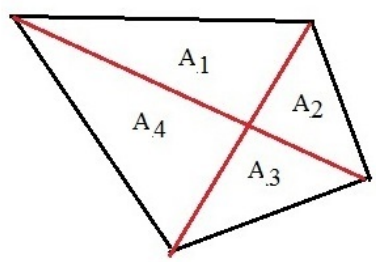
NUMBER OF DIAGONALS AND SUB-AREAS ONE CAN CREATE INSIDE POLYGONS

An interesting problem in geometry and topology concerns the number of unique diagonals and corresponding areas one can create inside polygons of N sides. We can define such diagonals as any line cutting any convex polygon which connects any two non-neighboring vertices. A simple case would be a quadrangle with four, not necessarily equal length, sides (N=4). Here one has two possible diagonals (D=2) which form four sub-areas (A=4) as indicated in the following sketch-

QUADRANGLE WITH N=4, D=2, AND A=4

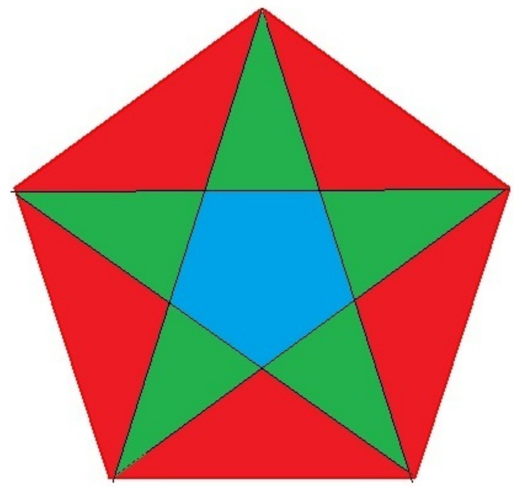


D=N(N-3)/2=4(1)/2=2

We want here to examine the general problem for convex polygons of N sides in order to determine the corresponding unique diagonals which may be drawn and the number of corresponding sub-areas created by them.

We begin by looking at a pentagon where N=5. Here there are two diagonals created for vertex one and two diagonals created at the neighboring vertex two. In addition a vertex three starting point creates one more diagonal. The remaining vertexes do not produce any new diagonal. Hence we get D=5 unique diagonals as shown-

PENTAGON WITH D=5 DIAGONALS AND A=11 SUB-AREAS

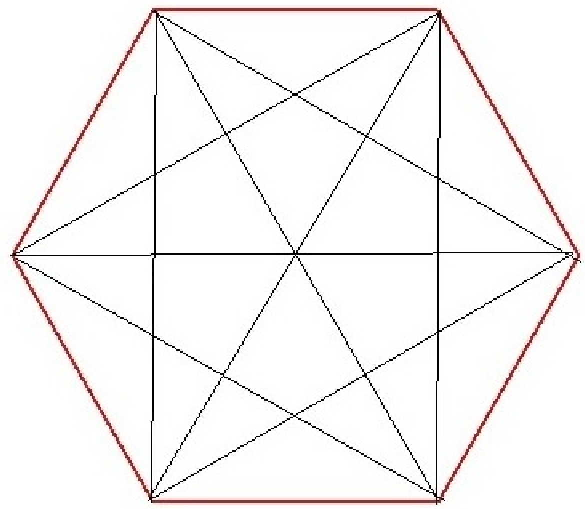


Number of Areas =5 red + 5 green + 1 blue=11

These diagonals lead to a total of eleven (A=11) sub-areas as shown. A distortion of this pentagon to one where the sides are not all equal does not alter the D=5 value. Note the smaller inner pentagon(shown in blue) created by this procedure.

Contiuing on to a hexagon (N=6), we find D=9 and A=24.This fact is clearly shown via the following diagram-

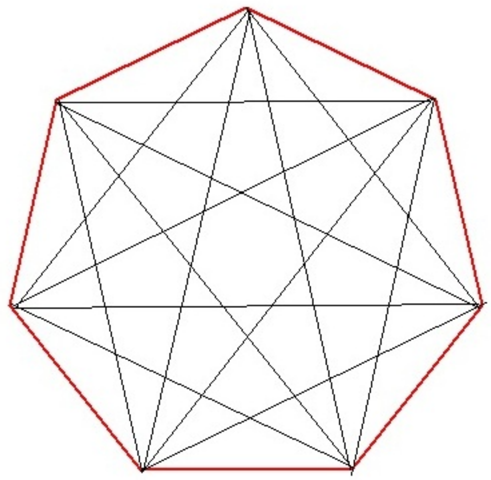
HEXAGON WITH ITS 6 SIDES, 9 DIAGONALS, AND 24 SUB-AREAS



The determination of the number of areas becomes particularly simple when N is an even number such as it is for the hexagon. In such cases on simply counts the number of sub-areas contained in the isosceles triangle formed by connecting two neighboring vertices with the polygon center and then multipling this number by N. For the hexagon we have A=4 x 6=24. Again note the interior smaller hexagon created by the procedure. In this case the sub-areas consist of six qudrangles, six larger isosceles triangles and 12 small right triangles.

