EVALUATIOM OF INTEGRALS USING CONTOUR INTEGRATION

In our lectures on integral solutions to differential equations using Laplace kernels ,we encountered integrals of the type-

$$y(x) = \oint_C \frac{f(t)\exp(xt)}{t^{n+1}}$$

where t=γ+iτ and C is a closed contour within the complex plane. To evaluate this type of integrals under conditions where the curve partially lies at infinity, one makes use of complex variable methods and in particular the Cauchy Integral Theorem. We present here this evaluation method using several specific examples.

Our a starting point is the Cauchy Integral Theorem-

$$\oint_C F(z)dz = 0 \text{ when } F(z) \text{ is analytic everywhere within } C$$

and Cauchy's Integral Formula-

$$\frac{d^n G(z_0)}{dz^n} = \frac{n!}{2\pi i}\oint_C \frac{G(z)dz}{(z-z_0)^{n+1}}$$

,where G(z) is analytic everywhere in C but the integrand has an (n+1) order pole at $z=z_0$ provided n is a positive integer.

Consider first the integral-

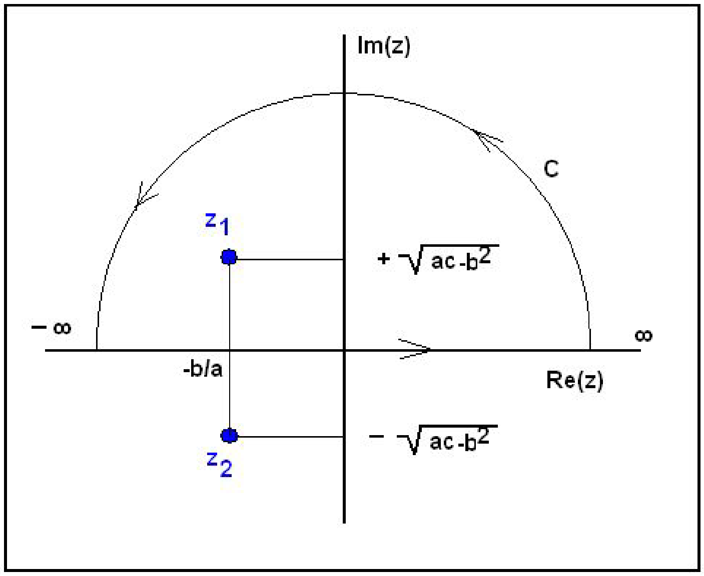
$$H(a,b,c) = \oint_C \frac{dz}{(az^2+2bz+c)} = \oint_C \frac{dz}{a[z-z_1][z-z_2]}$$

which has simple poles at-

$$z_1 = -\frac{b}{a} + i\sqrt{ac-b^2} \quad and \quad z_2 = -\frac{b}{a} - i\sqrt{ac-b^2}$$

where we are assuming a, b, and c are real and that $ac>b^2$.

We consider integration around the closed line integral defined as the infinite radius semicircle and the line z=x along the real z axis from minus to plus infinity as shown-



Here only the first order pole at $z=z_1$ lies within the closed contour C, so that the Cauchy Integral Formula reads-

$$\frac{1}{a}\oint_C \frac{(1/(z-z_2)dz}{z-z_1} = 2\pi i\frac{1}{a(z_1-z_2)} = \frac{\pi/a}{\sqrt{ac-b^2}}$$

From this last result and the fact that the integrand $1/(az^2+bz+c)$vanishes along the semicircle, leads to the result-

$$H(a,b,c) = \int_{-\infty}^{+\infty}\frac{dx}{ax^2+bx+c} = \frac{(\pi/a)}{\sqrt{ac-b^2}}$$

The simplest special case occurs for a=c=1 and b=0. There H(1,0,1)=π.

