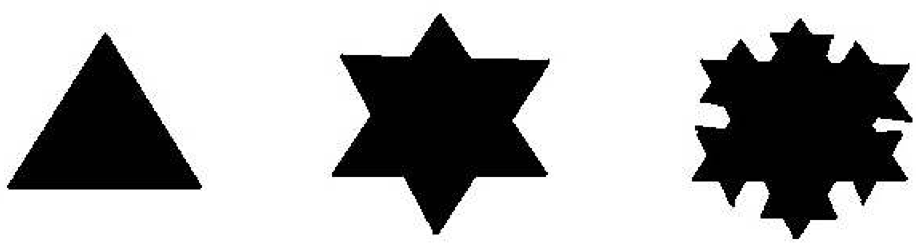
GENERATING A FRACTAL SQUARE

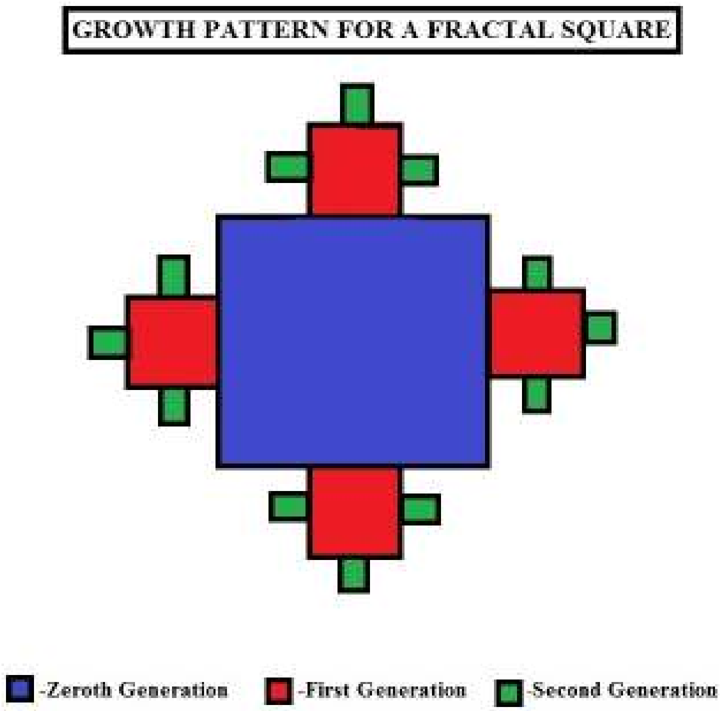
In 1904 the Swedish mathematician Helge von Koch(1870-1924) introduced one of the earliest known fractals, namely, the Koch Snowflake. It is a closed continuous curve with discontinuities in its derivative at discrete points. The simplest way to construct the curve is to start with an equilateral triangle of unit side-length and then break each of its sides into three equal parts of length 1/3 and next add a fourth element of length 1/3 to construct a smaller equilateral triangle plus two bounding straight lines. Continuing this type of breakup indefinitely leads to the Koch Snowflake. We can think of the process as one in which successive generations form smaller similar triangles and connecting lines form a closed curve. The development through generations 0, 1, and 2 look as follows-

CONSTRUCTION OF THE KOCH SNOWFLAKE



zeroth generation **first generation** **second generation**

There is of course no reason that only triangles can be used to construct a Koch curve. One such construction which we have come up with and which differs from other extant constructions is one based on a unit side-length square onto whose edges are placed smaller and smaller self-similar smaller squares. Drawing such a curve we find that that generations 0-1-2 produce the following figure-



The scaling between generations is fixed at a constant value of f lying in the range 0<f <1 such that the n+1 generation has a square side-length smaller than the nth generation by a factor of f. This reduction in size between generations allows us to maintain self similarity. You will notice that the construction here allows attachment of n+1 generation squares <u>only</u> to exposed edges of the nth generation elements. This procedure differs from a standard Koch curve.

