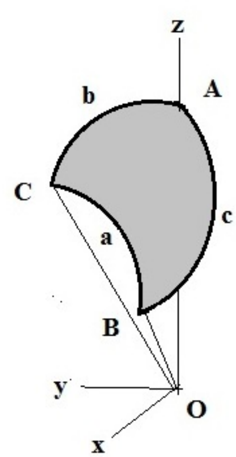
DERIVATION AND APPLICATION OF THE LAWS FOR SPHERICAL TRIANGLES

One of the more important topics for astronomers, mathematicians, solar energy enthusiasts, architects, etc is to be familiar with the laws governing spherical triangles. It is our purpose here to derive these laws in as simple a manner as possible and then apply them to several different specific problems.

Our starting point is to look at the following schematic for a spherical triangle and all its components-

GEOMETRY FOR A SPHERICAL TRIANGLE



The triangle is constructed by drawing three great circles on a unit radius sphere centered at O[0,0,0] . the spherical triangle has vertices at A, B, and C and its sides have lengths of a, b, and c measured in radians. We place A along the z axis such that its location in Cartesian coordinates is A[0,0,1]. Point B is placed to lie in the plane defined by the x and z axis such that we have B[x_B,0,z_B]. Finally, vertex C has coordinates C[x_C,y_C,z_C]. We draw three straight lines of unit length each from the origin at O to the three vertices and designate these lines as vectors **AO**, **BO**,and **CO** . Note that the dot product of any two of these just equals cos(a), cos(b), or cos(c). From the geometry we can now read off-

x_B=sin(c), z_B=cos(c), x_C= sin(b)cos(A) , y_C= sin(b)sin(A) , and z_C=cos(b)

The coordinates of the vertices of the spherical triangle can thus be written down explicitly as-

A[0.0.1],   B[sin(c), 0, cos(c)],   and C[sin(b)cos(A), sin(b)sin(A), cos(b)]

Next taking the dot product between **BO** and **CO**, we get-

cos(a)=sin(b)sin(c)cos(A)+cos(b)cos(c)

This represents essentially the law of cosines for a spherical triangle . The other two versions follow at once by interchanging the a,b,c,A,B,Cs. They are-

cos(b)=sin(a)sin(c)cos(B)+cos(a)cos(c)

and

cos(c)=sin(a)sin(b)cos(C)+cos(a)cos(b)

A law of sines law follows from manipulating cos(A) and cos(B) in the above formulas. We find-

$$\sin(A)^2 = \frac{[\sin(b)\sin(c)]^2 - [\cos(a) - \cos(b)\cos(c)]^2}{[\sin(b)\sin(c)]^2}$$

and

$$\sin(B)^2 = \frac{[\sin(a)\sin(c)]^2 - [\cos(b) - \cos(a)\cos(c)]^2}{[\sin(a)\sin(c)]^2}$$

Taking the root of the quotient produces-

$$\frac{\sin(A)}{\sin(B)} = \frac{\sin(a)}{\sin(b)}\sqrt{\frac{1-\cos(c)^2 - \cos(b)^2 - 2\cos(a)\cos(b)\cos(c)}{1-\cos(c)^2 - \cos(b)^2 + 2\cos(a)\cos(b)\cos(c)}} = \frac{\sin(a)}{\sin(b)}$$

Thus one has the Law of Sines for oblique spherical triangles.

Using the above laws allows us to calculate distances along a great circle between any two points A[(LAT_A, $LONG_A$]and B[LAT_B, $LONG_B$] on Earth. We do so by defining the third vertex of a spherical triangle being the North Pole at C[LAT_C=π/2, $LONG_C$=anything]. In this case angle C=$LONG_B$-$LONG_A$. The fact that the sphere radius is R=3960 miles instead of unity makes no difference since all quantities are expressed in radians. The distance c between A and B is then determined by using the formula-

$$\cos(c) = \sin(COLAT_A)\sin(COLAT_B)\cos(LONG_B - LONG_A) + \cos(COLAT_B)\cos(COLAT_A)$$

On expanding the terms in this expression, one arrives at the final distance between points A and B of-

$$c = \cos^{-1}\{\cos(LAT_A)\cos(LAT_B)\cos(LONG_B - LONG_A) + \sin(LAT_B)\sin(LAT_A)\}$$

Since c is expressed in radians, one can say that the fraction traversed along the great circle in going from A to B is f=c/2π.

