PROPERTIES OF THE GOLDEN RATIO AND FIBONACCI NUMBERS

The Golden Ratio is defined as the solution of the positive root of the algebraic equation $\varphi^2=\varphi+1$ and equals the irrational number-

$$\varphi = \frac{(1+\sqrt{5})}{2} = 1.61803398874989484820458683437...$$

At the same time one also has the Fibonacci Sequence-

$$F[n] = \{1, 2, 3, 5, 8, 13, 21, 34, ...\}$$

whose elements are generated by-

F[n+2)=F[n+1]+F[n]   subject to   F[1]=1 and F[2]=2

We want here to re-derive some oif the better known relations between the Golden Ratio and the Fibonacci Sequence plus add a few new observations of our own.

Let us begin by taking the first seven powers of φ. One finds-

$$\varphi^1 = \frac{(1+\sqrt{5})}{2}$$

$$\varphi^2 = \frac{(3+\sqrt{5})}{2}$$

$$\varphi^3 = \frac{(4+2\sqrt{5})}{2}$$

$$\varphi^4 = \frac{(7+3\sqrt{5})}{2}$$

$$\varphi^5 = \frac{(11+5\sqrt{5})}{2}$$

$$\varphi^6 = \frac{(18+8\sqrt{5})}{2}$$

$$\varphi^7 = \frac{(29+13\sqrt{5})}{2}$$

In looking at these results it becomes clear that the integer multiplying sqrt(5) is the Fibonacci number F[n-1]. Also when n>4, the first number in the numerator equals 2F[n-1]+1.
From this one can surmise that-

$$\varphi^n = \frac{1}{2}\left\{(2+\sqrt{5})F[n-1] + F[n-4]\right\} \quad \textit{provided} \quad n > 4$$

This produces our first relation between the Golden Ratio and the Fibonacci numbers. To test out the equality let us ask what is the value of the 12th power of φ. We have F[11]=144 and F[8]=34, so we get-

$$\varphi^{12} = \frac{(322 + 144\sqrt{5})}{2}$$

with very little effort. A computer evaluation carrying out the product of φ twelve times confirms this result.

Using the definition of the Fibonacci sequence we can evaluate the F[n-4] term to get the more compact form-

$$\varphi^n = \frac{(-1+\sqrt{5})F[n-1] + 2F[n]}{2} \quad \textit{provided} \quad n > 2$$

We can also invert things by noting that–

$$\varphi^n = F[n-2] + F[n-1]\varphi$$

So we have that-

$$F[2] = \frac{(\varphi^3 - 1)}{\varphi}$$

$$F[3] = \frac{(\varphi^5 - \varphi^3 + 1)}{\varphi^2}$$

$$F[4] = \frac{(\varphi^7 - \varphi^5 + \varphi^3 - 1)}{3}$$

This shows that-

$$F[n]=\frac{(-1)^{n+1}+\sum_{k=1}^{n-1}(-1)^{k+1}\varphi^{2(n-k)+1}}{\varphi^{n-1}}=\frac{(-1)^{n+1}+\varphi^{2n-1}\sum_{k=0}^{n-2}\frac{(-1)^k}{\varphi^{2k}}}{\varphi^{n-1}}=$$

But one can sum the finite geometric series , to obtain the result-

$$S[n]=\sum_{k=0}^{n-2}\frac{(-1)^k}{\varphi^{2k}}=\frac{(\varphi^{2(n-1)}+(-1)^{n-2})}{\varphi^{2(n-2)}(\varphi^2+1)}$$

Combining this information then produces a formula relating F[n] to powers of φ. It reads-

$$F[n]=\frac{(-1)^{n+1}(1-\varphi)+\varphi^{2n+1}}{(2+\varphi)\varphi^{n-1}}=\frac{1}{(3-\varphi)}\left\{\varphi^n+\frac{(-1)^n}{\varphi^{n+2}}\right\}$$

To show that it works, consider F[12]=233. A computer evaluation of the right hand side of this last equation yields F[12]= 233.00000000001 when evaluated to 15 decimal places. This result is identical to the famous classical formula of Binet which reads-

$$F[n]=\frac{(\varphi^{n+1}-(-\varphi)^{-(n+1)})}{(2\varphi-1)}$$

Both forms for F[n] produce integer Fibonacci Numbers by evaluating the indicated quotients involving the Golden Ratio.

