PROPERTIES OF AN EULER SQUARE

About 1780 the mathematician Leonard Euler discussed the properties an n x n array of letters or integers now know as a Euler or Graeco-Latin Square. Such squares have the property that every row and column contains each element just once and that all rows and columns add up to n(n+1)/2 when using the first n integers. Here is an elementary example-

$$\begin{vmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{vmatrix}$$

Notice that the sum of the elements equals exactly 3(3+1)=6. The simplest way to construct such a square is to start with a single row containing all elements n in random order. Next one shifts an identical second row to the left by one space and continues this procedure down to the last row n continuing the shifting of one space to the left for each row shift. The result is an Euler Square. Here is an example of a constructed 4 x 4 square when the first row reads [1 2 3 4]-

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$$

You will note each row and column contain all four integers just once and the sum of the elements in any row or column is exactly n(n+1)/2=10. The sum of the elements in the two diagonals will generally not be equal. By interchanging rows and/or columns in this Euler Square, one can create numerous other possibilities such as the following -

$$\begin{vmatrix} 3 & 1 & 4 & 2 \\ 1 & 4 & 2 & 3 \\ 4 & 2 & 3 & 1 \\ 2 & 3 & 1 & 4 \end{vmatrix} \quad and \quad \begin{vmatrix} 2 & 3 & 1 & 4 \\ 4 & 1 & 3 & 2 \\ 3 & 4 & 2 & 1 \\ 1 & 2 & 4 & 3 \end{vmatrix}$$

This shows that there are multiple possibilities for any n x n square but only one unique form when the first row is represented by the ascending values [1, 2, 3, 4,…n]. We call this special form a standard Euler Square. Its form is given by the matrix-

$$M = \begin{vmatrix} 1 & 2 & 3 & 4 & . & n \\ 2 & 3 & 4 & . & n & 1 \\ 3 & 4 & . & n & 1 & 2 \\ 4 & . & n & 1 & 2 & 3 \\ . & . & . & . & . & . \\ n & 1 & 2 & 3 & . & n-1 \end{vmatrix}$$

Notice this form satisfies all the properties for a standard n x n Euler Square including that the sum in any row or column equals precisely-

$$S(n) = \frac{n(n+1)}{2}$$

For even n this square matrix M has the trace(sum of the elements along the main diagonal)-

$$Tr(M) = 2\sum_{k=1}^{n/2}(2k-1)$$

For odd n the trace has the still simpler form-

$$Tr(M) = S = \frac{n(n+1)}{2}$$

so that it equals the column and row sums. The sum of the elements along the right secondary diagonal is always n^2 regardless of whether n is even or odd. Below you will find a table summarizing the properties of different sized standard Euler Squares starting with n=2 through n=9-

n	Trace	Absolute Value of Determinant
2	2	3

3	6	18
4	8	160
5	15	1875
6	18	27216
7	28	470596
8	32	943184
9	45	215233605

Both the trace and the absolute value of the determinant det(M) increase with increasing n with the rapid determinant growth becoming especially noticeable.

The Euler Squares can be thought of as precursors to certain mathematical puzzles such as Sudoku. Indeed it will usually be sufficient to construct a full (but not necessarily standard) n x n square by just starting with a set of n non-repeating integers . Consider the very simple case of a 3 x 3 Euler Square-

$$\begin{vmatrix} 1 & - & 2 \\ - & - & - \\ 3 & - & - \end{vmatrix}$$

where the dashes represent the location of the remaining six unknowns. We know that each row and column must add up to 6. So one sees at once that element $a_{2,1}$=2 and $a_{1,2}$=3. These results in turn force $a_{3,2}$=2, $a_{2,2}$=1, $a_{3,3}$=1 and $a_{2,3}$=3, producing the completed square-

$$\begin{vmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{vmatrix}$$

We could also have produced this square starting with elements 1, 3, and 2 in the first row and then moving things one unit to the right in row two, and an additional unit to the right in row three. Since this is not a standard Euler Square, the above trace law will no longer hold. Also the initial placement of the known elements cannot be random but will allow a solution only when placed in a certain pattern.

