

AN APPROXIMATION FOR N!

Several years ago we came up with a modified Pascal Triangle given by-

$$\begin{array}{cccccccc}
 & & & & 1 & & & \\
 & & & & 1 & & 1 & \\
 & & & 1 & & 4 & & 1 \\
 & & 1 & & 11 & & 11 & 1 \\
 & 1 & & 26 & & 66 & & 26 & 1 \\
 1 & & 57 & & 302 & & 302 & 57 & 1 \\
 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & &
 \end{array}$$

This configuration has the important property that the number of elements $D[n,m]$ in any given row n equals the number of columns m for that row. Also the sum of the elements in a given row are symmetric and always sum to $n!$. Furthermore one observes that the value of the elements along a given row n approach the form of a Gaussian as n gets large. This Gaussian will have a maximum at $m=(n+1)/2$. These facts suggest that a good approximation to $n!$ should become possible by calculating the area underneath an appropriately adjusted Gaussian. It is the purpose of this article to obtain such an approximation for $n!$.

Our starting point will be to sum all m elements in given row n . This produces the sum-

$$n! = \sum_{m=1}^n D[n,m]$$

, where the coefficients $D[n,m]$ are given by-

$$D[n,m] = \sum_{k=1}^m \frac{(-1)^{k-1} (n+1)! (n-k+1)^n}{(k-1)! (n+2-k)!}$$

Although this expansion is more complicated than that for a standard Pascal Triangle, it is still easy to evaluate by taking one column m at a time. We have-

$$D[n,1] = 1$$

$$D[n,2] = \frac{1}{0!} 2^n - \frac{(n+1)1}{1!}$$

$$D[n,3] = \frac{1}{0!} 3^n - \frac{(n+1)}{1!} 2^n + \frac{(n+1)n}{2!} 1^n$$

$$D[n,4] = \frac{1}{0!} 4^n - \frac{(n+1)}{1!} 3^n + \frac{(n+1)n}{2!} 2^n - \frac{(n+1)(n)(n-1)}{3!} 1^n$$

The pattern is obvious allowing us to expand the above Modified Pascal Triangle to any value of positive integer n . Here is the expanded result through $n=10$ -

