INVERSE FUNCTIONS

It is known that-

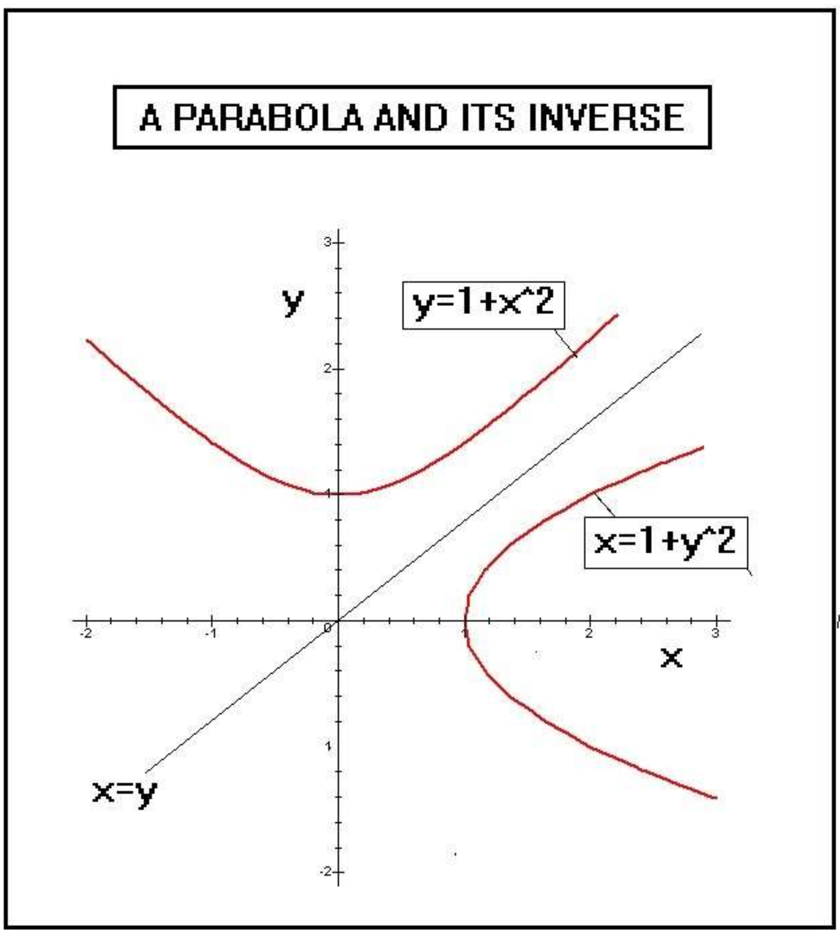
$$y=f(x)$$

can represent any 2D function such as a parabola y=x^2 or exponential function y=exp(x). What is less well understood is that f(x) has in many cases a unique inverse gotten by replacing x by y. It is our purpose here to obtain several inverses where these inverses exist.

Let us begin with the simple case of a parabola y=1+x^2. Replacing x by y produces- the inverse