Another interesting property of Euler squares becomes clear by looking at the square formed by the product of two sets [1, 2, 3] and [A, B, C] -

$$\begin{vmatrix} 1A & 2B & 3C \\ 2B & 3C & 1A \\ 3C & 1A & 2B \end{vmatrix}$$

In this square we notice that both-

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} \quad and \quad \begin{vmatrix} A & B & C \\ B & C & A \\ C & A & B \end{vmatrix}$$

already represent standard Euler Squares. Their product also is. Take the case of A=2, B=4, and C=1. This produces an Euler Square-

$$\begin{vmatrix} 2 & 8 & 3 \\ 8 & 3 & 2 \\ 3 & 2 & 8 \end{vmatrix}$$

It also works when the elements are added together such that-

$$\begin{vmatrix} 3 & 6 & 4 \\ 6 & 4 & 3 \\ 4 & 3 & 6 \end{vmatrix}$$

Notice that in both cases the sum of the rows and columns have the new value of 1+2+3+2+4+1=13.

To see if this procedure works for larger n squares, we start with the case-

$$\begin{vmatrix} 1A & 2B & 3C & 4D \\ 2B & 3C & 4D & 1A \\ 3C & 4D & 1A & 2B \\ 4D & 1A & 2B & 3C \end{vmatrix}$$

And let A=3, B=2, C=4, and D=5. By first multiplying the elements together, we retrieve the Euler Square-

$$\begin{vmatrix} 3 & 4 & 12 & 20 \\ 4 & 12 & 20 & 3 \\ 12 & 20 & 3 & 4 \\ 20 & 3 & 4 & 12 \end{vmatrix}$$

We note however if D had been 1 or some other numbers, the procedure would not have worked for then the first row would have contained the same element twice. Adding elements for the values of A, B, C, and D given will not work. So we can state that –

"Generating new n x n squares from two sets of elements which individually lead to Euler Squares will work when multiplying or adding the elements together only if none of the resultant elements repeat themselves in any given row or column".

If we took A=1, B=2, C=3, and D=4 things would work fine, producing-

$$\begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 1 \\ 9 & 16 & 1 & 4 \\ 16 & 1 & 4 & 9 \end{vmatrix} \quad and \quad \begin{vmatrix} 2 & 4 & 6 & 8 \\ 4 & 6 & 8 & 2 \\ 6 & 8 & 2 & 4 \\ 8 & 2 & 4 & 6 \end{vmatrix}$$

In this case the sum of the rows and columns would be 30 and 20, respectively.

There is nothing restricting the elements of the matrix representing Euler Squares containing more than two components. Thus one has the 3 x 3 standard Euler Square-

$$\begin{vmatrix} ABC & DEF & GHI \\ DEF & GHI & ABC \\ GHI & ABC & DEF \end{vmatrix}$$

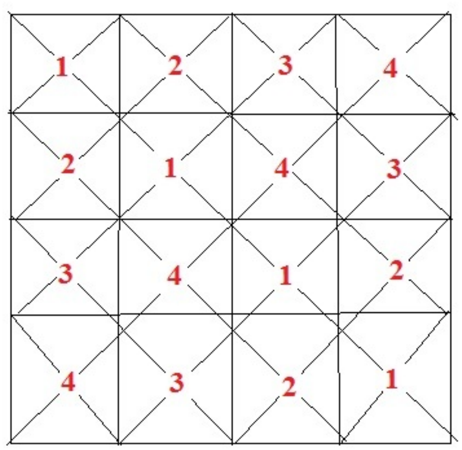
From this one also has the squares-

$$\begin{vmatrix} A & D & G \\ D & G & A \\ G & A & D \end{vmatrix} \qquad \begin{vmatrix} B & E & H \\ E & H & B \\ H & B & E \end{vmatrix} \qquad and \qquad \begin{vmatrix} C & F & I \\ F & I & C \\ I & C & F \end{vmatrix}$$

