EULER AND THE FUNCTION SIN(X)/X

In the early 17 hundreds the great Swiss mathematician Leonard Euler working alternatively at the Russian and the Prussian Academy of Sciences examined the function –

$$F(x)=\frac{\sin(x)}{x}=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n+1)!}=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\cdots.$$

It has a value of F(0)=1 and an even symmetry property F(x)=F(-x). Furthermore there are an infinite number of equally spaced zeros along the x axis at x=±nπ with n=1,2,3,4,.. These facts suggest that one can try to express F(x) as a product form-

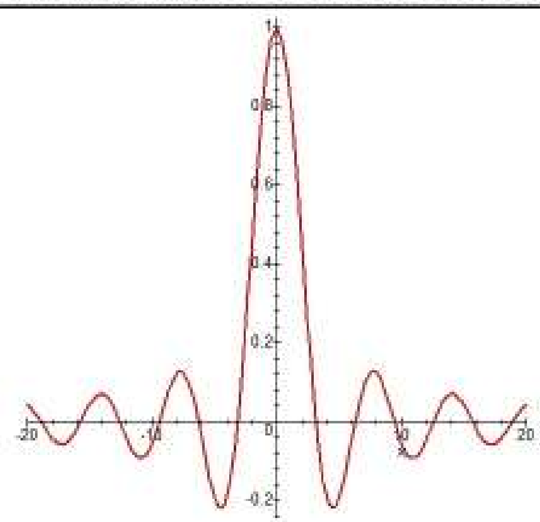
$$F(x)=(1-A_1x^2)(1-A_2x^2)(1-A_3x^2)\ldots$$

which has F(0)=1. The constants A_n are to be adjusted to match the location of the zeros of F(x). Taking $A_1=1/\pi^2$, $A_2=1/4\pi^2$, $A_3=1/9\pi^2$, etc , the function F(x) will indeed vanish at all of its zeros and one has the symmetric product expansion-

$$F(x)=\prod_{n=1}^{\infty}[1-\left(\frac{x}{n\pi}\right)^2]$$

A plot of this function follows-

FUNCTION F(X)=SIN(X)/X AS EVALUATED FROM EITHER ITS INFINITE SERIES OR INFINITE PRODUCT FORM



Matching the infinite product expansion of F(x) with the above given series, one is able to obtain several interesting identities. Looking at the coefficient of the x^2 terms , one finds-

$$\frac{1}{3!}=\frac{1}{\pi^2}\left\{\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right\}$$

This produces the interesting, but very slowly convergent, series for π -

$$\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

By choosing x=mπ , where m is a non-integer, one can get rid of the π terms in the product function to get-

$$F(m\pi)=\prod_{n=1}^{\infty}[1-(\frac{m}{n})^2]$$

If we in turn let m=1/2 and m=1/4 and look at the resulting ratio , one finds ,after a bit of manipulation, that-

$$\sqrt{2}=\prod_{n=1}^{\infty}[1-\frac{1}{(3+16n+16n^2)}]=\left(1+\frac{1}{3}\right)\left(1+\frac{1}{35}\right)\left(1+\frac{1}{99}\right)\ldots$$

This series indeed approaches the root of two but does so very slowly requiring the first hundred terms in the product just to generate a result good to three places. Also one can derive the famous Wallis Formula as follows. Let x=$\pi/2$ to obtain-

$$\frac{2}{\pi}=(\frac{3}{4})(\frac{15}{16})(\frac{35}{36})(\frac{63}{64})(\ldots=(\frac{1\cdot3}{2\cdot2})(\frac{3\cdot5}{4\cdot4})(\frac{5\cdot7}{6\cdot6})(\frac{7\cdot9}{8\cdot8})(\ldots$$

Inverting this result produces the Wallis Formula-

$$\frac{\pi}{2}=\frac{1\cdot3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot9\cdot}{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8\cdot}$$

The English mathematician John Wallis (1616-1703) derived this result about the time of the invention of calculus.