$$x=1+y^2$$

A graph of this parabola and its inverse follows-



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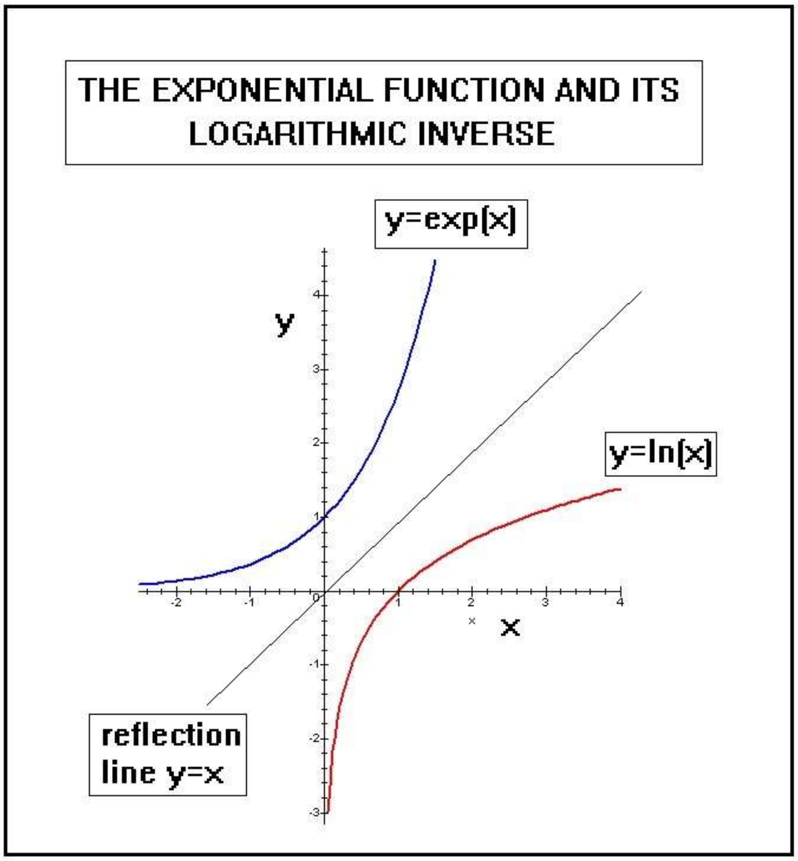
Note graphically that the inverse is achieved by reflecting things through the diagonal x=y.

Next we look at the exponential function y=f(x)=exp(x). Replacing x by y produces-

x=exp(y)   which is equivalent to   y=ln(x)

So exp(x) has ln(x) as its inverse. Here is the picture-



Historically this inverse was the first found after the invention of calculus by Newton and Leibnitz. The symmetry between the exponential and its inverse is obvious.

Consider next the tangent function y=f(x)=tan(x). Flipping things about the line x=y produces-

x=tan(y)   which is equivalent to   y=arctan(x)

A graph of these two functions follows-

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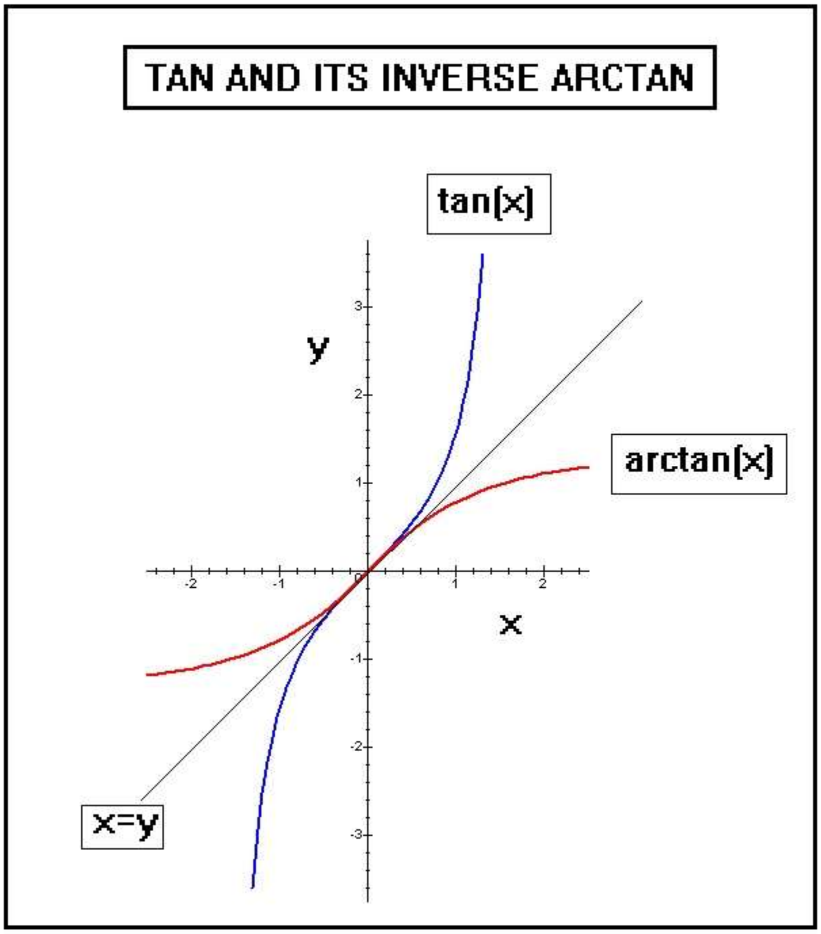