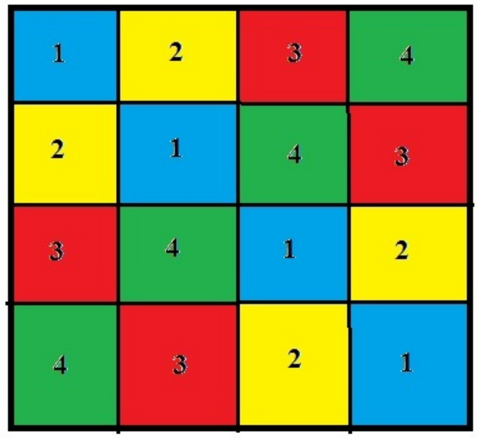
As long as the elements in a given row or column not equal each other, the squares will be standard Euler Squares. For example, [A,B,C,D,E,F,G,H,I]=[1,2,3,4,5,6,7,8,9] will work.

Finally we ask if the Euler squares encountered above can take on a more symmetric form. The answer is in the affirmative for n x n squares <u>provided</u> n is even. Such squares (which we will refer to as Sym-Squares) are easiest to construct graphically as shown in the following figure for a 4 x 4 square-

CONSTRUCTION OF A 4 X 4 SYM-SQUARE



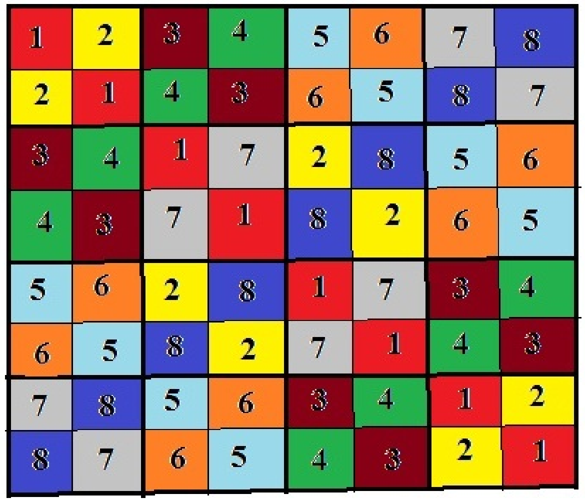
One starts with a large square box and divides it into $n^2=4^2=16$ equal size smaller boxes which are in turn broken into four equal sized right triangles by the crossing of two diagonal lines. Numbers 1 through 4 are then placed at the crossing points of the outer twelve boxes in the manner shown. The square is completed by placing the numbers 1 and 4 at the crossings in the remaining four inner boxes. The result is a Sym-Square. The numbers along the left diagonal of the original box are all ones and along the right diagonal are fours. A colorized version of a 4 x 4 Sym-Square follows-



If one now goes to a 2n x 2n Sym-Square, its matrix form will have a trace of 2n with all elements along this diagonal being 1. Along the right diagonal each element will have the value 2n so that the total sum is $4n^2$. Here is the result for a 8 x 8 Sym-Square for which n=4-

1	2	3	4	5	6	7	8
2	1	4	3	6	5	8	7
3	4	1	7	2	8	5	6
4	3	7	1	8	2	6	5
5	6	2	8	1	7	3	4
6	5	8	2	7	1	4	3
7	8	5	6	3	4	1	2
8	7	6	5	4	3	2	1

The elements are easily recognized once the main diagonals and the values in the outer two rows and columns have been recorded. Note the perfect symmetry of this square about the main (left) diagonal along which all the elements are one. Here the sum of the elements along any row or column equals 8(9)/2=36 and is thus the same as for the corresponding standard 8 x 8 Euler Square. A colorized version of this square looks like this-



Here there are eight small squares for each of the eight colors used. The symmetries are easy to recognize from the color pattern. Each row and column contains a given color only once.

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