If we now examine the total area contained within the curve terminating with the nth generation, we find-

$$A_n = 1 + 4f^2 + 4\cdot 3\cdot f^4 + 4\cdot 3^2 f^6 + ... + 4\cdot 3^{n-1} f^{2n-2} = 1 + 4f^2 \sum_{k=0}^{n-1} (3f^2)^k$$

It is recognized that this finite series represents an incomplete geometric series whose closed form solution is known. A little manipulation produces the result-

$$A_n = 1 + 4f^2 \frac{\{1 \; - (3f^2)^n\}}{(1 - 3f^2)}$$

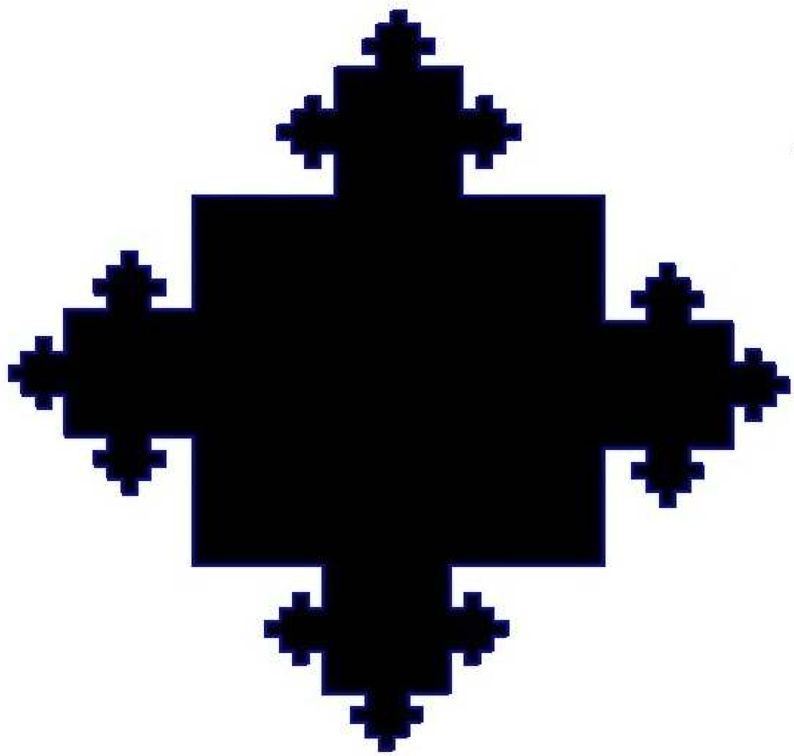
Thus-

$$A_\infty = 1 + \frac{4f^2}{(1-3f^2)}$$

provided that f<1/sqrt(3) and overlap is allowed. Notice that the ratio of the area of the n+1 generation compared to the nth generation for n≥1 is-

$$Area\ Ratio = \frac{4\cdot 3^n \cdot f^{2n}}{4\cdot 3^{n-1}\cdot f^{2n-2}} = 3f^2$$

Let us now look at the special case of f=1/3. This produces the following attractive configuration -



I call the figure the Black Snowflake. Notice that it has the interesting property that tangent diagonal lines with slope ±1 touch each generation. For this case the ratio of side-length of the squares from one generation to the next equals exactly three. We find A_0=1, A_1=13/9, and A_2=43/27. The total area when adding together all generations produces the finite value A_∞=5/3. This result makes sense since the rotated square composed of the four tangent lines mentioned has an area of two . We can also establish the A_∞ value by noting that the area ratio between generations is 1/3 for this case and hence-

$$A_\infty = 1 + \sum_{k=0}^{\infty} (\frac{1}{3})^k = 1 + \frac{3}{2} = \frac{5}{3}$$

When looking at the Black Snowflake the shortest distance between generation n+1 and n-1 remains finite so there is no possibility of overlap. As we increase f above 1/3 a point will be reached when an overlap can occur. Let us see what the limitations on f are to prevent such an overlap. Clearly to avoid overlap it is necessary for the smallest gap Δ between the 2nd (green)and zeroth(blue) generation in the above colorized figure be more than $f^3+f^4+f^5+\ldots$ This means-

$$\Delta = \frac{f}{2} - \frac{f^2}{2} \geq f^3 \sum_{k=0}^{\infty} f^k$$

On solving we find –

$$f \leq \sqrt{2} - 1 = 0.41421356...$$

Thus the restriction on f will be-

$$0 < f < \sqrt{2} - 1$$

In order to have a fractal square with an infinite number of generations. If one is interested in closed curves showing only the first n generations than this restriction can be relaxed.