The arctan(x) function (shown in red) has found numerous applications including in the description of changes across a shock wave and in signal processing. A highly accurate approximation for arctan(x) can be found at-

https://mae.ufl.edu/~uhk/ARCTAN-APPROX-PAPER.pdf

Another function of interest in the literature is the LambertW(x) Function. Its inverse is f(x)=xexp(x). That is-

LambertW(xexp(x))=x

Here is the picture-

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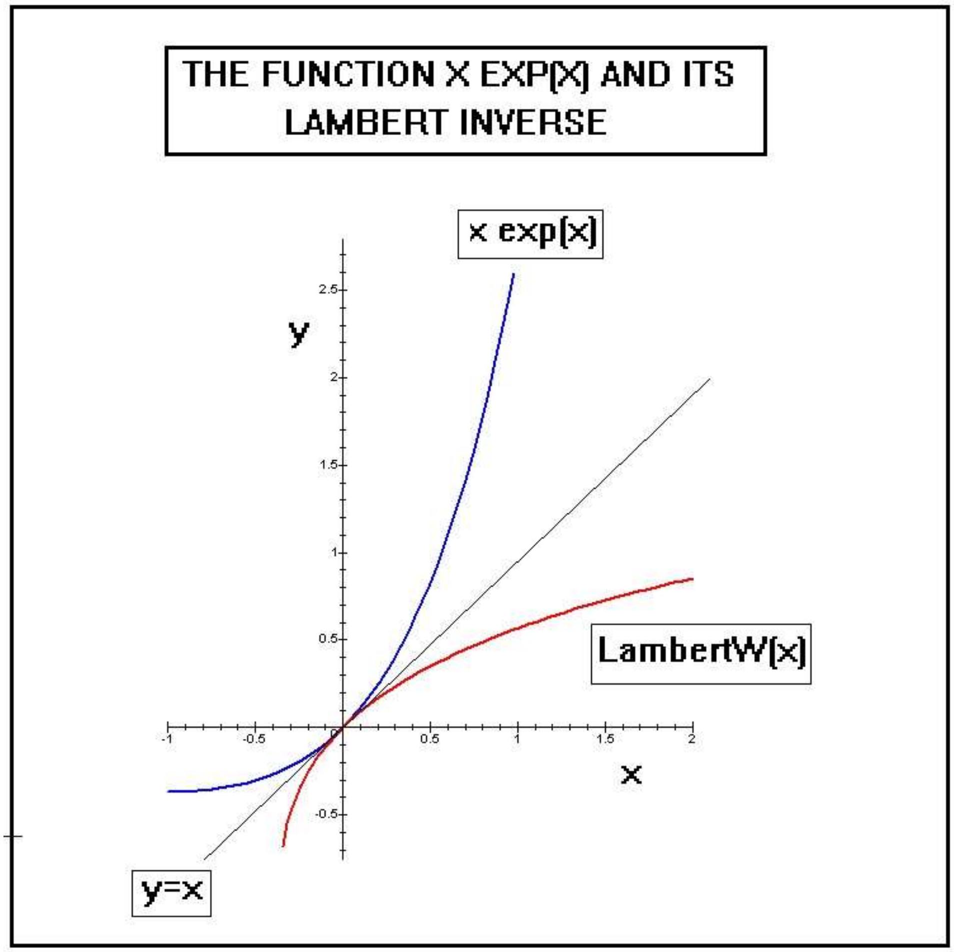

The Lambert Function is of interest in solving certain non-linear algebraic equations such as x^a=b^x.

