# FINDING ALL ROOTS Of f (z) USING CONTOUR TECNIQUES

# AND THE NEWTON-RAPHSON METHOD

**INTRODUCTION:**

The roots of a complex function f(z) can be obtained via the complex version of the Newton-Raphson Iteration Technique , starting with an initial guess of $z_0$= a+ib lying near a given root. The iteration used reads-

$$z_{n+1}=z_n-[f\{z_n)]/f'(z_n)$$

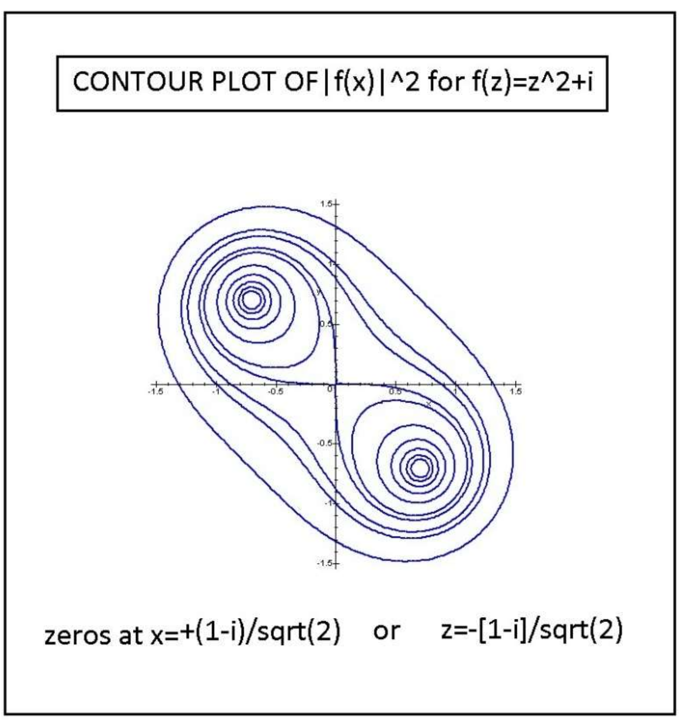
Unlike the case where f(z) is real, finding the value of $z_0$ becomes more complicated. We have found that one of the best ways to locate $z_o$ is to draw a contour map of the square of the amplitude of f(z) and then picking a point lying within the smallest of the contour value near a zero. It is our purpose here to carry out such iterations for several different f(z)s.

**ZEROES OF F(Z)=Z^2+i:**

Here we have-

F(z)=(x^2-y^2) +i(2xy+1) with z=x+iy=r exp(iθ)

On contour plotting |f(z)|^2 we have the picture-



It looks like a good starting point for the complex Newton-Raphson Iteration will be

$z_0$=[-1+i]/0.7 or [1-i]/0.7. To find the zero within the 4th quadrant we iterate-

$z_{n+1}$=$z_n$-[$z_n$^2+i]/(2$z_n$) subject to $z_0$=[1-i]*0.7

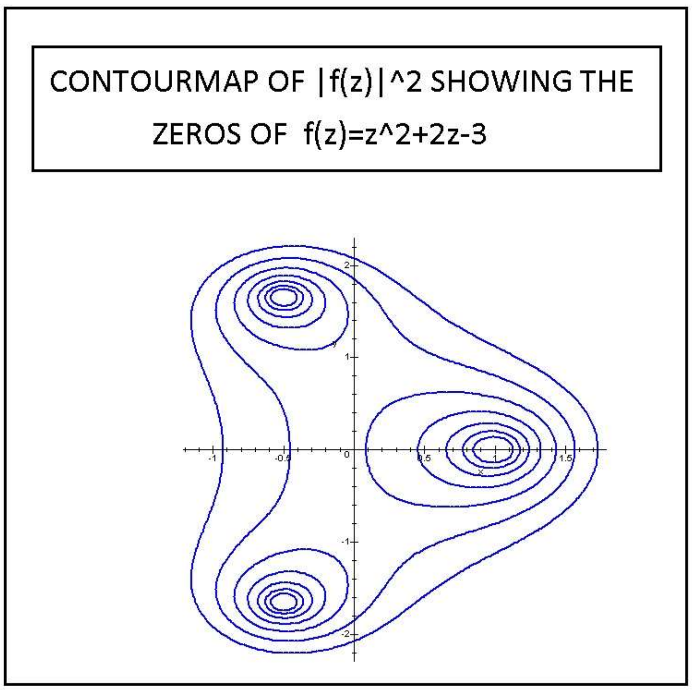
this yields-

$z_1$=0.7(1-i), $z_2$=0.7071(1-i), and $z_3$=0.70710678(1-i)

