GENERATING AND EVALUATING INFINITE SERIES

There are an infinite number of infinite series which are either convergent or divergent. Classic examples of the two types are the harmonic series-

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ... = \sum_{n=1}^{\infty} \frac{1}{n}$$

which diverges to infinity and the geometric series-

$$S = 1 + r + r^2 + r^3 + ... = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad provided \quad that \quad |r| < 1$$

The standard way to test whether a series converges or not is the ratio test one learns about in elementary calculus. This test says a series is convergent if-

$$\frac{\lim}{n \to \infty} \left[\frac{|S_{n+1}|}{|S_n|} \right] < 1$$

, where the subscript n refers to the nth term in the series and the vertical bars indicate absolute value. So the particular value of the following geometric series-

$$S = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + ... = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} = \frac{2}{3}$$

converges by the ratio test .

If we convert the harmonic series to an alternating series form-

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - ... = \sum_{n=1}^{\infty} \frac{(=1)^n}{n}$$

it is also convergent. Looking at the sum of the first 40 and 41 terms we get the bounds-

$$0.6808 < S < 0.7052$$

So the series should sum close to the average of 0.6930.The exact value of this last sum follows from the series expansion of ln(1+x) which reads-

$$\ln(1+x)=\sum_{n=1}\frac{(-1)^n}{x^n}$$

This means S=ln(2)=0.69314718…

The use of an analytic function to sum the series gives one an important clue for generating a myriad of other infinite series via a MacLaurin series expansions of certain functions F(x) whose series are evaluated at x=0 or some other chosen point. Consider the function sin(x)/x. This expands as the series-

$$F(x)=\frac{\sin(x)}{x}=1-\frac{1}{3!}x^2+\frac{1}{5!}x^4+\frac{1}{7!}x^6+...=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}$$

At x=1 we get that-

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}=\sin(1)=0.84147...$$

Also one notices that sin(x)/x has zeros at x=±(nπ), so, as first noted by Leonard Euler, we can also write-

$$F(x)=\frac{\sin(x)}{x}=[1-(\frac{x}{\pi})^2][1-(\frac{x}{2\pi})^2][1-(\frac{x}{3\pi})^2]...=\prod_{n=1}^{\infty}[1-(\frac{x}{n\pi})^2$$

Hence at x=π/2 we get-

$$\frac{2}{\pi}=(1-\frac{1}{4})(1-\frac{1}{16})(1-\frac{1}{36})...=\prod_{n=1}^{\infty}(1-\frac{1}{(2n)^2})=\sum_{n=1}^{\infty}\frac{(-1)^n\pi^{2n}}{2^{2n}(2n+1)!}$$

We can also re-write this last expression as-

$$\frac{\pi}{2}=[\frac{2^2\cdot 4^2\cdot 6^2\cdot 8^2\cdot ...}{1^2\cdot 3^2\cdot 5^2\cdot 7^{2\cdot}\cdot ...}]$$

This form is known as the Wallis Formula.

Take next the Gaussian function and its MacLaurin series-

$$\exp(-x^2)=1-\frac{1}{1!}x^2+\frac{1}{2!}x^4-\frac{1}{3!}x^6+...=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^{2n}$$

At x=1 it produces the identity-

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n:}=\exp(-1)=0.367879...$$

We also have the arctan function which expands as the series-

$$\arctan(x)=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+...$$

At x=1 it generates the well known Gregory Formula-

$$\arctan(1)=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}...=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)}=\frac{\pi}{4}=0.785398..+$$

Let I us ask next what is the value of the infinite series-

$$\frac{1}{0!0!}+\frac{1}{2!2!}+\frac{1}{3!3!}+\frac{1}{4!4!}+...=\sum_{n=1}^{\infty}\frac{1}{(n)!^2}$$

It clearly converges because of the factorials in the denominator. To get its exact value we recall that the modified Bessel Function of the first kind of order zero reads-

$$I_0(x)=\sum_{n=0}^{\infty}\frac{(x/2)^{2n}}{(n!)^2}$$

So, on setting x=2 we get the sum to be $I_0(2)=2.27958..$

Some infinite series do not have corresponding F(x)s. In that case one must attack the series directly to gets its values. Take, as an example, the series-

$$S=\sum_{n=1}^{\infty}\frac{n^{-n}}{n!}=1+\frac{1}{8}+\frac{1}{162}+\frac{1}{6144}+...=1.131338296600626371...$$

This is an extremely rapidly converging series with no obvious F(x) equivalent. However, by just adding up the first ten terms in the series one already produces an 18 digit accurate value for the infinite series. The ratio test reads

$$R=\frac{n^n}{(n+1)^{n+2}}$$

It goes as $1/n^2$ as n approaches infinity.

Additional infinite convergent series and their values follow-

$$\sum_{n=0}^{\infty}\frac{\exp(-n)}{(n+1)^2}=1.11110935...$$

$$\sum_{n=1}^{\infty}\frac{(n-1)^2}{(2n+1)!}=1.7182818...$$

$$\sum_{n=1}^{\infty}\frac{(n^2+n+1)^2}{n(n!)^3}=12.3313389...$$

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