As a final inversion let us look at the function-

f(x)=y=(x+1)/(x-1)

It has a singularity at x=1 and so we will confine our attention to the well behaved range of -infinity<x<1. Replacing x by y produces the inverse-

x=(y+1)/(y-1)

Here both functions are seen to coincide as the following graph indicates-

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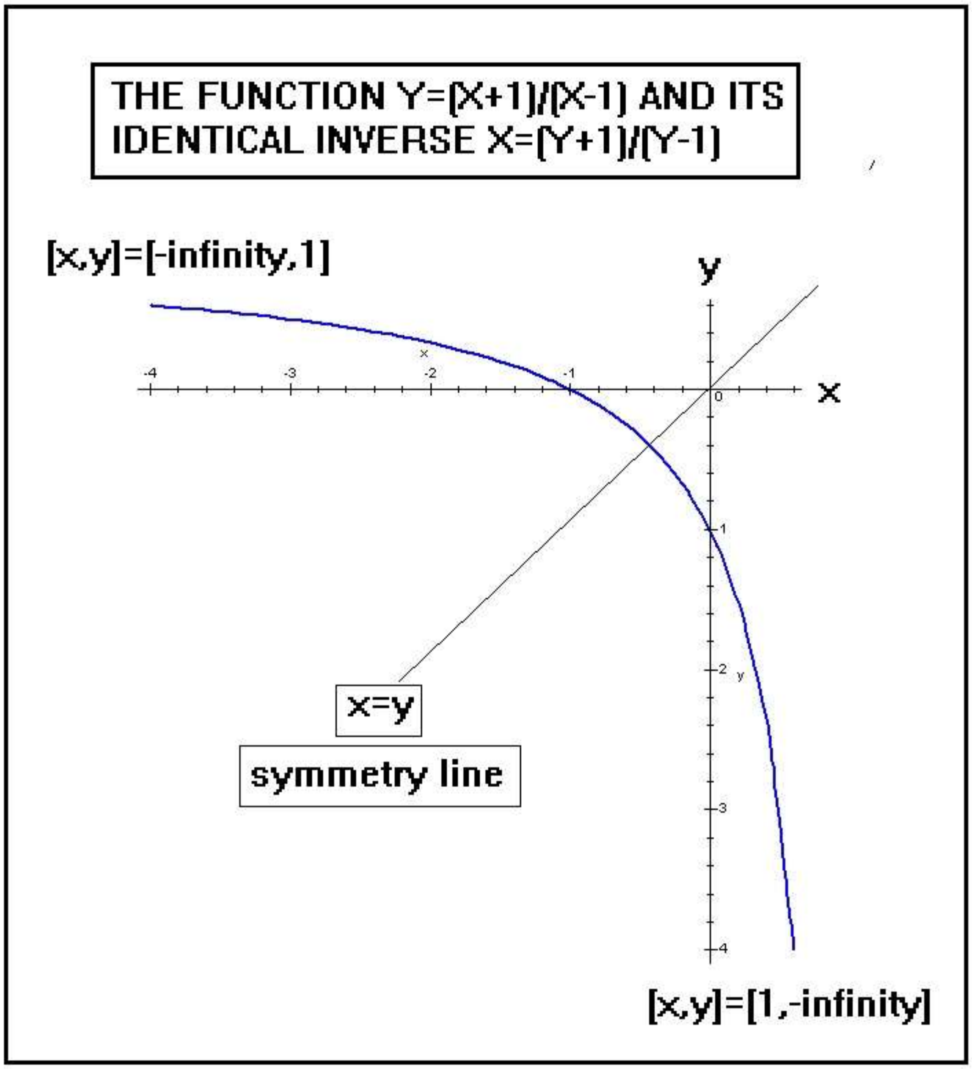

To find such functions where f(x)=f(x)^(-1) requires that the interchange of x with y does not alternate the original equation. The functions y=1/x and (x+y)^2=xy have this property. The function y=(x^2+1)/x however does not since its inverse is xy=y^2+1.

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