Look next at the integral-

$$K(a,n) = \oint_C \frac{dz}{(z^2+a^2)^n}$$

about the same semicircle contour C shown above. Here we have nth order poles at z=ia and z=-ia provided again that n is a positive integer . Again applying the Cauchy Integral Formula, one finds that-

$$K(a,n)=2\int_{x=0}^{\infty}\frac{dx}{(x^2+a^2)^n}=\frac{2\pi i}{(n-1)!}\left[\frac{d^{n-1}(1/(z+ia)^n}{dz^{n-1}}\right]_{z=ia}$$

Thus for a=1 and n=2 we have –

$$\int_{x=0}^{\infty}\frac{dx}{(z^4+2x^2+1)}=\frac{\pi i}{1!}\left[\frac{d}{dz}\left(\frac{1}{(z+i)^2}\right)\right]_{z=i}=\frac{\pi}{4}$$

Another definite integral which can be solved by complex variable methods is-

$$K(a)=\oint_C\frac{\exp(iz)}{(z^2+a^2)}dz=\oint_C\frac{[\cos(z)+i\sin(z)]}{(z-ia)(z+ia)}dz$$

where C is again the same semicircle contour shown above. Here we have simple poles at ±ia and need only consider the one lying along the positive Im(z) axis. One finds that-

$$K(a)=\frac{2\pi i}{0!}\left[\frac{\exp(iz)}{(z+ia)}\right]_{z=ia}=\frac{\pi}{a}\exp(-a)$$

so that-

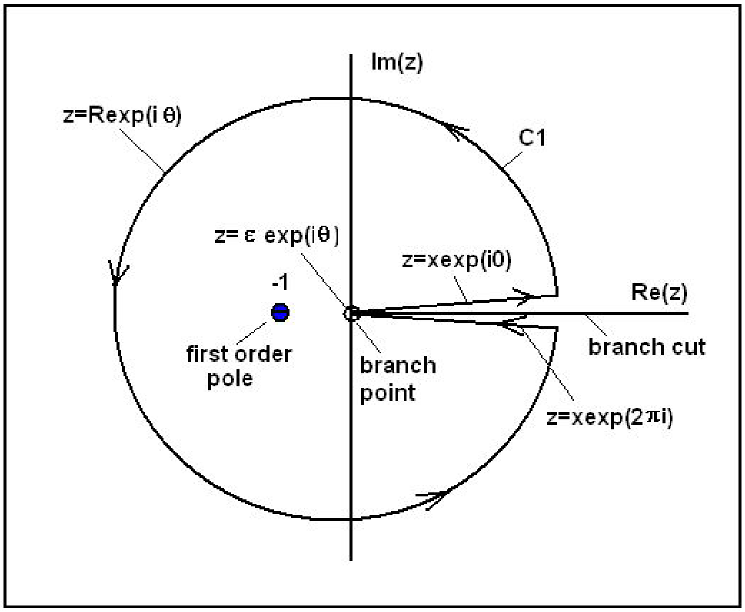
$$\int_{-\infty}^{+\infty}\frac{\cos(x)}{x^2+a^2}dx=\frac{\pi}{a}\exp(-a)\quad and\quad\int_{-\infty}^{+\infty}\frac{\sin(x)}{x^2+a^2}dx=0$$

The vanishing of the second integral can also be seen directly by noting the odd symmetry of sin(x).

Next consider an integral involving a multi-valued function requiring the use of a branch cut in its integration. Specifically we consider-

$$L = \oint_{C1} \frac{z^{\alpha-1}}{z+1} dz \text{ where } 0 < \alpha < 1$$

This time $z^{\alpha-1}$ is multivalued and so requires a branch cut to avoid non-uniqness. The contour chosen looks as follows-



Now according to the Cauchy theorem the value of the closed contour C1 is just $2\pi i(-1)^{\alpha-1}$. Furthermore if we let R become infinite and ε go to zero, the only non-zero contributions remain the contributions along the two straight lines z=x exp(i0) and z=x exp(2πi). One has-

$$2\pi i(-1)^{\alpha-1} = \int_{x=\infty}^{0} \frac{x^{\alpha-1}}{1+x} dx + \int_{x=0}^{\infty} \frac{[x\exp(2\pi i)]^{\alpha-1}}{1+x} dx$$

or-

$$\int_{x=0}^{\infty} \frac{x^{\alpha-1}}{1+x} dx = \frac{2\pi i\,(-1)^{\alpha-1}}{\exp[(\alpha-1)2\pi i]-1} = \frac{\pi}{\sin(\alpha\pi)}$$

