## **INFINITE PRODUCTS**

One defines an infinite product as-

$$F(x) = \prod_{n=1}^{\infty} [F_n] = (F_1)(F_2)(F_3)...$$

Taking the natural logarithm of each side one has-

$$\ln[F(x)] = \sum_{n=1}^{\infty} \ln(F_n) = \ln(F_1) + \ln(F_2) + \ln(F_3) + \dots$$

So that the initial infinite product will converge only if the sum of the log terms converges. One of the earliest infinite products is the Wallis product of 1655. It reads–

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{4n^2}{(4n^2 - 1)} = \left(\frac{4}{3}\right) \left(\frac{16}{15}\right) \left(\frac{36}{35}\right) \dots = \frac{(2 \cdot 2)(4 \cdot 4)(6 \cdot 6)\dots}{(1 \cdot 3)(3 \cdot 5)(5 \cdot 7)\dots}$$

and is easily established by noting that-

$$\int_{0}^{\pi/2} \sin(x)^{2n} dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{\pi}{2} \quad , \quad \int_{0}^{\pi/2} \sin(x)^{2n+1} dx = \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n+1)}$$

and realizing that as n goes to infinity the two integrals should be equal. This yields the Wallis result-

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n+1)(2n-1)} = 1.57079632....$$

Looking at the log convergence test, one finds that the log series for the Wallis product sums to  $\ln(\pi/2)$  and thus the Wallis product converges but, unfortunately, does so very slowly.

The next important historical contributions to infinite products were those of the Swiss mathematician Leonard Euler (1707-1783). In particular he looked at the function  $F(x)=\sin(\pi x)/\pi x$  and made use of the fact that this function has simple zeroes at all integer values between minus and plus infinity. Accordingly he postulated that one should be able to expand things as follows-

$$\frac{\sin(\pi x)}{\pi x} = \prod_{n=1}^{+\infty} (1 - A_n x^2)$$

and recognized that  $A_n$  must equal  $1/n^2$  since the function must vanish at x=1, 2, 3, 4, etc. As a result one has the convergent infinite product-

$$\frac{\sin(z)}{z} = \prod_{n=1}^{\infty} \left[ 1 - \left(\frac{z}{n\pi}\right)^2 \right] = (1 - \left(\frac{z}{\pi}\right)^2)(1 - \left(\frac{z}{2\pi}\right)^2)(1 - \left(\frac{z}{3\pi}\right)^2)\dots$$

On setting  $z=\pi/2$ , we have the infinite product-

$$\frac{2}{\pi} = \prod_{n=1}^{\infty} \left(1 - \frac{1}{4n^2}\right) = 0.636619772...$$

On setting z to  $\pi/6$ , one finds-

$$\frac{\pi}{3} = \prod_{n=1}^{\infty} \left[ \frac{n^2}{n^2 - (\frac{1}{6})^2} \right] = \frac{36}{35} \cdot \frac{144}{143} \cdot \frac{324}{323} \cdot \frac{576}{575} \cdot \frac{900}{899} \cdot \dots$$

and by equating the coefficients of the  $x^2$  terms in the equality, one has his famous infinite series result-

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2)$$

Following similar arguments to those used by Euler, one can also show that the zeroth order Bessel function which has roots at  $\lambda_1$ ,  $\lambda_2$ ,  $\lambda_3$ ,...yields the infinite product-

$$\frac{J_0(x)}{x} = \prod_{n=1}^{\infty} \left( 1 - \left( \frac{x}{\lambda_n} \right)^2 \right)$$

and that  $\cos(x)$ , which has its zeros at  $(2n+1)(\pi/2)$ , satisfies-

$$\cos(x) = \prod_{n=1}^{\infty} \left[ 1 - \frac{4x^2}{((2n-1)\pi)^2} \right]$$

Another infinite product, again going back to Euler, is that for the Riemann zeta function. It can be expressed in terms of prime numbers  $p_n$  as-



A good discussion of how this result is derived can be found in the book "Prime Obsession" by John Derbyshire and likely will play a role in the final proof of the Riemann conjecture.