One can go on and look at the x^4 terms to find that-

$$\frac{7\pi^4}{360}=\sum_{n=1}^{\infty}\frac{\Psi(1,n)}{n^2}$$

Here Ψ(1,n) is the polygamma function defined by –

$$\Psi(1,n) = \frac{d^2}{dx^2}[\ln(\Gamma(x)]|_{x=n}$$

The first few values are Ψ(1,1)=0, Ψ(1,2)=1, Ψ(1,3)=5/4, and Ψ(1,4)=49/36. The sum converges rather slowly toward the value-

$7\pi^4/360$ =1.8940656589944918351530064689470438 29856…

A recasting of the coefficients of the x^4 term also allows one to obtain the even simpler formula-

$$\frac{\pi^4}{90} = \sum_{k=1}^{\infty} \frac{1}{k^4} = 1.0823232337111381915160036965411679027 75 \ldots$$

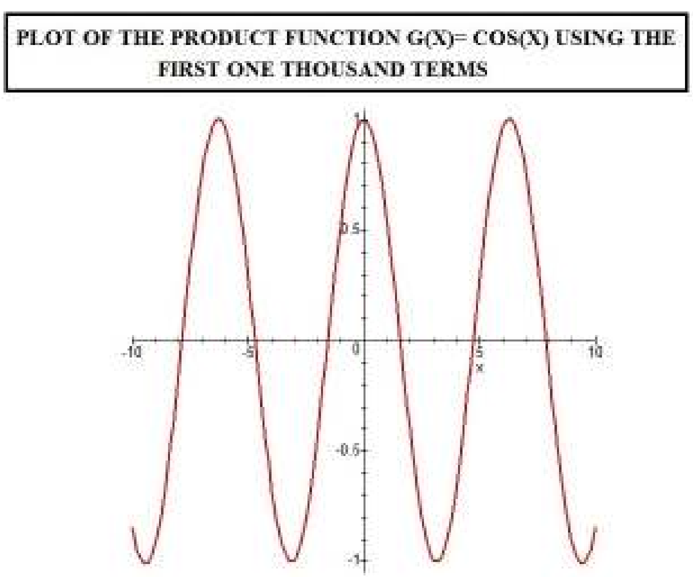
as first shown by Euler. His approach can be extended to other periodic functions. For example, we can expand the cosine function as –

$$G(x)= \cos(x)=(1-B_1x^2)(1-B_2x^2)(1-B_3x^2)(\ldots$$

The constant B_n must be chosen so that G(x) vanishes at $x=\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}\ etc.$ This implies the product function-

$$G(x)=\cos(x)=\prod_{n=1}^{\infty}\left[1-(\frac{2x}{(2n-1)\pi})^2\right]$$

Here is a plot of this product in -10<x<10 using the first 1000 terms in the product. The agreement with the standard cosine function is seen to be excellent-



At x=π/4 we find that-

cos(π/4)=1/sqrt(2)=[1-(1/4)][1-(1/36)][1-(1/100)][1-(1/196)][…

The tangent function may be expressed as a product function by simply taking the ratio of the sine and cosine product expressions. We have that-

$$\tan(x) = \frac{x[1-(\frac{x}{\pi})^2][1-\left(\frac{x}{2\pi}\right)^2][\pi^2-\left(\frac{x}{3\pi}\right)^2][}{[1-\left(\frac{2x}{\pi}\right)^2[1-(\frac{2x}{3\pi})^2][1-(\frac{2x}{5\pi})^2][}$$

The function is seen to have zeros at x=nπ and poles at x=(2n+1)π/2. At x=π/4 this tangent expression produces a Wallis like formula-

