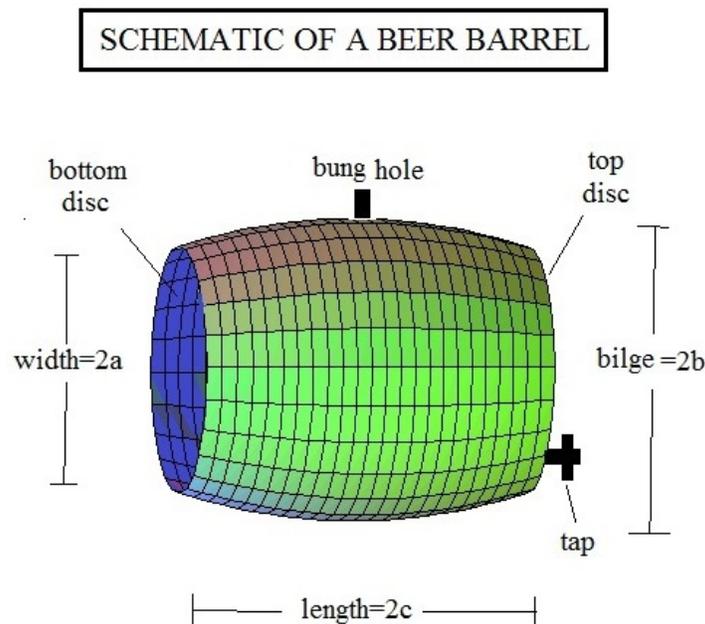


DETERMINING THE VOLUME OF BEER IN A BARREL

One of the classic problems in mechanics is determining the amount of fluid contained in a barrel using dip stick measurements. This problem was first looked at from a mathematical viewpoint by the famous astronomer Johannes Kepler in the sixteen hundreds. He wanted to check the accuracy of standard dip-stick measurements being used by brewers and merchants. His studies were carried out in pre-calculus days and so gave only an approximate results for beer volume in a partially empty keg. We want here to re-look at the problem using a full calculus approach.

Our starting point is the following schematic of a beer barrel-



Such barrels are typically made of 24 to 36 oak staves held together by six metal hoops with the top and bottom of the barrel closed by flat wooden discs. The side of the barrel is curved with a shape close to that of a parabola. The barrel is filled with beer through the bung hole located at the widest diameter referred to as the bilge and distributed by the tap shown.. We denote by $2a$, $2b$, and $2c$ the barrel width, the bilge, and the length of the barrel. The axis of the barrel is denoted by z and the radial direction by r . Typically $b/a=1.2$ and $c/a=1.5$.

We begin our calculations by first looking at volume of a filled barrel. As already stated earlier the barrel side has a shape fairly close to that of a parabola. By adjusting this shape to the radii of $r=b$ at $z=0$ and $r=a$ at $z=\pm c$, we find –

$$r = b - [(b - a) / c^2] z^2 = b - kz^2$$

Next we slice the barrel into discs of radius r and thickness dz . As the thickness is allowed to go to zero we get the calculus result-

$$V_{full} = 2\pi \int_0^c (b - kz^2)^2 dz = 2\pi c \left\{ b^2 - 2bkc^2 / 3 + k^2 c^4 / 5 \right\}$$

If we take the typical values of $a=10$ inches, $b=12$ inches, and height equal $2c=30$ inches, the filled barrel will contain a volume of $V= 3864 \pi$ cubic inches or about 199 liters.

We next continue on to a partially filled beer barrel. Here integral for the amount of beer in the barrel becomes extremely complex. To avoid this difficulty we approximate the barrel by a cylinder of length $2c$ and mean radius-

$$R = \sqrt{V_{full} / (2\pi c)}$$

That is $R=11.35$ inches in the above specific case. Now if the beer level in the equivalent horizontal cylindrical tank is at vertical distance d from the z axis, we find its volume to be-

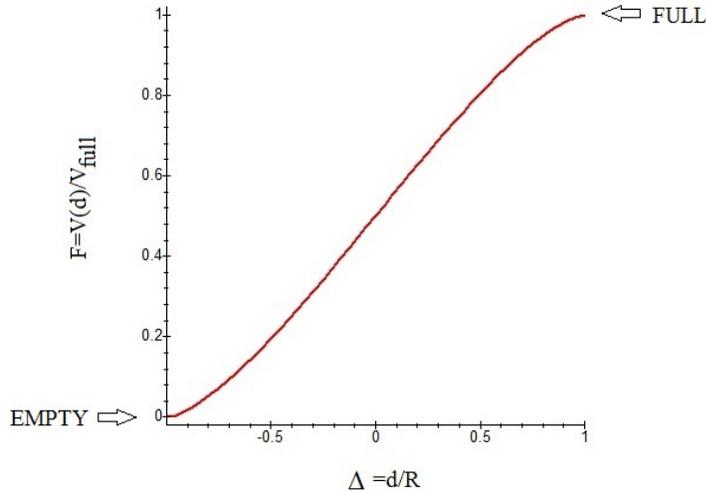
$$V(d) = \frac{1}{2} V_{full} + 4c \int_{y=0}^d \sqrt{R^2 - y^2} dy = \frac{1}{2} V_{full} + 2cR \left\{ d \sqrt{1 - \left(\frac{d}{R}\right)^2} + R \arcsin\left(\frac{d}{R}\right) \right\}$$

Here $V(d)$ ranges from empty at $d=-R$, to half-full when $d=0$, to full at $d=R$. Plotting $F=V(d)/V_{full}$ versus $\Delta=d/R$ we get the non-linear relation-

$$F(\Delta) = \frac{1}{2} + \frac{1}{\pi} \left[\Delta \sqrt{1 - \Delta^2} + \arcsin(\Delta) \right]$$

It plots as shown-

VOLUME OF BEER VERSUS FLUID DEPTH IN TANK



Except near the ends one has an almost linear behavior of fluid level versus fluid volume. This is the reason why beer brewers historically have relied on dip-stick measurements to measure fluid volume in a barrel of beer. The procedure is to insert a dip stick at the bung hole and have its lower end hit the bottom of the barrel at either end. Except for small errors when the barrel is almost empty or almost full, a linear measure of the dip level along the dip stick will quickly yield a good volume estimate. If we consider the special case of $R=11.35''$ and $c=15''$ discussed earlier, a full barrel will yield a tilted dip stick immersion length of –

$$l = 2R / \cos(\theta) = 2R \sqrt{1 + \left(\frac{c}{2R}\right)^2} = 27.21''$$

A reading of $l=13.6''$ means the container is half full and $l=6.8''$ means its quarter full.

Kepler had the feeling that he was being cheated by the beer merchant supplying him with kegs of beer for his second wedding. It turns out that he really had no worries since dip stick measurements of fluids in barrels are quite reliable even near the ends if properly graduated. He should have been more concerned with watering down of the beer since accurate hydrometers for density measurements had not yet been invented and one rather depended on taste.