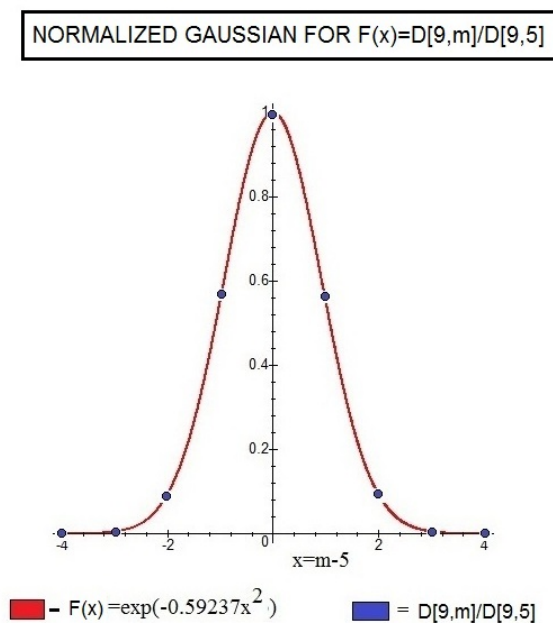
1
 1 1
 1 2 1
 1 3 3 1
 1 4 6 4 1
 1 5 10 10 5 1
 1 6 15 20 15 6 1
 1 7 21 35 35 21 7 1
 1 8 28 56 70 56 28 8 1
 1 9 36 84 126 126 84 36 9 1
 1 10 45 120 210 252 210 120 45 10 1
 1 11 55 165 330 462 462 330 165 55 11 1
 1 12 66 252 504 792 924 840 504 252 66 12 1
 1 13 78 351 858 1456 1716 1596 858 351 78 13 1
 1 14 91 462 1287 2431 3003 3003 2431 1287 462 91 14 1
 1 15 105 595 1773 3543 5005 5005 3543 1773 595 105 15 1
 1 16 120 768 2380 4862 7429 8734 7429 4862 2380 768 120 16 1
 1 17 136 969 3060 6188 9698 11628 9698 6188 3060 969 136 17 1
 1 18 153 1197 3951 8160 12870 15309 12870 8160 3951 1197 153 18 1
 1 19 171 1456 5005 10287 15399 19448 15399 10287 5005 1456 171 19 1
 1 20 190 1753 6188 12870 19448 24310 19448 12870 6188 1753 190 20 1
 1 21 210 2107 7429 15309 23184 27132 23184 15309 7429 2107 210 21 1
 1 22 231 2520 8580 17710 26834 33067 26834 17710 8580 2520 231 22 1
 1 23 253 2967 10287 21470 32685 39907 32685 21470 10287 2967 253 23 1
 1 24 276 3462 12066 24752 37937 46206 37937 24752 12066 3462 276 24 1
 1 25 300 4012 14060 28845 43868 53130 43868 28845 14060 4012 300 25 1
 1 32 423 5808 20134 41601 62770 77925 62770 41601 20134 5808 423 32 1
 1 40 600 8486 31364 63523 94247 113486 94247 63523 31364 8486 600 40 1
 1 48 816 12167 46188 93784 141120 171230 141120 93784 46188 12167 816 48 1
 1 56 1078 16796 64699 131281 196878 231864 196878 131281 64699 16796 1078 56 1
 1 64 1380 22399 87343 177109 268416 315266 268416 177109 87343 22399 1380 64 1
 1 72 1728 28695 111630 227925 345636 408808 345636 227925 111630 28695 1728 72 1
 1 80 2112 37179 145680 298803 452760 539796 452760 298803 145680 37179 2112 80 1
 1 88 2540 48188 186480 377837 561698 666192 561698 377837 186480 48188 2540 88 1
 1 96 3012 62976 244641 493783 727712 859986 727712 493783 244641 62976 3012 96 1
 1 104 3528 81840 316224 632448 933696 1092960 933696 632448 316224 81840 3528 104 1
 1 112 4080 105408 403904 807808 1201712 1399680 1201712 807808 403904 105408 4080 112 1
 1 120 4776 135232 500736 1001472 1492208 1743040 1492208 1001472 500736 135232 4776 120 1
 1 128 5616 173184 646592 1293184 1939776 2266624 1939776 1293184 646592 173184 5616 128 1
 1 136 6512 220416 835200 1670400 2496000 2896000 2496000 1670400 835200 220416 6512 136 1
 1 144 7472 278976 1064320 2128640 3192960 3727360 3192960 2128640 1064320 278976 7472 144 1
 1 152 8504 350016 1336960 2673920 4010880 4692480 4010880 2673920 1336960 350016 8504 152 1
 1 160 9608 435072 1664000 3328000 5001600 5856000 5001600 3328000 1664000 435072 9608 160 1
 1 168 10784 535296 2052480 4104960 6106880 7142400 6106880 4104960 2052480 535296 10784 168 1
 1 176 12032 650880 2503040 5006080 7359360 8576000 7359360 5006080 2503040 650880 12032 176 1
 1 184 13360 782976 3016320 6032640 8848000 10368000 8848000 6032640 3016320 782976 13360 184 1
 1 192 14768 932640 3603200 7206400 10507520 12288000 10507520 7206400 3603200 932640 14768 192 1
 1 200 16256 1101120 4275200 8550400 12288000 14336000 12288000 8550400 4275200 1101120 16256 200 1
 1 208 17824 1289600 5030400 10060800 14336000 16640000 14336000 10060800 5030400 1289600 17824 208 1
 1 216 19472 1498560 5878400 11769600 16640000 19200000 16640000 11769600 5878400 1498560 19472 216 1
 1 224 21200 1729600 6819200 13696000 18880000 21760000 18880000 13696000 6819200 1729600 21200 224 1
 1 232 23008 2003200 7953600 15856000 21760000 25088000 21760000 15856000 7953600 2003200 23008 232 1
 1 240 24896 2300800 9283200 18256000 24960000 28800000 24960000 18256000 9283200 2300800 24896 240 1
 1 248 26864 2623360 10817600 20912000 28800000 33600000 28800000 20912000 10817600 2623360 26864 248 1
 1 256 28912 3000960 12566400 23840000 33600000 39200000 33600000 23840000 12566400 3000960 28912 256 1
 1 264 31040 3444160 14537600 27072000 39200000 45760000 39200000 27072000 14537600 3444160 31040 264 1
 1 272 33248 3963840 16752000 30720000 45760000 53280000 45760000 30720000 1675

Adding up the elements in row $n=10$ produces 3628800 which is precisely $10!$.