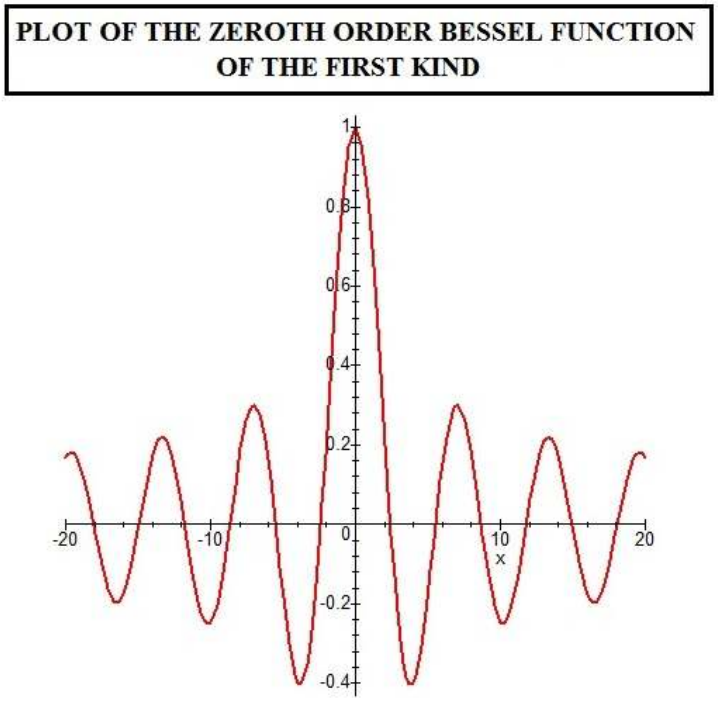
$$\frac{\pi}{4}=\prod_{n=0}^{\infty}\left[\frac{[(2+4n)^2-1]\cdot[(4+4n)^2]}{[(2+4n)^2]\cdot[(4+4n)^2-1]}\right]$$

Writing out the first six terms we get-

$$\frac{\pi}{4} \approx \left(\frac{4}{5}\right)\left(\frac{80}{81}\right)\left(\frac{324}{325}\right)\left(\frac{832}{833}\right)\left(\frac{1700}{1701}\right)\left(\frac{3024}{3025}\right)$$

This approximation yields a three digit accurate result as can be seen by looking at the last ratio on the right which equals 0.999669… .

It is also possible to generate infinite product expressions for other functions who do not have their zeros spaced at constant intervals. A prime example of such a function is the Bessel Function of the First Kind of Order Zero. It looks as follows-



Note the even symmetry about x=0, that $J_0(0)=1$, and that the distance between neighboring zeros is not constant. The standard infinite series representation for this function is-

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!^2 2^{2n}} = 1 - \frac{x^2}{4} + \frac{x^4}{64} - O(x^6)$$

The zeros of $J_0(x)$ can be obtained from mathematical handbooks or more simply from canned math programs on one's PC. The first four zeros are-

Zero Number	μ_1	μ_2	μ_3	μ_4
X Value	2.404825558…	5.520078110…	8.653727913…	11.79153444…

Notice that the distance between neighboring roots are not constant but that the spacing slowly approaches a value of π when |x|gets large.

Now following the same argument that Euler used for finding an infinite product representation of F(x)=sin(x)/x, we can represent $J_0(x)$ as-

$$J_0(x) = \prod_{n=1}^{\infty} \left[1 - \frac{x^2}{{\mu_n}^2}\right]$$

This product function clearly vanishes at all roots of $J_0(x)$ and has $J_0(0)=1$. Lets test it at x=1 where $J_0(1)=0.7651976866..$. The right hand side reads-

$$\left(1-\frac{1}{\mu_1{}^2}\right)\left(1-\frac{1}{\mu_2{}^2}\right)\left(1-\frac{1}{\mu_3{}^2}\right)\left(1-\frac{1}{\mu_4{}^2}\right)(\quad)$$

Working out the first ten products, we find a value of .772793… . This indicates both that the above product representation for $J_0(x)$ is valid but also that it is very slowly convergent coming within only one percent of the answer at x=1 when using the first ten terms in the product.

On expanding the infinite product definition of Jo(x) in powers of x^{2n}, we find , on equating the coefficients of the x^2 in the series with the x^2 term in such an expansion, that-

$$\frac{1}{4}=\sum_{n=1}^{\infty}\frac{1}{\mu_n{}^2}$$

which means that $\mu_1>2$.

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