FRESNEL INTEGRALS AND THE CORNU SPIRAL

In the PDE section we discussed the fundamental solution to the Helmholtz equation and showed that the diffraction of light by an aperture is governed by this relation in terms of the wave function

$$\phi(P) = Const.\iint_S \exp[-ik(r+R)]dS$$

provided that the light wavelength $\lambda=2\pi/k$ is small compared to the aperture dimension sqrt(S). Here R is the distance from a point source to a point within the aperture and r the distance from that point to a point P on the screen. By expanding r+R in a Taylor series based on the distances r_0 and R_0 from the aperture center and retaining only the first non-vanishing term beyond the first, one can find the diffraction pattern expected at the screen. For a slit aperture one has the already discussed Fraunhofer diffraction pattern. For a circular aperture, however, the Taylor expansion has its first non vanishing term beyond the first going as the square of the distance from the aperture center. This leads to Fresnel diffraction and is described by the integral

$$\phi(x) = Const.\int_0^x \exp - i(\pi/2)t^2 dt$$

Taking the real and imaginary parts of this integral yields the well known Fresnel integrals

$$C(x) = \int_0^x \cos[(\pi/2)t^2]dt \quad , \quad S(x) = \int_0^x \sin[(\pi/2)t^2]dt$$

The actual diffraction pattern produced by a circular aperture at the screen will go as the square of φ and yields the intensity pattern

$$I(x) = |\phi|^2 = [C(x)^2 + S(x)^2]$$

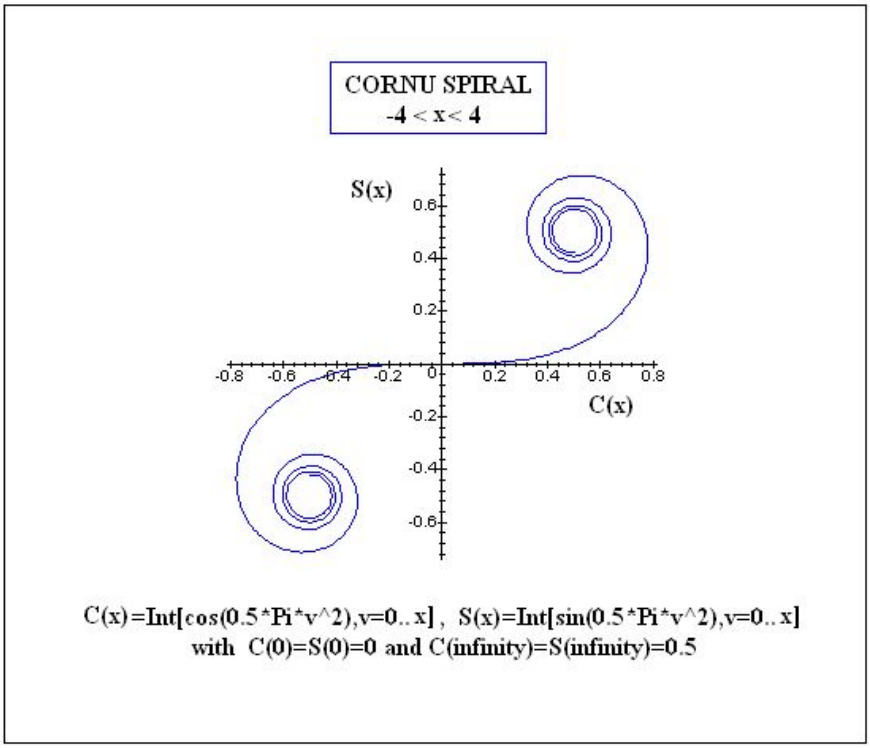
To evaluate the Fresnel integrals one simply expands the cosine and sine terms in infinite series and then integrates term by term. This produces the following

$$C(x) = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n(\pi/2)^{2n}t^{4n}}{(2n)!}dt = \sum_{n=0}^{\infty} \frac{(-1)^n(\pi/2)^{2n}t^{4n+1}}{(2n)!(4n+1)}$$

and

$$S(x)=\int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n(\pi/2)^{2n+1}t^{4n+2}}{(2n+1)!}dt=\sum_{n=0}^{\infty}\frac{(-1)^n(\pi/2)^{2n+1}t^{4n+3}}{(2n+1)!(4n+3)}$$

A parametric plot of these Fresnel integrals F(x)=C(x)+iS(x) produces the famous Cornu spiral shown here



One sees at once that both C(x) and S(x) vanish at x=0 and that they approach the value ½ as x becomes infinite. The variable substitutions p=sqrt(pi/2)*t and q=(pi/2)*t^2 lead (for x=∞)to

$$\int_0^{\infty}\cos(p^2)dp=\int_0^{\infty}\sin(p^2)dp=\frac{\pi}{2\sqrt{2}}$$

,

$$\int_0^{\infty}\frac{\cos(q)}{\sqrt{q}}dq=\int_0^{\infty}\frac{\sin(q)}{\sqrt{q}}dq=\frac{\pi}{\sqrt{2}}$$

It is also possible to relate the Fresnel integrals to other known functions. For example, looking at the confluent hypergeometric(Kummer) function M(a,b,z), we have

$$xM[\frac{1}{2},\frac{3}{2},\frac{i\pi x^2}{2}] = x + i\frac{\pi x^3}{6} - \frac{\pi^2 x^5}{40} - i\frac{\pi^3 x^7}{336}... = C(x) + iS(x)$$

Also one can can show that the Fresnel integrals are expressible as an error function erf(z) , namely

$$\frac{(1+i)}{\sqrt{\pi}} \int_{w=0}^{\sqrt{\pi}(1-i)(x/2)} \exp - w^2 dw = \frac{(1+i)}{2} \int_0^{\sqrt{\pi}(1-i)x/2} [1 - w^2 + (w^4/2!) - ....]dw = C(x) + iS(x)$$

so that one has the special case

$$C(\frac{2}{\sqrt{\pi}}) + iS(\frac{2}{\sqrt{\pi}}) = \frac{(1+i)}{2} erf(1-i) = 0.75330237... + i0.56284890...$$

Finally, the Laplace transform of C(x) has the somewhat complicated form

$$\int_0^\infty C(\sqrt{\frac{2t}{\pi}}) \exp(-st) dt = \frac{1}{2s(\sqrt{s^2+1})(-s+\sqrt{s^2+1})^{1/2}}$$

There are many additional definite integrals expressible in terms of C(x) and S(x) functions. We refer the reader to the **"Handbook of Mathematical Functions"**by Abramowitz and Stegun for some of these.