PROPERTIES OF THE DIVISOR FUNCTION SIGMA

One of the more important numbers encountered in number theory is the divisor function designated as σ(N) and equal to the sum of all its divisors of N including 1 and N. It is also related to our own number fraction f(N) via the equality-

$$\sigma(N)=1+N[1+f(N)]$$

It is the purpose of this note to find and discuss the properties of σ(N).

We begin by looking at the special forms of N when this number is a prime. We already know from previous notes on this page that f(p) is always zero for any prime p. Hence it must be true that σ(p)=1+p. From it follows that-

$$\sigma(p^2)=1+p+p^2$$
$$\sigma(p^3)=1+p+p^2+p^3$$
$$\sigma(p^4)=1+p+p^2+p^3+p^4$$

This means-

$$\sigma(p^n)=\sum_{k=0}^{n} p^k = [p^{(n+1)}-1]/(p-1)$$

We have used the geometric series to get the right hand side of this last equality.

Let us test things out for the prime p=127 with n=2. Here we have-

$$\sigma(16129) =(127^3-1)/(127-1)=16257$$

You will notice that sigma(p) lies just a little above p. This is to be expected as p gets large.

Consider next the product-

$$F=\sigma(p^n)*\sigma(q^m)=\sigma[p^n*q^m]$$

, where p and q are primes and n and m positive integers. We find, for p=2, q=3, n=2 and m=1, that-

$$F=\sigma(2^2)*\sigma(3^1)=\sigma(4*3)=28.$$

Next we see what are the values of sigma for any integer not just primes. To get the value of σ(N) we use the identity that any integer can be expressed as-

$$N=p1^a*p2^b*p3^c*\ldots$$

, where a,b, and c are integers and pn are primes. So we get-

$$\sigma(N)=\sigma(p1^a)*\sigma(p2^b)*\sigma(p3^c)\ldots$$

This means we need to only find sigma (pn^k) for different n and ks. Let us demonstrate with the composite number N=1459=2^1*3^6. We have-

$\sigma(1459)=\sigma(2^1)*\sigma(3^6)=3*1093=3279$

Another even larger composite number is N=13243=17*19*41. It has-

$$\sigma(N)=18*20*42=15120$$

So we can find σ(N) for any integer by evaluating the products σ(pn^a).Typically the number of such products will be relatively small(less than five or so). Thus N=47821436=2^2*13*491*1873 yields

$$\sigma(N)=7*14*492*1874=90356734$$

Finally we point out that once σ(N) is known, we can write down at once the value of the number fraction. The relation is-

$$f(N)=[\sigma(N)-N-1]/N$$

The advantage of f(N) over σ(N) is that f(N) is a relatively small fraction near unity compared to a large integer for σ(N. Also f(p)will always be zero.

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