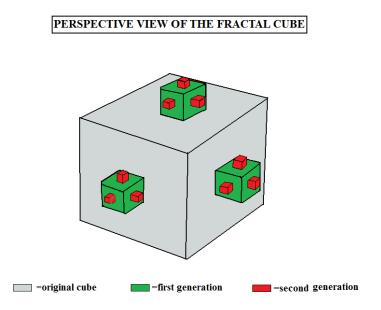# GENERATIONAL DEVELOPMENT OF A FRACTAL CUBE

Consider a cube of side length $L_0=1$, surface area $A_0=6$ and volume $V_0=1$. Let it generate a total of six smaller cubes of $L_1=1/3$, $A_1=6/9$ and $V_1=1/27$ each. These are attached to the center of the six faces of the original cube. We refer to these cubes as the first generation offsprings. Continuing on to a second generation, we have still smaller cubes of $L_2=1/9$, area $A_2=6/81$, and $V_2=1/729$ each. What is being generated is a 3D version of something like a Koch curve but where each generation is generated only by the previous generation. A drawing of the resultant structure looks as follows-

PERSPECTIVE VIEW OF THE FRACTAL CUBE

☐ =original cube   ▇ =first generation   ▇ =second generation

Following the standard Mandelbrot designation, this three dimensional fractal containing an infinite number of ever smaller self-similar cubes has the unit cube as its **initiator** . The **generation law** is that a new smaller cube grows centered on each open faced surface of the preceding generation.. This is a very simple reproduction law which leads to a quite complicated three dimensional structure.

On generalization, this fractal cube is seen to add $6(5^{n-1})$ new cubes of side length $S_n=1/3^n$ at each new generation . One has the surprising result that the total volume of the fractal cube remains finite and sums to $V_{total}=14/11$ despite that fact that one is dealing with an infinite number of cubes. In addition the total surface area remains finite unlike what happens in 2D Koch area problems where the perimeter becomes infinite. Let us prove these properties. The total fractal cube volume , summing up the contribution of the original cube and its subsequent generations, looks like this-

$$V_{total} = 1 + 6[\frac{1}{3^3} + \frac{5}{3^6} + \frac{5^2}{3^9} + ...] = 1 + \frac{6}{5}\sum_{k=1}^{\infty}(\frac{5}{27})^k = \frac{14}{11}$$

Note in this calculation one has taken into account the fact that the unit cube has 6 offspring but all subsequent generations have only 5 offspring since the remaining surface is blocked. Also in this type of volume calculation one must be careful that subsequent generations do not bump into the wall of the original cube . This turns out to not be a problem since the distance from the surface of any of the cubes of the second generation facing the surface of the original unit cube is $1/9^{th}$. While the sum $1/27+1/81+1/243+\ldots$sums to $1/18$. However note that there would be a problem with wall collision if the generating law chose the subsequent generation to have ½ of the previous generation side-length. There one of the second generation cubes facing the original cube wall would have its surface separated by $1/8$ while the sum $1/8+1/16+1/32+\ldots$ adds up to ¼. Thus growth would not be possible toward the six walls of the original cube after $n=2$. Generalizing this result to the generic case of side length $L_1=a <1$, we see that to avoid wall contact the gap spacing $\Delta_n=a^{n-1}/2-a^n/2$ between the nth generation and the n-2 generation should exceed the infinite series sum $a^{n+1}/(1-a)$. That is-

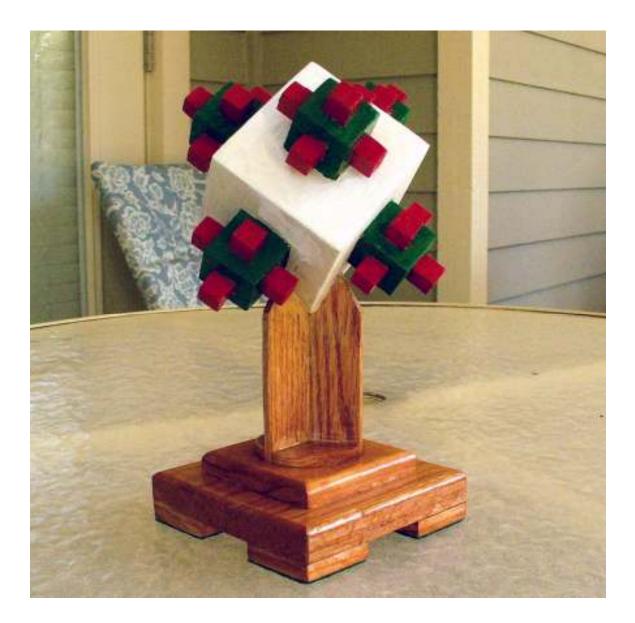$$\Delta_n = \frac{a^{n-1}}{2}(1-a) > a^{n+1}(1+a+a^2+...) = \frac{a^{n+1}}{1-a}$$

That is –

$$a < [\sqrt{2}-1] = 0.41421356...$$

The total external surface area is a bit more tricky to calculate since part of the previous generation surface is taken away by the offspring. After some detailed analysis for $a=1/3$, we find the total external surface area to be-

$$A_{total} = 6(\frac{8}{9})\left\{1+(\frac{5}{9})+(\frac{5}{9})^2 + \right\} = 12$$

Thus the surface area is finite and twice the value of the original unit cube surface without first generation blocking.

Below, you will find a photo of a finished painted and stained wood sculpture I constructed of a cube plus its first (green) and second (red) generation when $a=0.4<[sqrt(2)-1]$.

If magnified by a factor of 100 or so it would probably make an interesting public art display. The size of the first generation cube could be increased to say $S_1=1/2$ if one is not interested in adding details beyond n=2. The Hausdorff Dimension based on the original cube and its first and second generation is-

$$D = \frac{\ln(813)}{\ln(17)} = 2.36506...$$

so that is not a very compacted 3D structure. See-

http://en.wikipedia.org/wiki/List_of_fractals_by_Hausdorff_dimension

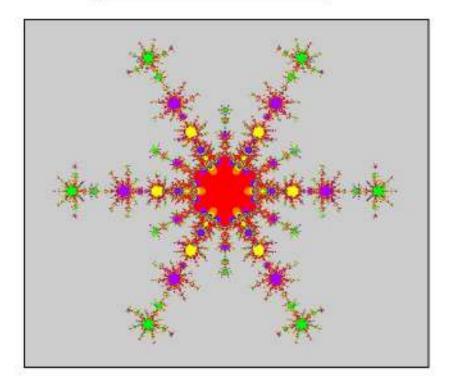to look at the non-integer dimensions of other fractals.

We have shown that 3D fractals exist and that they have the property that each subsequent generation maintains the same self-similar 3D shape of the initiator. Other starting solids such as the dodecahedron and the icosahedron could also serve as initiators but their growth would be limited because of expected generational overlap in space of the nth generation.

We have also attempted without much success to find a 3D fractal structure based on the 2D type generated by the very simple iteration

$$Z_{n+1} = Z_n^{\ 6} - 1.11$$

In 2D, this iteration yields the amazingly intricate six-fold symmetric pattern-

ITERATION Z[n+1]= Z[n]^6-1.11

If one could convert this to its 3D analogue, then, perhaps, it could be successful in mimicking certain virus shapes which clearly must be produced by very simple replication formulas.