For a seven-sided polygon (heptagon) we find D=14 and A=50. Here is the corresponding diagram-

HEPTAGON AND ITS FOURTEEN DIAGONALS

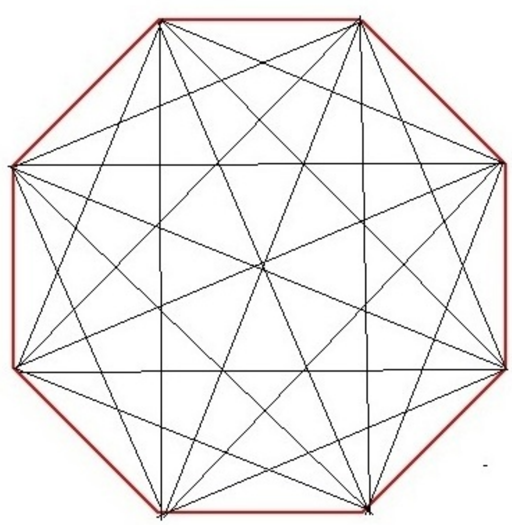


diagonals=4+4+3+2+1=14

Note the small heptagon found surrounding the polygon center . For this odd N case no diagonal passes through the center making it more difficult to count the number of sub-areas. Here the sub-areas consist of triangles, quadrangles, pentagons, and a heptagon.

For an octagon (N=8) we find D=20 and A=80. Here is its diagram-

OCTAGON WITH ITS 8 SIDES, 20 DIAGONALS, AND 80 SUB-AREAS



The counting of the number of sub-areas is this time quite easy since some of the diagonals pass through the center. This requires one to simply count the number of areas within a single pie shaped piece and multiply the result by N=8. That is A=10 x 8=80.

We have now accumulated sufficient information to generate the following table and use it to obtain some generalizations-

Polygon Name	N	D	A
Quadrangle	4	2	2
Pentagon	5	5	11
Hexagon	6	9	24
Heptagon	7	14	50
Octagon	8	20	80

Looking at column D we see that the first difference goes as 3-4-5-6 and the second difference as 1-1-1 . This suggests that D relates as a quadratic to N. That is-

$$D = aN + bN^2$$

, with the coefficients a and b to be determined. Using the values for the quadrangle (N=4, D=2) and the hexagon(N=6, D=9) we have-

$$\begin{bmatrix} 4 & 16 \\ 5 & 25 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

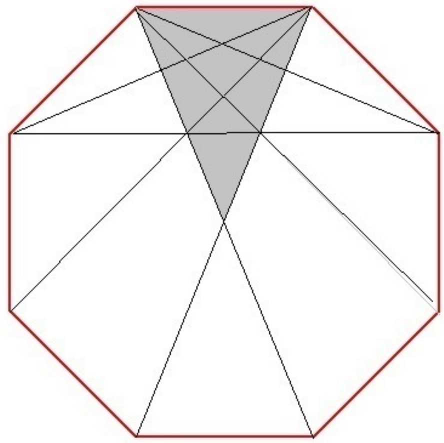
This solves as a=-3/2 and b=1/2. Hence we have the polygon side number N related to the total number of allowed unique diagonals D as-

D=N(N-3)/2

From this formula we can predict that any convex decagon (N=10) has D=35 diagonals.

To find a relation involving the number of sub-areas is a bit more difficult. Let us begin with even number Ns. These all have N vertexes with having diagonals passing through the polygon center. As a result there are N isosceles triangles formed. For the N=8 case the top isosceles triangle and the lines that cross it look like this-

TEN SUB-AREAS WITHIN THE UPPER ISCOCELES TRIANGLE OF AN OCTAGON



for entire octagon N=8, D=20, and A=80

We see from the graph that there are five unique crossings (C=5) of the sub-isosceles triangle by the diagonals which produce 10 distinct sub-areas. The total number of subareas created by all the allowed diagonals (D=20) is thus A=8 x 10=80.