Take now the case of flying from A at London(LAT=51.5072degN, LONG=0.1275degW) to B at New York City (LAT=40.7127N, LONG=74.0059W). In this case we have –

C=(74.0059-0.1275)=73.8784 deg

b=90-51.5072=40.71227=38.4928deg

c=90-40.7127=49.2873deg

So solving the cos(c) equation , we find-

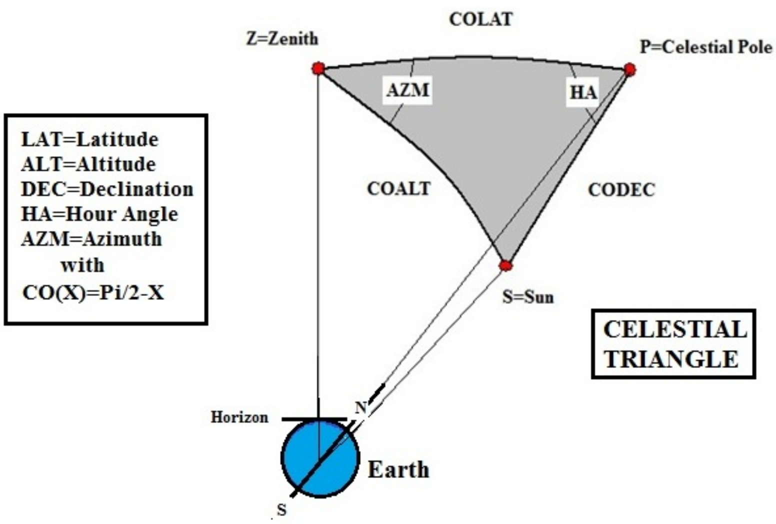
cos(c)=0.64157 or c=50.091deg=0.87425rad

This means the distance between London and New York City on a great circle route is-

L=[0.8742/(2π)](2πR)=3462 miles

The usual number quoted is 3461 miles .

With the above Law of Cosines and Law of Sines for spherical triangles it is also possible to use them to describe the position of the sun, moon, and other heavenly bodies on any date and time. For example, we can use the formulas for determining the sun's position from any LAT and LONG observation point in the Northern Hemisphere. Our starting point for such an analysis is the following astronomical spherical triangle lying on an imaginary celestial sphere as shown-



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We have the three vertices of the triangle lying at the Zenith Z directly above the observer, the north celestial pole at P, and the sun position at S. The curved sides of this spherical triangle are equal to the co-altitude COALT=π/2-ALT, the co-latitude COLAT=π/2-LAT and the co-declination CODEC=π/2-DEC. The two angles of interest are the hour Angle HA and the azimuth AZM. Here HA refers to the number of hours away from local noon and azimuth is the angle in the observers plane away from a north-south line. At sunrise or sunset the altitude ALT is zero so that COALT=π/2 there.

Making the substitutions A=AZM, B=HA, a=CODEC, b=COALT, and c=COLAT into the spherical triangle equations derived above, we get two of the most important equations in celestial mechanics, namely,-

$$\frac{\sin(AZM)}{\sin(HA)}=\frac{\sin(CODEC)}{\sin(COLAT)}$$

and-

$$\cos(AZM)=\frac{\cos(CODEC)-\cos(COALT)\cos(COLAT)}{\sin(COALT)\sin(COLAT)}$$

These equations are sufficient to precisely locate the sun anywhere in the Northern Hemisphere knowing the values of ALT, LAT, and DEC on the day in question. As a demonstration, let us ask the question at what azimuth will the sun rise a few days from now on my birthday (September 16th) here in Gainesville, Florida? The latitude here is LAT=29.56deg, the ALT at sunrise is zero, and the nautical almanac gives a declination of DEC=+2.88deg for that day. Accordingly the sun will rise at an azimuth of-

$$AZM=\cos^{-1}\{\sin(DEC)/\cos(LAT)\}=86.688deg$$

Thus the sun rises almost to the east being just 3.312deg north of an east-west line. The hour angle for that day will be-

$$HA=\sin^{-1}\left\{\frac{\cos(LAT)\sin(AZM)}{\cos(DEC)}\right\}=60.398886\deg$$

That is, the sun will rise at 60.39886/15=4.026hrs before local noon. Our local noon will occur at LONG/15=82.45deg/15=5.4956 hrs Greenwich mean time(GMT). So sunrise that day will occur in Gainesville at 5.4956-4.026=1.469 pm GMT.

It should be possible to construct a mechanical sun-tracker based on the above equations and the input of LAT, LONG, and DEC information. This way one would have a sun follower completely independent of weather conditions for solar energy conversion requiring continuous concentrated sunlight.

It is interesting that the azimuth at sunrise or sunset depends only on DEC and LAT. So at the Spring and Fall Equinoxes , where DEC=0, we have the sun rise precisely in the east regardless of LAT. At the Neolithic monument of Stonehenge where LAT=51.1788deg, the sun will rise at the Summer Solstice (DEC=+23.5deg) at an azimuth of 50.4976 deg. So the sun rises 90-50.4976=39.50deg north of an east-west line that day. This happens to be the exact orientation of a symmetry line through that monument(see http:///www2.mae.ufl.edu/~uhk/STONEHENGE.pdf).