Several other properties involving the Golden Ratio follow from the above results. For example, if we let n->∞ in the sum S[n], we find-

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{\varphi^{2k}}=1-\frac{1}{\varphi^2}+\frac{1}{\varphi^4}-\frac{1}{\varphi^6}+....=\frac{(\varphi+1)}{(\varphi+2)}=\frac{5+\sqrt{5}}{10}=0.72360679...$$

Also it is obvious when using the same approach as that used for finding the values of an infinite geometric series, that-

$$\sum_{k=0}^{\infty}\frac{1}{\varphi^{2k}}=1+\frac{1}{\varphi^2}+\frac{1}{\varphi^4}+...=\varphi=1.61803398...$$

$$\sum_{k=0}^{\infty}\frac{1}{\varphi^{k}}=1+\frac{1}{\varphi}+\frac{1}{\varphi^2}+...=\varphi+1=2.618033988...$$

and

$$\sum_{k=0}^{\infty}\frac{1}{\varphi^{4}}=\frac{\varphi(\varphi+1)}{(\varphi^2+1)}=1.170820393...$$

This is just a small collection of numerous other sums involving the Golden Ratio which can be found.

Let us next go back and look at the ratio of F[n+1]/F[n]. We have from the above formula for F[n] that-

$$\frac{\lim}{n \to \infty} \frac{F[n+1]}{F[n]} = \frac{(\varphi+1)\varphi^{n+1}}{(\varphi+1)\varphi^{n}} = \varphi$$

That is, the ratio of the n+1 and nth Fibonacci number approaches the Golden Ratio as n becomes large. Running through a few of these ratios we find=

2/1-2, 3/2=1.5, 5/3=1.6667, 8/5=1.6000, 13/8=1.62500, 21/13=1.61538, 34/21=1.61904

So clearly the ratio oscillates about the Golden Ratio and eventually will zero in on this value. If we take F[101]/F[100] we get-

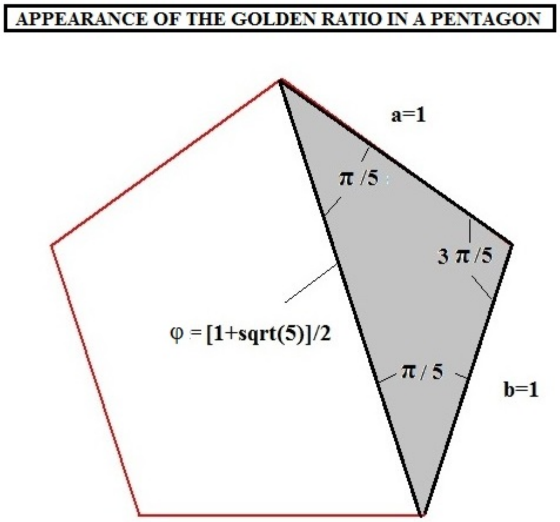
$$\frac{F[101]}{F[100]} = 1.6180339887498948482045868343656381177203$$

which is accurate to 43 decimal places.

We next look at some of the ways φ enters into 2D geometric problems. As Johannes Kepler first noted, the definition equation $\varphi^2=\varphi+1$ represents essentially the Pythagorean Theorem for a right triangle with sides 1 and sqrt(φ) and hypotenuse φ. The angle between the shortest side and the hypotenuse is given by-

$$\theta = \cos^{-1}(\frac{1}{\varphi}) = \cos^{-1}(\frac{\sqrt{5}-1}{2}) = 57.58804..\deg$$

This angle is just slightly larger than 1 radian. The term {sqrt(5)-1}/2=1/φ=φ-1 is often designated by Φ and also satisfies φΦ=1. The presence of sqrt(5) in the definition of the Kepler triangle and its relation to φ suggests that there should be many other geometric figures where the Golden Ratio plays a role. One of these involves the regular pentagon of unit side-length as shown-



Applying the Law of Cosines to the shaded triangle yields-

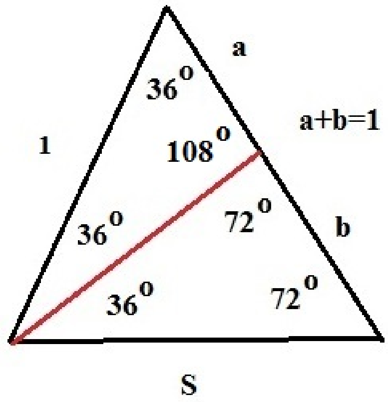
$$\varphi^2 = 2\{1-\cos(\frac{3\pi}{5})\} = 2\{1+\sin(\frac{\pi}{10})\} = 2.618033989... = \varphi + 1$$

So the length of the diagonal line is just equal to the Golden Ratio. From this last equality we also see that-

$$\sin(\frac{\pi}{10}) = \frac{(\sqrt{5}-1)}{4} = \frac{(\varphi-1)}{2} = \frac{\Phi}{2} \quad \text{and} \quad \cos(\frac{\pi}{10}) = \frac{\sqrt{2+\varphi}}{2}$$