Now what is very interesting about the values of the elements in any row is that they are symmetric about $m=(n+1)/5$ and that their point plot approaches the following Gaussian-

$$G=a \exp \{-b[m-(n+1) / 2]^2\}$$

, where the constants a and b are adjusted to match $D[n,m]$ at two different ms near its peak. Already for $n=9$ we get very close agreement as shown on the following graph-



Here the constants are adjusted using the Modified Pascal Triangle values for D[9,5] and D[9,3]. This produces $a=D[9,5]=156190$ and $b=(1/4)\ln\{D[9,5]/D[9,3]\}=0.59237597$. The agreement between the Gaussian and the nine D[9,m] points is seen to be excellent. Adding up the area underneath the Gaussian we should get a good estimate for 9!. That is-

$$n! \approx D[9,5] \int_{m=1}^9 \exp[-0.59237597(m-5)^2] dm = 359686.083$$

This result indeed lies within 0.88 % of the exact value of $9!=362880$. Going to even higher values of n will decrease the error further.

Now getting back to the generic form, we have the area underneath the Gaussian can be expressed analytically as-

$$Area(n) = a \int_{m=1}^n \exp\{-b[m - (n+1)/2]^2\} dm$$

Letting $z = \sqrt{b}[m - (n+1)/2]$, we can write things as-

$$n! \approx Area(n) = \left(\frac{a}{\sqrt{b}}\right) \int_{-c}^c \exp(-z^2) dz = \left(a \sqrt{\frac{\pi}{b}}\right) \{erf(c)\}$$

Here $c = \sqrt{b}(n-1)/2$ and $erf(x)$ is the standard error function defined by-

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_{x=0}^x \exp(-x^2) dx$$

The values of a and b are determined by matching the Gaussian at two integer values of $D[n,m]$ for fixed n . Since c has a value much greater than one for large n (so that the error function approaches unity), we also have the slightly weaker approximation-

$$n! \approx a \sqrt{\frac{\pi}{b}}$$

We are now in a position to approximate $n!$ for any positive integer value of n . Take the case of the large value $n=40$. Here we find-

$$b = (1/6) \ln\{D[40,20]/D[40,18]\} \quad , \quad c = 39\sqrt{b}/2 \quad \text{and} \quad a = D[40,20] \exp(b/4)$$

This produces the approximation-

$$n! \approx := (0.8171672474 \dots) \cdot 10^{48}$$

which compares to the exact value of $n! = (0.8159152832) \cdot 10^{48}$. Thus there is an error of just 0.1534%. The classic Stirling approximation has a slightly larger error than the $D[n,m]$ approach

We can summarize the above results as-

$$n! \approx a \sqrt{\pi/b}$$

, where for odd n we have-

$$a = D[n, \frac{n+1}{2}] \quad \text{and} \quad b = \frac{1}{4} \ln \left\{ \frac{D[n, \frac{n+1}{2}]}{D[n, \frac{n-3}{2}]} \right\}$$

and for even n the coefficients become-

$$a = D[n, n/2] \left\{ \frac{D[n, n/2]}{D[n, (n-4)/2]} \right\}^{1/24} \quad \text{and} \quad b = \left(\frac{1}{6}\right) \ln \left\{ \frac{D[n, n/2]}{D[n, (n-4)/2]} \right\}$$

In this approximation n is fixed and the integer m is chosen as near as possible to the peak of the Gaussian at $m=(n+1)/2$.

Finally we point out that , since the present approach to finding an approximate value for n! relies on knowing at least two values for D[n,m] along a given row n, one needs to carry out some calculations already involving large factorials. Hence, very often this effort will be comparable to just adding up the elements in the modified triangle to get an exact value for n!. An important new result found here is that one can now partition any n! into a precise Gaussian parts. Who would have guessed that the Gaussian form of 7! can be written as-

$$1+120+1191+2416+1191+120+1=5040 \quad ?$$

U.H.Kurzweg
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Gainesville, Florida