SHORTEST DISTANCE BETWEEN TWO POINTS ON A SPHERE

It is known that the shortest distance between point A and point B on the surface of a sphere of radius R is part of a great circle lying in a plane intersecting the sphere surface and containing the points A and B and the point C at the sphere center. Let us use the calculus of variations and spherical coordinates to define this great circle and show how to calculate the geodesic distance between points A and B on the surface. One starts with the definition of length between points A and B along the great circle. Mathematically one has-

$$L = R\int_{\theta=\theta_A}^{\theta=\theta_B} d\theta\sqrt{1+\sin^2\theta(\frac{d\varphi}{d\theta})^2}$$

where one is using spherical coordinates defined as-

$$x = R\sin\theta\cos\varphi\,,\ y = R\sin\theta\sin\varphi\,,\ z = R\cos\theta$$

With θ being the polar angle and φ the azimuthal angle. In terms of latitude and longitude on the earth one has LAT=(π/2-θ) and LONG=φ , when measured relative to Greenwich. Now according to the Euler-Lagrange equation one has that the integrand is an extremum(here a minimum) when-

$$\frac{d}{d\theta}\left[\frac{\partial\sqrt{\ }}{\partial\varphi'}\right]=0 \quad so\ \ that\ \ (\frac{d\varphi}{d\theta})^2 = \frac{c^2 1}{\sin^2\theta[\sin\theta^2 - c^2]}$$

where c is an adjustable constant. The geodesic distance between point A and B is thus-

$$L = R\int_{\theta_A}^{\theta_B} \frac{\sin\theta}{\sqrt{\sin^2\theta - c^2}}d\theta$$

On integrating one finds-

$$L = \frac{R}{2}\arctan\left[\frac{2i(\delta-\gamma)}{\gamma\delta - 4}\right] \quad with\ \delta = \frac{[2a-(1-c^2)]}{\sqrt{a^2 - a(1+c^2)+c^2}}\ and\ \gamma = \frac{[2b-(1+c^2]}{\sqrt{b^2 - b(1+c^2)+c^2}}$$

and $a=\sin(\theta_a)^2$, $b=\sin(\theta_b)^2$. The constant c^2 is determined by solving the first order equation for $d\varphi/d\theta$ which brings in the values of φ_a and φ_b.

To test out this result consider point A at $\theta=\pi/4$ and $\varphi=0$ and point B at to $\theta=\pi/2$ and $\varphi=\pi/2$. Here we have $a=\sin(\pi/4)^2=1/2$ and $b=\sin(\pi/2)^2=1$ so that $\gamma=\infty$ and $\delta=2c^2/\mathrm{sqrt}(2c^2-1)$. This in turn implies that-

$$L=\frac{R}{2}\arctan\left[\frac{2i(-\infty)}{(-\infty)2c^2/\sqrt{2c^2-1}}\right]=\frac{R}{2}\arctan\left[\frac{i\sqrt{2c^2-1}}{c^2}\right]$$

and will have a minimum positive value of $L=\pi R/2$ when $c^2=0.5$, a value consistent with the latitudes of the end points of the geodesic. That this result must be correct is seen by visualizing a globe and noting that the distance from A to B will be just equal to one fourth of the global circumference. In general to find c^2 one needs to first solve the $d\varphi/d\theta$ equation and then plug into the above solution for L. This can be a rather lengthy procedure and instead one usually uses an alternate route based upon spherical geometry(see our earlier discussion on spherical geometry). The law of cosines for a spherical triangle having corners at A and B plus a third corner at the pole P of the globe leads to the simple formula-

$$L=R\cos^{-1}\{\cos[LAT(A)-LAT(B)]-[\cos(LAT(B)\cos(LAT(A)][1-\cos(LONG(B)-LONG(A))]\}$$

where LAT and LONG refer to the latitude and longitude of the end points A and B , respectively.

Thus for the same end points considered above we have $LAT(A)=\pi/4$, $LONG(A)=0$, $LAT(B)=0$, and $LONG(B)=\pi/2$. This leads (with much less effort) to the same result-

$$L=R\cos^{-1}\{\cos[\pi/4-0]-[\cos(0)\cos(\pi/4)][1-\cos(\pi/2-0)]\}$$
$$=R\cos^{-1}\{1/\mathrm{sqrt}(2)-1/\mathrm{sqrt}(2)\}=\pi R/2$$

The length of the geodesic going from the north pole to the south pole is calculated using $LAT(A)=\pi/2$, $LAT(B)=-\pi/2$ with $LONG(A)=LONG(B)=0$ say. This yields at once that –

$$L=R\cos^{-1}\{\cos(\pi/2+\pi/2)-[\cos(-\pi/2)\cos(\pi/2)][1-\cos(0-0)]=\pi R$$

We conclude our discussion by looking at Erastosthenes measurement of the earth's circumference $C=2\pi R$. He noted that at the summer solstice the sun in Aswan at noon was directly overhead while the sun at his home in Alexandria, Egypt was at 7deg from the zenith at noon of the same day He also knew the distance between Alexandria and Syene(Aswan)was 500 miles directly south. Although he had no knowledge of extremum principles and the formulas of spherical trigonometry, he

nevertheless came up with the first (and surprisingly accurate) measure of the earth's circumference. In terms of the above discussion one has at Alexandria LAT= 31degN and LONG(A)=30degE and at Aswan LAT=24degN and LONG=30degE. Also L=500miles. Plugging these numbers into the above equation for L one has approximately-

$$\cos(500/R)= \{\cos(7deg)-[\cos(24deg)\cos(31deg)][1-\cos(0)]$$

or

$$500/R=\cos^{-1}\{\cos(7/57.29)\}=0.12218 \text{ or } R=4092 \text{ miles}$$

A not a bad result compared to the accurate value of R=3960 miles, especially since it was carried out some2249 years ago. Columbus would have done well to study Eratosthenes results so that he would have had a better estimate of the distance to East Indies when sailing west from Spain.

June 2009