. Consider the case of θ=π/24 radians=7.5 deg. The construction for this angle goes as follows-



Using the Pythagorean Theorem and the Law of Cosines, this leads to the following triangle and its side-lengths-



(4)-CONCLUSION: It is possible to construct right triangles and their smallest angle by using only a compass and ruler. We specifically work out the angles π/n for n=2,3,4,6,8,10,12, 20, and 24. For these θs all trigonometric functions are expressible as integers and their roots. Thus, for example, we find-

sin(Pi/20)=[1/sqrt(2)]sqrt{1-(1/4)sqrt[10+*sqrt(5)]}=0.15643…

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