Thus one converges very rapidly toward the constant 1/sqrt(2)=0.707106781… The two exact solutions become the complex conjugates z=±(1-i)/sqrt(2)

**ZEROS OF THE CUBIC F(Z)-Z^3+2Z-3:**

Here the countour-map of |f(Z)|^2 looks like this-


CONTOURMAP OF |f(z)|^2 SHOWING THE ZEROS OF f(z)=z^2+2z-3

It indicates three zeros with one of these being z=1. The other two are complex conjugates given by solving z^2+z+3=0. The roots are z=(-1/2)[1±isqrt(11)] . Applying the Newton-Raphson Method to the zero in the 2nd quadrant, we start with $z_0$=0.5+i1.6 and then iterate-

$Z_{n+1}$=$z_n$-[$z_n$^3-2$z_n$-3]/[3$z_n$^2+2]

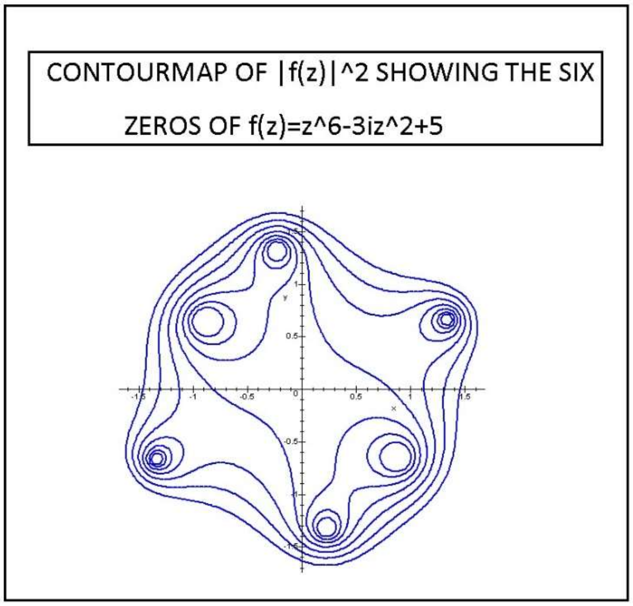
The first iteration already yields-

$z_1$=0.4988562654+i1.660549711

The exact value is z=-0.5+i1.658312.

**ZEROS OF F(z)=Z^6-3iZ^2+5:**

Here we know from Gauss that the absolute value of f(z) will have six roots. Where they lie in the z plane follows from the following contour map-




Here all roots are non-real with two falling in both the 2nd and 4th quadrant and one each in the first and 3rd quadrant. One could probably obtain the exact solutions by using the new variable $Psi = z$^2 and then solve the Psi equation as a cubic. This however would require considerable effort and therefore not worth pursuing. The Newton-Raphson Method however is ideal for such higher order complex number polynomials. Let us show how to find the zero in the fourth quadrant near $z_0$=0.9-0.6i. The iteration reads-

$$z_{n+1}=z_n-[G(z_n)/G(z_n)']$$

with –

G= x^6+6ix^5y-15x^4y^2-20ix^3y^3+15x^2y^4+6ixy^5-y^6-3ix^2+6xy+3iy^2+5

Starting with $z_o$=0.9-0.6i, we find $z_1$=0.8903+0.6550i, and $z_2$=0.89524-65441i. A computer solution places this root at z=0.895211-0.654438i. This means the second iteration already gives the root location to four place accuracy.

**CONCLUSION:**

We have shown that any continuous function f(z) can have multiple roots which can be both real or complex. By first using a contour plot of the square of the absolute value of |f(z)| one can determine near what point f(z) is to be evaluated to quickly locate a root via iteration using the complex version of the Newton-Raphson Method. These days most advanced mathematics programs, such as MAPLE or MATHEMATICA, have built in programs which quickly find zeros of any complex function f(z) by iteration in split seconds.

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