Let us next calculate the perimeter P of the fractal square under consideration. For the zeroth generation we have –

$$P_0 = 4(1-f)$$

When the first generation is included we find-

$$P_1 = 4(1-f) + 4 \cdot 3f(1-f() = 4(1-f)[1+3f]$$

and the inclusion of the second generation produces-

$$P_2 = 4(1-f)[1+3f+9f^2\}$$

Thus we have that the total perimeter of the fractal square through the nth generation becomes-

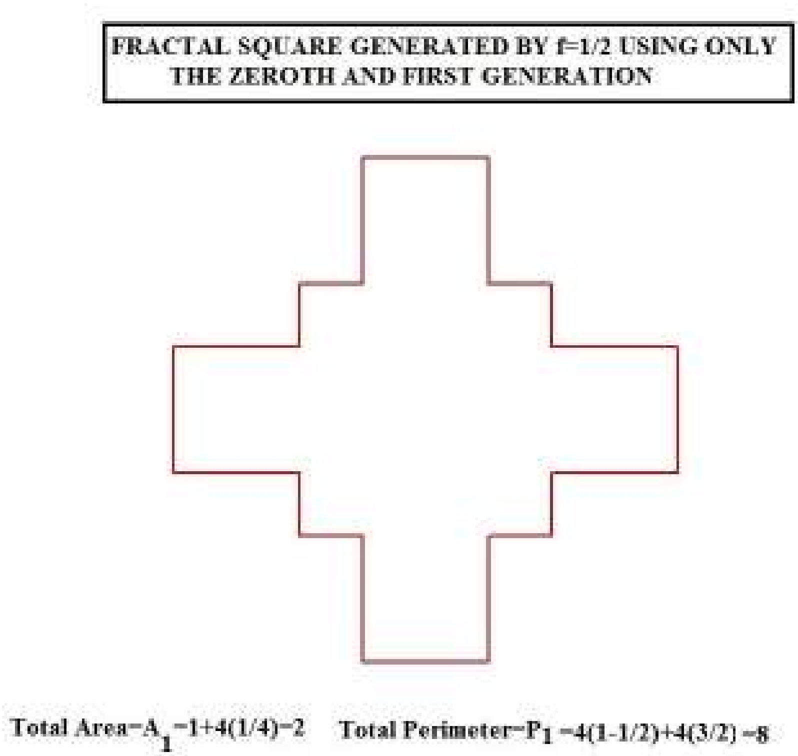
$$P_n = 4(1-f)\sum_{k=0}^{n-1}(3f)^k + 4(3f)^n$$

Notice the last term in this equality arises from the fact that the parts of the perimeter blocked by the n+1 generation no longer exist. We thus see that like for a standard Koch curve the perimeter becomes infinite at $n=\infty$ when $f>1/3$. However, you will also note the unexpected result that the perimeter remains finite for $n=\infty$ and $f<1/3$. It shows, for example, that $P_\infty=12$ and $A_\infty=17/13$ for $f=1/4$ while $f=1/3$ produces $P_\infty=\infty$ and $A_\infty=5/3$. As we have done for the Black Snowflake, one can use computer graphics to quickly draw the fractal square through any desired number of generations for a fixed f. Consider drawing the figure using only the zeroth and first generation when the starting point is a unit square and f=1/2. Our computer program using MAPLE reads-

```
with(plots);

listplot([[1,1],[1/2,1],[1/2,2],[-1/2,2],[-1/2,1],[-1,1],[-1,1/2],[-2,1/2],[-2,-1/2],[-1,-1/2],[-1,-1],[-1/2,-1],[-1/2,-2],[1/2,-2],[1/2,-1],[1,-1],[1,-1/2],[2,-1/2],[2,1/2],[1,1/2],[1,1]],
color=red, scaling=constrained, axes=none);
```