You will notice that this result is also equal to the product Γ(α) Γ(1-α) of the Gamma function so that one has-

$$\Gamma(1/2)^2 = \pi = \int_{x=0}^{\infty} \frac{dx}{(1+x)\sqrt{x}} = 2\int_{x=0}^{\infty} \frac{du}{\cosh(u)} \; where \; x = \sinh(u)^2$$

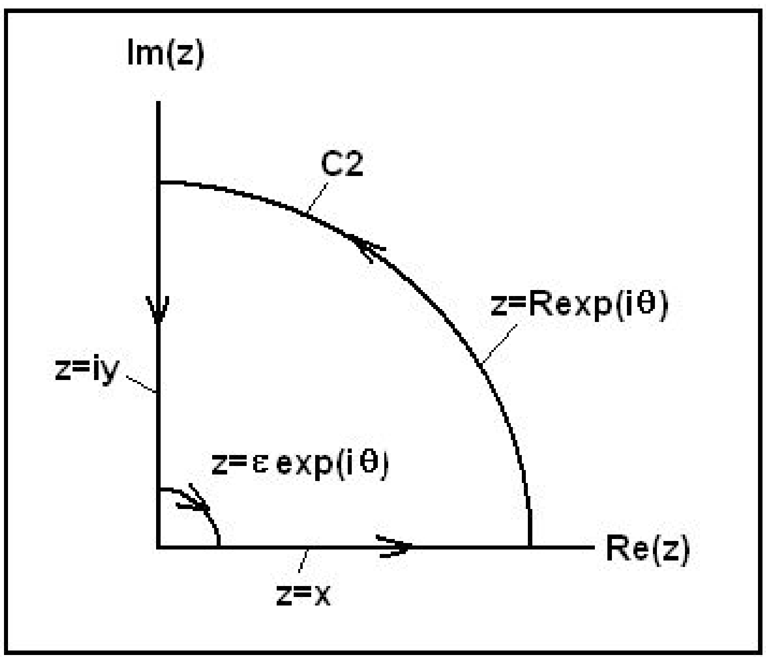
Another example of contour integration is that used to prove the identity-

$$P = \int_{x=-\infty}^{\infty} \frac{\sin(x)}{\sqrt{x}} dx = \sqrt{\frac{\pi}{2}}$$

Here we look at the related contour integral-

$$\oint_{C2} \frac{\exp(iz)}{\sqrt{z}} dz = \oint_{C2} \frac{\cos(z) + i\sin(z)}{\sqrt{z}} dz$$

where C2 is the contour shown-



Since there are no poles(or branch points)within the contour shown, one has, according to Cauchy's Theorem –

$$0=\int_{\varepsilon}^{R}\frac{\exp(ix)}{\sqrt{x}}dx+i\int_{\theta=0}^{\pi/2}\exp[iR\exp(i\theta)]\sqrt{R\exp(i\theta)}\,d\theta+\sqrt{i}\int_{y=R}^{\varepsilon}\frac{\exp(-y)}{\sqrt{y}}dy$$
$$+i\int_{\theta=\pi/2}^{0}\exp[i\varepsilon\exp(i\theta)]\sqrt{\varepsilon\exp(i\theta)}d\theta$$

If one now lets R become infinite and ε approach zero, the theta integrals vanish and we are left with-

$$\int_{x=0}^{\infty}\frac{[\cos(x)+i\sin(x)]}{\sqrt{x}}dx=\sqrt{i}\int_{y=0}^{\infty}\frac{\exp(-y)}{\sqrt{y}}dy=\sqrt{i}\,\Gamma(1/2)=\sqrt{\frac{\pi}{2}}(1+i)$$

from which we have our result .

Another example which can be nicely treated by contour integral methods is the integral-

$$M=\int_{x=-\infty}^{+\infty}\frac{dx}{\cosh(x)}=\pi$$

This time one notices that the complex function 1/cosh(z) has simple poles along the imaginary axis at z=iπ(2n+1)/2 and that the function vanishes at z=±∞. This suggest the use of a rectangular contour with sides along the z=x axis, along the line z=x+iπ, plus the two remaining sides at z=±∞+iy. This contour encloses the simple pole at z=iπ/2 and by the Cauchy Integral Formula leads to-

$$\int_{x=-\infty}^{\infty}\frac{dx}{\cosh(x)}+\int_{y=0}^{\pi}\frac{idy}{\cosh(\infty+iy)}+\int_{x=\infty}^{-\infty}\frac{dx}{\cosh(x+i\pi)}+\int_{y=\pi}^{0}\frac{idy}{\cosh(-\infty+iy)}=\frac{2\pi i}{\sinh(\frac{i\pi}{2})}$$

Now. noting that the two integrals involving y vanish and that the third integral just equals the first, we find our desired result that M=π.