As another example consider an isosceles triangle where two sides have unit length and a third side of length S as shown-

GEOMETRY FOR FINDING THE RATIO b/a



The angles are a specified and a red line bisects the lower left corner. The object is to find the ratio b/a, expresing the result as a function of φ. We have that a+b=1. To solve this problem we first use the Law of Cosines to show that the third side has length-

$$S = \sqrt{2[1-\cos(\pi/5)]} = 2\sin(\frac{\pi}{10}) = \frac{1}{\varphi}$$

Next we apply the Law of Sines to the two inner oblique triangles formed by the red line bisector. This produces-

$$a = \frac{\sin(\frac{\pi}{5})}{\sin(\frac{3\pi}{5})} = \frac{\varphi}{1+\varphi} \quad and \quad b = \frac{S\sin(\frac{\pi}{5})}{\sin(\frac{2\pi}{5})} = \frac{1}{1+\varphi}$$

, so that-

$$\frac{b}{a} = \frac{1}{\varphi} = \Phi = \frac{(\sqrt{5}-1)}{2} = 0.61803398...$$

The final question we want to clarify is how does one calculate accurate values for the irrational number φ? There are three basic ways for doing so. These are (a) continued fractions, (b) series expansion, and (c) an iteration procedure. Let us look a bit at each of these.

To generate accurate values of the Golden Ratio we can begin with the basic definition $\varphi^2=\varphi+1$ and rewrite it as-

$$\varphi = 1 + \frac{1}{\varphi} = 1 + \frac{1}{1+\frac{1}{\varphi}} = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{\varphi}}} = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{\varphi}}}}$$

That is, by continues re-substitution, we generate the simple continued fraction expansion shown. The more steps one includes in the evaluation the more accurate the approximation to the Golden Ratio becomes. Taking the fourth term and neglecting the 1/φ term in the denominator, we get φ≈5/3=1.6666…Such a continued fraction converges very rapidly and will do so even more if the fraction is worked around a number closer to φ .

A second approach for finding φ is to rewrite it as-

$$\varphi = \frac{1}{2} + \frac{1}{2}\sqrt{5}$$

and then expand the root of five in an infinite series about a point b close to sqrt(5). We find after application of a Taylor series about b=4, that-

$$\varphi = \frac{1}{2}\left\{3 + \frac{1}{4} - \frac{1}{64} + \frac{1}{512} - \frac{5}{16384} + \frac{7}{131072} - \frac{21}{2097152} + \frac{33}{16777216} - \right\}$$

The series converges but rather slowly. Taking the first seven terms in the series yields φ≈1.618033171 which is accurate to six places.

Finally we look at an iteration procedure . This will turn out to be the most efficient of our evaluation techniques. Here we start with the identity-

$$\sqrt{5} = 1 + \frac{4}{(\sqrt{5}+1)}$$

and then write down the iteration procedure-

$$f[n+1] = 1 + \frac{4}{1+f[n]} \quad subject \quad to \quad f[1] = 2$$

It produces f[2]=7/3, f[3]=11/5,f[4]=9/4, and f[5]=29/13. As n goes to infinity f[∞]=sqrt(5)=2.2360679. Thus we can say-

$$\varphi = \frac{\lim}{n\to\infty}\left(\frac{1}{2}\right)\{1+f[n]\}$$

Using the 20th iteration for f[n] we find-

$$\varphi \approx \{57314/35422\} = 1.61803399\ldots$$

which is accurate to seven places.

An alternate iteration approach is to use the third continued fraction expansion for φ and use it to write down the iteration-

$$G[n+1] = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{G[n]}}} = \frac{2+3G[n]}{1+2G[n]} \quad subject \quad to \quad G[1] = 3/2$$

As n gets large the rational number G[n] will approach the Golden Ratio. At the 20th iteration we find-

$$G[20] = \{2504730781961/1548008755920\} = 1.6180339887498948482045\ldots$$

This is a 24 place accurate result for the Golden Ratio. An even faster convergence occurs when using the sixth approximation which leads to-

$$G[n+1] = \frac{8+13G[n]}{5+8G[n]} \quad subject \quad to \quad G[1[= \frac{3}{2}$$

For this iteration formula, the 20th iteration produces the 49 digit accurate result-

$$\varphi = 1.618033988749894848204586834365638117720309179805\ldots$$

We notice that the mth iteration formula is given by-

$$G[n+1] = \frac{F[m-1] + F[m]G[n]}{F[m-2] + F[m-1]G[n]}$$

where F[m] is a Fibonacci Number. If G[1]=3/2 then G[2]=F[m+3]/F[m+2]. And so G[2] gets closer to φ as m increases.

Thanksgiving
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