Let us next demonstrate how one may convert a continued fraction into an infinite product. Consider the square root of two. One of its continuous fraction representation is-

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}}} = [1; 2, 2, 2, 2, 2, 2, \dots]$$

whose convergents are given by-

$$C_0 = 1, C_1 = \frac{3}{2}, C_2 = \frac{7}{5}, C_3 = \frac{17}{12}, \text{ with } \frac{C_{n+1}}{C_n} = \frac{(2+C_n)(1+C_{n-1})}{(1+C_n)(2+C_{n-1})}$$

We thus have

$$\sqrt{2} = \prod_{n=1}^{\infty} \frac{C_n}{C_{n-1}} = \left(\frac{3}{2}\right) \cdot \left(\frac{14}{15}\right) \cdot \left(\frac{85}{84}\right) \cdots = (1 + \frac{1}{2})(1 - \frac{1}{15})(1 + \frac{1}{84}) \cdots$$

The sequence of numbers 2, -15,84,...appearing in the denominator of this infinite product expansion is generated by-

$$N_n = \frac{(-1)^{n+1}(1+C_{n-1})(2+C_{n-2})}{(C_{n-1}-C_{n-2})}$$

so that the next numbers will be -493, +2870, -16731, etc.

As another example of constructing an infinite product, we look at exp(1)=2.718281828459045...Starting with its computer obtained simple continued fraction exp(1)=[2;1,2,1,1,4,1,1,6,1,1,8,1,1,10...] we generate the convergents  $C_n$  to find-

$$C_0 = 2, C_1 = 3, C_2 = \frac{8}{3}, C_3 = \frac{11}{4}, C_4 = \frac{19}{7}, C_5 = \frac{87}{32}$$

This time one can find no simple recurrence relation to generate the higher  $C_n s$  because the continued fraction pattern in the square bracket above has varying integer values. Nevertheless, one can continue finding higher values of  $C_n$  by brute force calculation so that the resultant <u>finite</u> product can approximate exp(1) to any desired order of accuracy. We will stop here with  $C_5$ . The result is the approximation-

$$\exp(1) = \prod_{n=1}^{\infty} (1 - B_n) \cong (1 + \frac{1}{2}) \cdot (1 - \frac{1}{9}) \cdot (1 + \frac{1}{32}) \cdot (1 - \frac{1}{77}) \cdot (1 + \frac{1}{608})$$

It would be nice if someone out there would find the general form of  $B_n$  in this last expression so that an exact infinite product for exp(1) could be given. It will probably involve the use of the semi-periodic form of the coefficients in [...] which go as [...(2n-2), 1, 1, (2n), 1, 1, (2n+2)...]. Alternatively one could search for a more symmetric form of a partial fraction for exp(1) which lends itself to general evaluation of  $B_n$ . Recall from above that the Wallis formula gives an exact value for  $\pi$  although its simple partial fraction array has the non-periodic form [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, ...].

To sum certain infinite products involving quotients of functions of n it is sometimes convenient to first expand the numerator and denominator term in the quotient and then proceed to cancel various terms in the resultant expansion. A good example of such a procedure involves the evaluation of the infinite product-

$$S = \prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1} = \prod_{n=2}^{\infty} \left[ \frac{(n-1)(n^2 + n + 1)}{(n+1)(n^2 - n + 1)} \right]$$

Expanding this product out produces-

$$S = \left[\frac{(1)(7)}{(3)(3)}\right] \cdot \left[\frac{(2)(13)}{(4)(7)}\right] \cdot \left[\frac{(3)(21)}{(5)(13)}\right] \cdot \left[\frac{(4)(31)}{(6)(21)}\right] \cdot \left[\frac{(5)(43)}{(7)(31)}\right] \cdot \dots$$

After cancelling terms we are left with-

$$S = \frac{2}{3} \cdot \left[\frac{1}{1}\right] \cdot \left[\frac{1}{1}\right] \cdot \left[\frac{1}{1}\right] \cdot \dots = \frac{2}{3}$$

Another example is-

$$\prod_{n=1}^{\infty} \left[ \frac{n+1+(-1)^n}{n+1} \right] = \left[ \frac{1}{2} \right] \cdot \left[ \frac{4}{3} \right] \cdot \left[ \frac{3}{4} \right] \cdot \left[ \frac{6}{5} \right] \cdot \left[ \frac{5}{6} \right] \cdot \ldots = \frac{1}{2}$$

For most cases the infinite product of a quotient of functions of n will be either zero or infinity. Thus-

$$\prod_{n=1}^{\infty} \frac{n^2}{n^3 + 1} = 0 \quad and \quad \prod_{n=1}^{\infty} \frac{(n+1)^2}{(n^2 + 1)} = \infty$$