AN APPROXIMATION FOR N!

Several years ago we came up with a modified Pascal Triangle given by-

1
1 1
1 4 1
1 11 11 1
1 26 66 26 1
1 57 302 302 57 1
1 120 1191 2416 1191 120 1

This configuration has the important property that the number of elements D[n,m] in any given row n equals the number of columns m for that row. Also the sum of the elements in a given row are symmetric and always sum to n!. Furthermore one observes that the value of the elements along a given row n approach the form of a Gaussian as n gets large. This Gaussian will have a maximum at m=(n+1)/2 . These facts suggest that a good approximation to n! should become possible by calculating the area underneath an appropriately adjusted Gaussian. It is the purpose of this article to obtain such an approximation for n!.

Our starting point will be to sum all m elements in given row n. This produces the sum-

$$n! = \sum_{m=1}^{n} D[n,m]$$

, where the coefficients D[n,m] are given by-

$$D[n,m] = \sum_{k=1}^{m} \frac{(-1)^{k-1}(n+1)!(n-k+1)^n}{(k-1)!(n+2-k)!}$$

Although this expansion is more complicated than that for a standard Pascal Triangle, it is still easy to evaluate by taking one column m at a time. We have-

$$D[n,1[=1$$

$$D[n,2] = \frac{1}{0!}2^n - \frac{(n+1)1}{1!}$$

$$D[n,3]] = \frac{1}{0!}3^n - \frac{(n+1)}{1!}2^n + \frac{(n+1)n}{2!}1^n$$

$$D[n,4] = \frac{1}{0!}4^n - \frac{(n+1)}{1!}3^n + \frac{(n+1)n}{2!}2^n - \frac{(n+1)(n)(n-1)}{3!}1^n$$

The pattern is obvious allowing us to expand the above Modified Pascal Triangle to any value of positive integer n. Here is the expanded result through n=10-

1

1 1

1 4 1

1 11 11 1

1 26 66 26 1

1 57 302 302 57 1

1 120 1191 2416 1191 120 1

1 247 4293 15619 15619 4293 247 1

1 502 14608 88234 156190 88234 14608 502 1

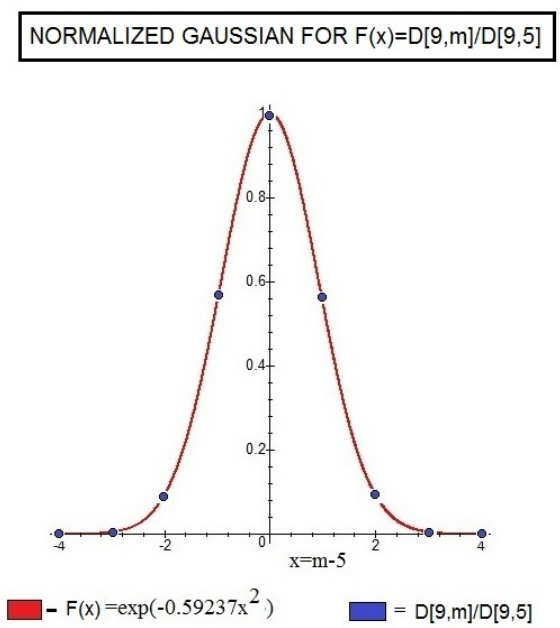
1 1013 47840 455192 1310354 1310354 455192 47840 1013 1

Adding up the elements in row n=10 produces 3628800 which is precisely 10! .

Now what is very interesting about the values of the elements in any row is that they are symmetric about $m=(n+1)/5$ and that their point plot approaches the following Gaussian-

$$G=a \exp\{-b[m-(n+1)/2]^2\}$$

, where the constants a and b are adjusted to match D[n,m] at two different ms near its peak. Already for n=9 we get very close agreement as shown on the following graph-



Here the constants are adjusted using the Modified Pascal Triangle values for D[9,5] and D[9,3]. This produces a=D[9,5]=156190 and b=(1/4)ln{D[9,5]/D[9,3]}=0.59237597. The agreement between the Gaussian and the nine D[9,m] points is seen to be excellent. Adding up the area underneath the Gaussian we should get a good estimate for 9!. That is-

$$n! \approx D[9,5] \int_{m=1}^{9} \exp[-0.59237597(m-5)^2]dm = 359686.083$$

This result indeed lies within 0.88 % of the exact value of 9!=362880. Going to even higher values of n will decrease the error further.

Now getting back to the generic form, we have the area underneath the Gaussian can be expressed analytically as-

$$Area(n) = a \int_{m=1}^{n} \exp\{-b[m-(n+1)/2]^2\}dm$$

Letting z=sqrt(b)[m-(n+1)/2], we can write things as-

$$n! \approx Area(n) = (\frac{a}{\sqrt{b}}) \int_{-c}^{c} \exp(-z^2)dz = (a\sqrt{\frac{\pi}{b}})\{erf(c)\}$$

Here c=sqrt(b)(n-1)/2 and erf(x) is the standard error function defined by-

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_{x=0}^{x} \exp(-x^2)dx$$

The values of a and b are determined by matching the Gaussian at two integer values of D[n,m] for fixed n. Since c has a value much greater than one for large n(so that the error function approaches unity) , we also have the slightly weaker approximation-

$$n! \approx a\sqrt{\frac{\pi}{b}}$$

We are now in a position to approximate n! for any positive integer value of n. Take the case of the large value n=40. Here we find-

b=(1/6)ln{D[40,20]/D[40.18]} , c=39sqrt(b)/2 and a=D[40,20]exp(b/4)

This produces the approximation-

$n! \approx :=(0.8171672474\ldots)\cdot 10^{48}$

which compares to the exact value of $n!=(0.8159152832)\cdot 10^{48}$. Thus there is an error of just 0.1534% . The classic Stirling approximation has a slightly larger error than the D[n,m] approach

We can summarize the above results as-

$n! \approx a\ \mathrm{sqrt}(\pi/b)$

, where for odd n we have-

$$a = D[n,\frac{n+1}{2}] \quad and \quad b = \frac{1}{4}\ln\{\frac{D[n,\frac{n+1}{2}]}{D[n,\frac{n-3}{2}]}\}$$

and for even n the coefficients become-

$$a = D[n,n/2]\left\{\frac{D[n,n/2]}{D[n,(n-4)/2]}\right\}^{1/24} \quad and \quad b = (\frac{1}{6})\ln\left\{\frac{D[n,n/2]}{D[n,(n-4)/2]}\right\}$$

In this approximation n is fixed and the integer m is chosen as near as possible to the peak of the Gaussian at m=(n+1)/2.

Finally we point out that , since the present approach to finding an approximate value for n! relies on knowing at least two values for D[n,m] along a given row n, one needs to carry out some calculations already involving large factorials. Hence, very often this effort will be comparable to just adding up the elements in the modified triangle to get an exact value for n!. An important new result found here is that one can now partition any n! into a precise Gaussian parts. Who would have guessed that the Gaussian form of 7! can be written as-

1+120+1191+2416+1191+120+1=5040 ?

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