Looking at the first few even sided polygons, the numbers of crossings C found for each sub-isosceles triangle, and the total sub-areas created divided by N, we find the following table-

N	4	6	8	10	12
C	0	2	5	9	14
A/N	1	4	10	22	37

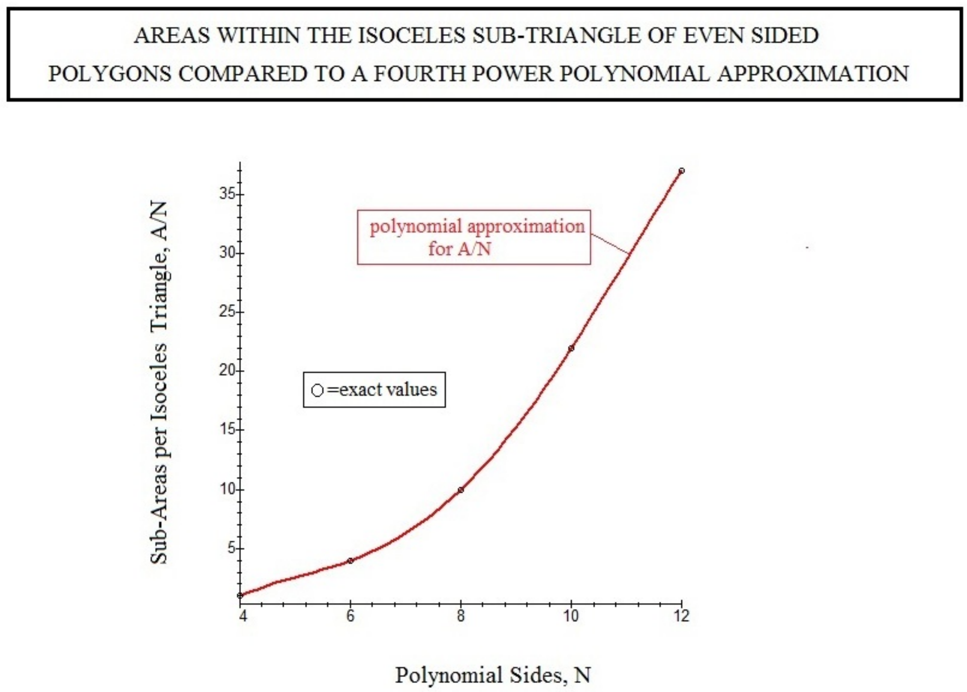
We notice from the table that the row containing the Cs has its second differences equal to one each. This suggests a quadratic relation between N and C. A bit of manipulation shows this to be-

$$C = \frac{1}{8}(N+2)(N-4)$$

This checks with all values in the table given for C. Unfortunately there is no easy way to relate C to A other than to say that each crossing C produces more than two areas. This however does not lead to a closed form solution of A versus N. Accordingly, we look for a polynomial approximation for A/N versus N using the five values for A,N given in the last table. This produces the approximation –

$$B=A/N:= -38+47N/2-83N^2/16+N^3/2-N^4/64$$

A graph of the exact values of B versus this approximation follows-



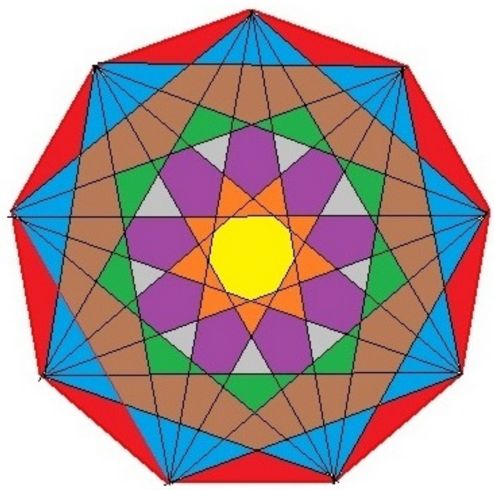
As expected the approximation coincides with the exact values of the sub-areas for even sided polygons as obtained by counting. However, no unique functional relation between N and A seems to exist. Hence one is forced to the resort of to counting the number of areas within the polygon no matter if the polygon has an even or odd number of sides.

We have carried out such a counting procedure and have obtained the following comprehensive table-

Number of Sides, N	Unique Diagonals, D	Total sub-areas, A
3	0	1=3(0)+1
4	2	4=4(1)
5	5	11=5(2)+1
6	9	24=6(4)
7	14	50=7(7)+1
8	20	80=8(10)
9	27	154=9(17)+1
10	35	220=10(22)
11	44	309=11(28)+1
12	54	444=12(37)

It took quite an effort to find the values of A for N=9 and N=11. In those cases we used a coloring method to conduct the count. The following is the procedure for the N=9 case-