Finally, it should be pointed out that one can also use contour integrations to obtain inverse Laplace transforms via the operation-

$$F(t)=L^{-1}[f(s)]=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}f(s)\exp(st)ds$$

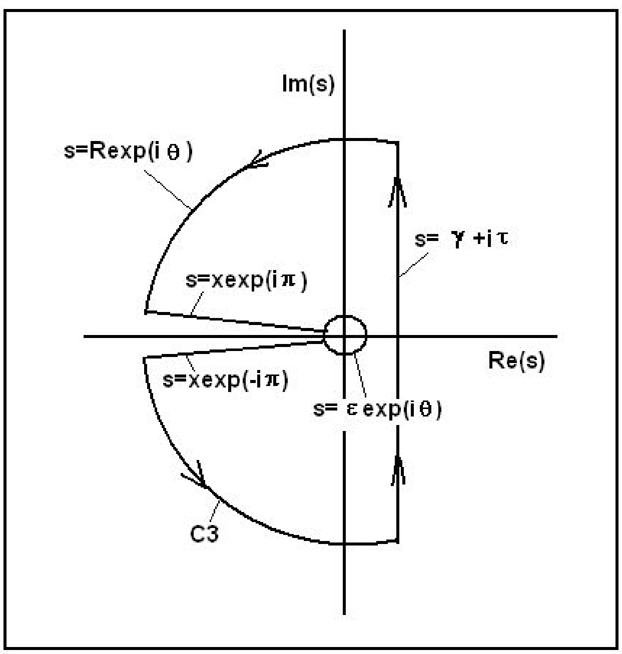
where the closed contour curve closes to the left of the line s=γ+iτ.
To demonstrate we consider f(s)=1/s^{n} so that one has a single nth order pole at s=0.
It then follows from the Cauchy Integral that-

$$F(t)=\frac{1}{2\pi i}\left[\frac{2\pi i}{(n-1)!}\right]\left[\frac{d^{n-1}\exp(st)}{ds^{n-1}}\right]_{s=0}=\frac{t^{n-1}}{(n-1)!}$$

A second example of inverting a Laplace transform we take-

$$L^{-1}f(s)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\exp(-k\sqrt{s})}{s}\exp(st)$$

This transform will be recognized as that obtained when dealing with time-dependent conduction into a semi-infinite bar whose initial temperature is zero and which has its end at x=0 maintained at constant temperature. Here one is dealing with a branch point at s=0 so one considers a contour C3 consisting of the line s=γ+i τ connected to a circle at abs(s)=∞ , two lines on the top and bottom of the negative Re(s) axis which serves as a branch cut, plus a small circle of radius ε about the origin. The contour is shown here-



Since there are no poles within the contour C3, the value of the integral about the entire closed contour must be zero. Furthermore the contribution around the outer circle vanishes as R goes to infinity and that about the inner circle is just 2πi as ε is allowed to go to zero. One is left with the remaining two contributions along the

lines s=x exp(iπ) and s=x exp(-iπ). After some manipulation (as shown in class) this results in-

$$F(t) = 1 - erf(\frac{k}{2\sqrt{t}}) = erfc(\frac{k}{2\sqrt{t}})$$

Typically one does not use the contour integration approach for simpler functions f(s) since these will be found in tables of Laplace transforms. However there are many more complex forms for which contour integration remains the only avenue of approach. For instance, I remember about ten years ago one of my students (Whitney Jones ,"Injection Cooling within a Micro-Heat Exchanger", MS thesis, University of Florida , 1999)was struggling with the inversion of Laplace transforms such as-

$$f(s) = \left(\frac{a}{s} + \frac{b}{s^2}\right)\left[1 - \frac{\cosh(c\sqrt{s})}{\cosh(d\sqrt{s}) + K\sinh(d\sqrt{s})\coth(ce\sqrt{s})}\right]$$

for which no inverses can be found in even the most extensive transform tables.

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