The graphic output is-

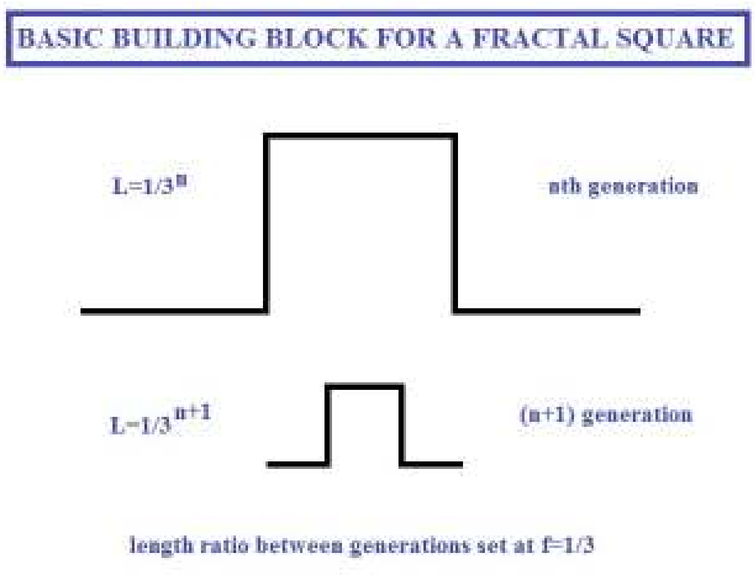


Note the total area is 2 as given by the above formula for A_1. The perimeter becomes $P_1=8$.

Finally we briefly mention an alternative way to generate these fractal square configurations by a genetic algorithm approach as discussed by us in an earlier note(http://www2.mae.ufl.edu/~uhk/GENETIC-CODES.pdf) and related to the Lindenmayer system. In this procedure any closed curve can be constructed from straight line segments defined only by their length L and the angle θ they make with regard to the next line segment. The designation of a line will be [L,θ]. For the fractal square the angles are restricted to either θ =π/2 for counterclockwise or θ=-π/2 for clockwise. Thus for the square fractal with f=1/3 we have the basic building block-

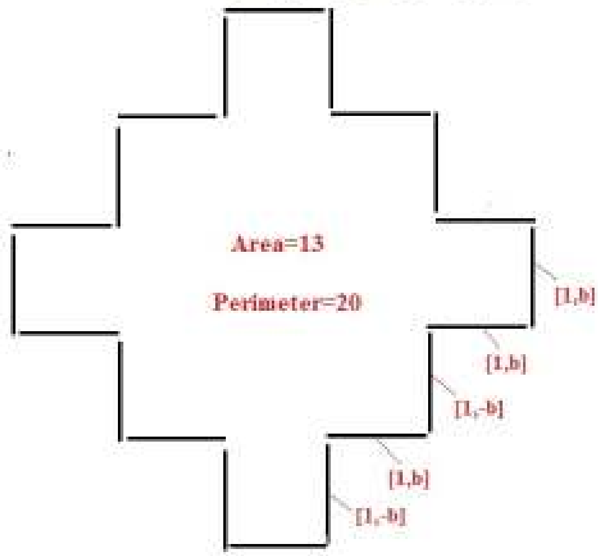
$$[\frac{1}{3^n},-\frac{\pi}{2}],[\frac{1}{3^n},+\frac{\pi}{2}],[\frac{1}{3^n},+\frac{\pi}{2}],[\frac{1}{3^n},-\frac{\pi}{2}],[\frac{1}{3^n},+\frac{\pi}{2}],[$$

A graph of this five element block look like this for generations n and n+1-



If we connect the building block for the n=1 generation four times the following figure results-

CONSTRUCTION OF A FRACTAL SQUARE WITH f=1/3
USING THE GENETIC CODE [1,-b],[1,b],[1,-b],[1,b],[1,b] with b=Pi/2



As is seen, the code generates three staircase functions which are hooked together to form the fractal square. If one is interested in defining a square fractal containing the next higher generation it is only necessary to increase n to n+1 in the $L=1/3^n$ lengths of the basic five element code and superimpose the result unto the exposed line edges of the nth generation squares.