154 SUB-AREAS CREATED BY 27 DIAGONALS OF A REGULAR NONAGON

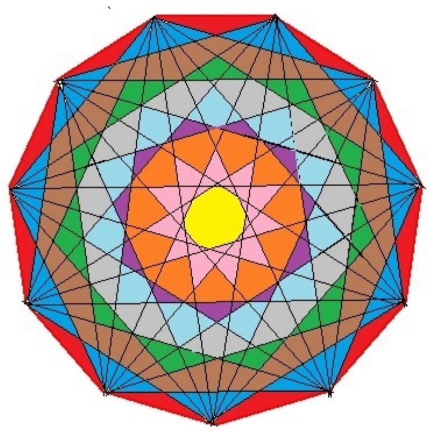


SUM OF AREAS CREATED = 9 red+45 blue+36 brown+27 green +9 gray+ 9 purple+18 orange +1 yellow = 154

You will note that the total number of sub-areas A for odd N goes as A=N(integer)+1 while for even N we have A=N(integer). The counting becomes much more difficult as N is increased further. This stems from the fact that when drawing in the diagonals by hand using paintbrush one looses accuracy as N increases.

Here is the color coding solution for N=11-

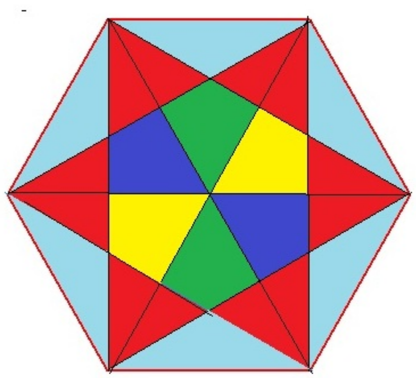
COLOR PATTERN CREATED BY D=44 DIAGONALS INSIDE A REGULAR N=11 POLYGON. TOTAL SUB-AREAS EQUAL TO A=309

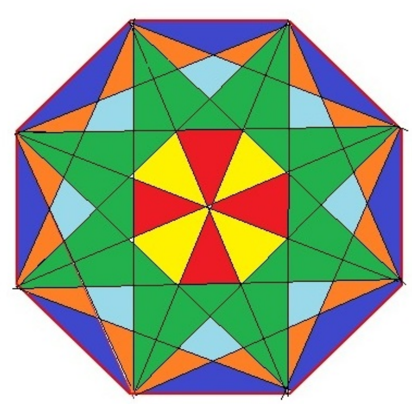


total sub-areas=A=308= red=11, blue=77, brown=66, green=55, gray=33, light-blue=33, purple=11, orange=22, and yellow=1

One sees that the counting becomes difficult in the grey and light blue areas. Very small variations in diagonal orientation can fake areas not really there.

An interesting result of the present discussion on sub-area creation inside a polygon is the creation of multiple computer images which start to approach in character that of a modern abstract design or painting. Here are two examples of colored images following from diagonals in a regular hexagon and a regular octagon, respectively-



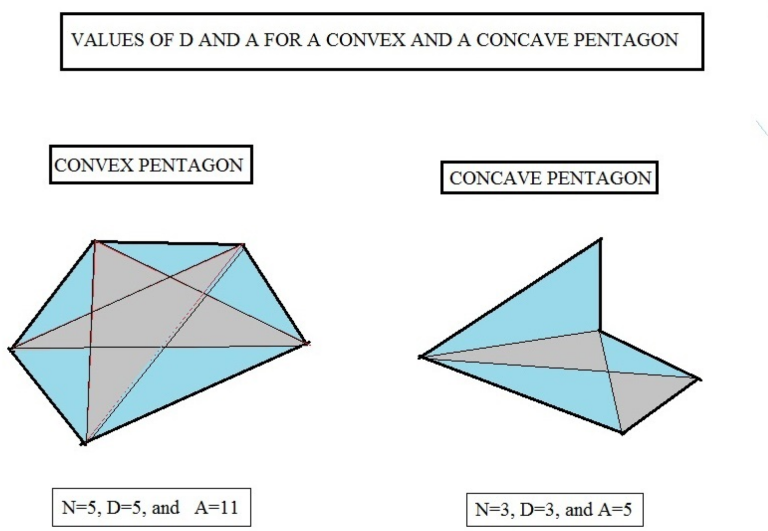


These type of patterns are often encountered in Amish decorations on plates and barns in the Lancaster, Pa. area.

A final point we wish to make concerns any changes one may expect to find when the polynomials considered become irregular. We have already shown in the first example in this article that the number of diagonals remain unchanged when dealing with any convex quadrangle. Such 2D figures always have just two diagonals no matter what the shape as long as every vertex has an unobstructed view of all the other vertices. Thus we can state that the number of unique diagonals one can create for any polygon N with un-obstructive view of all vertices remains the same. That is the number of unique diagonals D will always be related to the polygon side number N as-

$$D=N(N-3)/2$$

We demonstrate this point for two irregular pentagons in the following sketch-



The first polygon is concave and so satisfies the D versus N equation while the second one has a convex vertex angle blocking an un-obstructive view of all vertices through the pentagon interior. Hence the D(N) relation fails.

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